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STRESS TRANSMISSION THROUGH A COHESIONLESS MATERIAL

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Abstract. We apply a point load onto a bidimensional packing of elastic grains. The internal state of stress is accessed by means of photoelastic visualization. The region in which the stress response is confined exhibits a parabolic profile. This evidence supports the parabolic nature of the equations describing the stress transmission. Our results oppose classical elasto-plastic models of continuum mechanics, and other recent hyperbolic proposals.

1. INTRODUCTION

Within the frame of continuum mechanics, the behavior of a grain collection is described as elasto-plastic [1-3]. For soil mechanics or fundation problem purposes, a number of computational programs based on finite-element meshing have been devised, in which granular soils are considered as elastic below the Mohr-Coulomb yield criterion, and plastic at this threshold. Associated with the mechanical equilibrium equations $\partial_i\sigma_{ij} = \rho g_j$, where σ_{ij} is the stress tensor and ρg_j the weight per unit volume, the Mohr-Coulomb yield condition leads to a set of hyperbolic partial-differential equations for the stresses [4], whereas the stress field is described by elliptic equations in the elastic state.

Nevertheless, the insufficiency of continuum approaches has been recognized since a long time. The well-known photoelasticity measurements of Dantu [5,6] or of Drescher *et al.* [7] have emphasized the existence of privileged force paths for the force propagation in granular materials. More recently, a number of experimental works addressed the problem of stress fluctuations [8-11]. These observations encouraged the design of discrete models. Coppersmith *et al.* [12] introduced the "*q*-model", in which forces are taken to be scalar, and their transmission on the beads beneath is weighted by a random number *q* uniformly distributed between 0 and 1. Assuming the probability distribution function of the force to be stationary in the medium, they could successfully determine this distribution function analytically. The force distribution given by this ansatz appears to be in very good agreement with both experimental [8,10] and numerical [13] determinations in that it displays an exponential tail and the standard deviation of the force is of the same order of magnitude as the mean value. Nevertheless, the *q*-model of Coppersmith et al. is scalar, and therefore cannot ensure the balance of internal horizontal forces.

In order to account for the tensorial nature of the internal stresses, Socolar [14] introduced the so-called *a*-model, which deals with a lattice composed of square-cells in contact. Socolar adopted a micropolar approach, and therefore considered a torque field in addition to the force field. The stress equations are basically undetermined, and acceptable solutions (i.e. discarding tensile forces) are chosen at random. Then the series of equilibrium equations is sequentially solved downward from the top. Socolar's ansatz succeeds in accounting for the saturation of the pressure observed in deep silos, and leads to hyperbolic solutions, insensitive to the whole set of boundary conditions. Another interesting attempt to model stress propagation in granular materials is the "BCC-model" of Bouchaud et al. [15-18]. These authors *a priori* assumed that

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the horizontal (σ_{xx}) and vertical (σ_{yy}) stress components obeyed a relation σ_{yy}/σ_{xx} = const everywhere in the medium. Injecting this relation into the force balance equations $\partial_i\sigma_{ij} = \rho g_j$ straightforwardly leads to a set of hyperbolic partial-differential equations. Note that such models aimed to describe the nature of the stress equations below the onset of plastic failure, in the state classically qualified as elastic in the soil mechanics literature. Therefore their implications oppose to that of continuum theories, in which stress equations are elliptic, but they are (more or less) consistent with the continuum theory of Mohr-Coulomb plasticity, in which stress equations are hyperbolic.

Attempts to solve this inconsistency were recently made by Savage [2], and Tkachenko and Witten [19]. They emphasized the role played by the grain coordination number. Depending on the coordination, the number of balance equations is larger than, or equal to, the number of contact forces. Therefore the force system is either hyper- or isostatic. In the hyperstatic case, the present consensus is that it is necessary to introduce the grain compliance and the whole set of boundary conditions in order to solve the force indeterminacy, which would imply elliptic equations; this is in contrast to the isostatic case, which would entail the existence of a local relation between the contact forces (as in [20,21]) and therefore yield hyperbolic equations.

2. EXPERIMENTAL SETUP

In order to obtain insight in the nature of the stress equations far below the plastic deformation threshold, we performed point-punching experiments in a bidimensional packing. The point-punch test provides the Green function of the mechanical response of the medium to any external loading, and is the signature of the nature of the stress equations. The packing is made up of a collection of elastic squares in contact, and bounded by a metal frame. The grains are square-shaped, and their vertices have been trimmed. They are made of polycarbonate elastomer, and their Young modulus and Poisson coefficient are respectively $Y = 3.15\cdot10^5$ Pa and $\nu = 0.36$. The dimension of each square grain is 15 mm x 15 mm and the thickness is 5 mm. The edges of each grain are polished, in order to reduce the interparticle friction. The punch consists of a steel sphere of 3 mm diameter and of Young modulus $Y = 2\ 10^{11}$ Pa. The bidimensional packing is partially disordered in the following way: each linear array, constituting a layer, is regular, but there is a varying shift with the underlying layer (Fig. 1). Owing to tip effects, the real contacts experienced by each square cell are preferentially located at its vertices. Hence, the present tiling is convenient to model the stress propagation through different disordered contact networks, simply by varying the shift between adjacent layers. The sample is then positioned between two polarizers at right angles, in order to perform photoelastic visualization of the strain state.

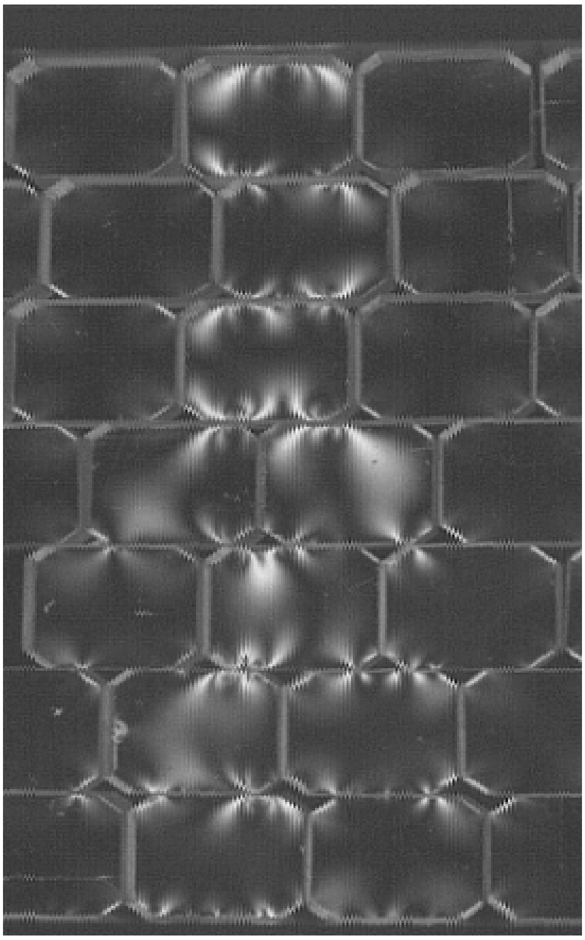

Fig. 1. Typical photoelastic view of a packing submitted to a point load of 20 *N*. Each packing is made up of regular arrays of square grains; there is an arbitrary shift between adjacent arrays. Owing to tip effects, the real contacts that grains experience are preferentielly located at their vertices.

3. RESULTS AND ANALYSIS

Fig. 1 shows a photoelastic view of the point-loaded packing: the load P is typically 20 *N*. When varying the packing structure we obtain different patterns for the internal stress. It is interesting to perform an ensemble average of such stress patterns. Fig. 2 presents the average of 10 digitized images corresponding to different packings. It clearly appears that the strained region is bounded by a parabola. This is a remarkable result, since it refutes both the classical elasticapproach and recent alternative hyperbolic proposals.

The response of a semi-infinite elastic medium to a point normal force was first given by Boussinesq [22]. In two dimensions, if the origin (of the polar coordinate system r, θ) is taken as the point of application of the load P, the stress is everywhere radial, and its magnitude is given by $\sigma_r(\boldsymbol{r}) = (2P/\pi)(\cos\theta/r)$. Hence the curves of constant stress magnitude are a set of circles passing through the point of application of the force (in two dimensions). In three dimensions, the constant stress surfaces approximately look like ellipsoids. On the other hand, hyperbolic models would imply the existence of two bright rays, arising from the point load, and propagating along two symmetrically inclined characteristic directions. Therefore our experimental observations rule out all models giving rise to elliptic or hyperbolic stress equations, and support a diffusive model for the stress transmission. From Fig. 2, we deduce a diffusion coefficient $D = 0.45 \pm 0.05$ in grain-size units.

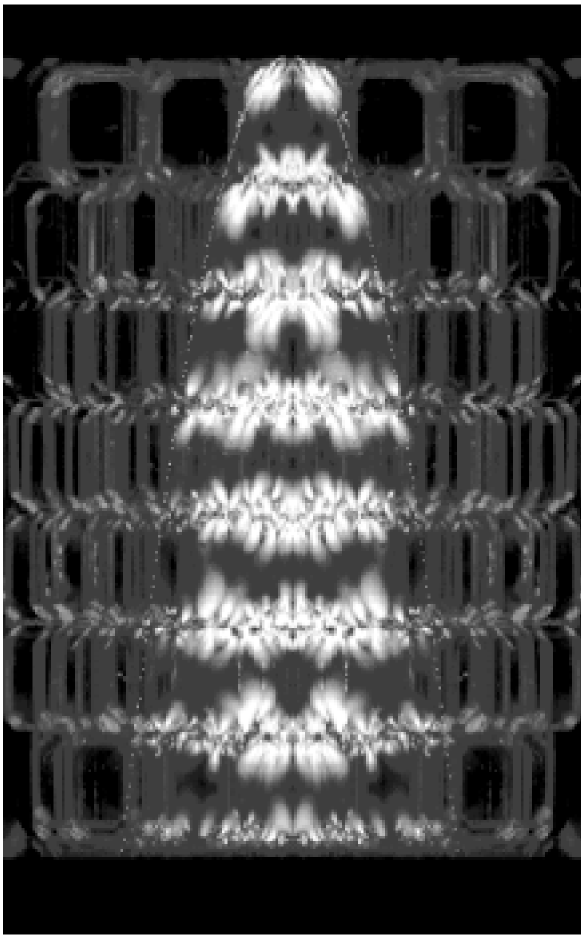

Fig. 2. Superimposition of ten digitized pictures. Each packing corresponds to a different arbitrary shift between horizontal adjacent layers. The strained region appears to be bounded by a parabola that we represent by a dashed line, and which is a guide for the eyes. The bounding parabola obeys the equation $x = \sqrt{2Dy}$ (with $D = 0.45 \pm 0.05$ grain size).

If the results obtained from our model system can be extrapolated to real geomaterials, one may be surprised that such a confusion between parabolic and elliptic behaviors has been going on for half a century*. It is nevertheless worth noting that in the case of granular layers submitted to a point load P, the elastic theory leads to a bell-shaped pressure, like the gaussian profile predicted by the diffusive model. This kinship is certainly responsible for the successes of computations based on elastic finite-element meshing and used in civil engineering.

The discrepancy of our results with both the classical elliptic description and hyperbolic models calls for some comments. First, the inadequacy of the classical elastic descriptions obviously arises from the alteration in the stress transmission introduced by the unilateral character of the contact force. Moreover, in cohesionless materials, forces are transmitted through point contacts, unlike the finite-element meshing of continuous media. Besides, the diffusion behavior that we evidenced from static measurements is fully consistent with the results of sound transmission experiments [23]. Lastly, it is worth recalling that the evidenced parabolic behavior corresponds to an ensemble average over numerous pile configurations. This result upholds stochastic descriptions such as the "q-model" [13]. One may be surprised by the success of such a simplified model, although it violates the balance of the horizontal force components in its original version [13]. Actually, we argue that this requirement is fulfilled when disordered packings are considered. Such topological disorder is well featured by the experimental system presented here, and is also an essential property of real geomaterials. For the sake of simplicity, consider an isostatic, frictionless, disordered system [24-26]. An unequal transmission factor q of the vertical force component naturally ensues, which is fully compatible with the

* Of course, a hyperbolic behavior is expected to be recovered when the Mohr-Coulomb plastic failure yield is attained.

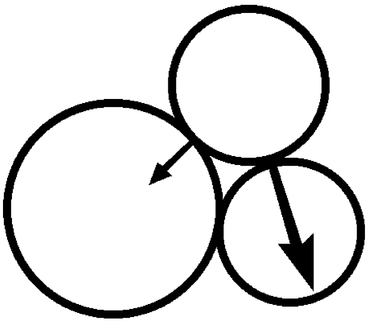

Fig. 3. Equilibrium of three frictionless spherical grains. The position disorder naturally yields an unequal transmission factor q on the grains beneath.The q-distribution is shown here to result from the topological disorder of the packing, and is compatible with the force equilibrium.

force balance (see Fig. 3). Of course, the validity of the q-model hypothesis, which states an uncorrelated series of coefficients q along the force transmission tree, has to be requestioned. In the particular case of textured materials, correlated and biased q-series might be envisaged.

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