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DYNAMIC GROWTH OF A SPHERICAL INCLUSION IN THERMOELASTIC MEDIUM

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Abstract. The paper deals with an initial value problem of dynamic uncoupled thermoelasticity concerning a moving spherical thermal inclusion in an infinite solid. An extended Kosevich' theory of continuously distributed defects due to prescribed plastic fields is used. Applying a generalization of the isothermal elastodynamics with continuosly distributed defects, the displacement and stress fields due to a spherical thermal inclusion growing linearly with time are obtained.

1. INTRODUCTION

The dynamical thermal stresses produced by a spherical temperature inclusion in an infinite body were analyzed by Piechocki and Ignaczak [1] 1960. These authors assumed that a temperature field is constant inside of a sphere and vanishes outside of the sphere; and it changes in time as the Dirac impulse function or as the Heaviside step function. For such a temperature they obtained a closed form of dynamical thermal stresses in the body.

The fact that a discontinuous temperature field can be treated as an inclusion, and the methods developed by the theory of defects can be applied to it, were observed by Muskhelishvili [2] in 1916. These ideas can be also found in the book by Maysel [3] and in a paper by Willis [4].

Willis [4] analyzed an isothermal process of elastodynamics corresponding to a given deformation field on a finite region of $\boldsymbol{E}^3$. In particular, he investigated the stress waves produced by a spherical strain inclusion in $\boldsymbol{E}^3$.

Wang, Sun and Xiao [5] analyzed a solution analogous to that of Willis in connection with discussion of the martensitic phase transitions.

In the present paper we discuss the displacement and stress fields due to the expansion of a spherical temperature inclusion in an infinite body.

The dynamical response of a body to an isothermal mechanical inclusion or a temperature inclusion can be described by the extended theory of defects due to given plastic fields developed by Kosevich [6-8].

If the total displacement field of a solid with defects is described by a vector u_i, then the distortion tensor w_{ij} is defined as, *cf.* Kosevich [6-8], Landau and Lifshits [9], Ignaczak and Rao [10]

$$w_{ij} \equiv u_{j,i}. \tag{1.1}$$

In general $w_{ij} \neq w_{ji}$ and the following equality holds true

$$\varepsilon_{ijm}\, w_{mk,j} = -\rho_{ik}, \tag{1.2}$$

where ρ_{ik} denotes the dislocation density tensor, satisfying the condition $\rho_{ik,i} = 0$ which for a single dislocation expresses the conservation of a Burgers vector along the dislocation line.

A total distortion tensor w_{ij} may be postulated in the form

$$w_{ij} = w_{ij}^{E} + w_{ij}^{P} + w_{ij}^{\Theta}, \tag{1.3}$$

where w_{ij}^{E}, w_{ij}^{P} and w_{ij}^{Θ} represent the elastic, plastic and thermal parts of the distortion, respectively, and neither w_{ij}^{E}, w_{ij}^{P}, or w_{ij}^{Θ} are given by the gradient of a vector field. Hence the motion of a dislocation is related not only to a plastic deformation but also to a thermal one.

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2. ELASTIC FIELDS DUE TO A THERMAL INCLUSION

Let a nonhomogeneous anisotropic linear thermoelastic body, occupying a three-dimensional region Ω, be subject to a dynamic thermomechanical load. Let the displacement be described by a vector $u_i = u_i(\boldsymbol{x},t)$ and the stress by $s_{ij} = s_{ij}(\boldsymbol{x},t)$. The temperature field is assumed to be known and is denoted by $\Theta = \Theta(\boldsymbol{x},t)$. If T denotes the absolute temperature, we have $\Theta \equiv T - T_0$ where T_0 denotes a reference temperature.

The equation of motion has the usual form

$$\rho \ddot{u}_i = s_{ik,k} + b_i \,, \tag{2.1}$$

where $\rho = \rho(\boldsymbol{x})$ and $b_i = b_i(\boldsymbol{x})$ denote the mass density and body force vector fields.

The following boundary condition is adjoined to (2.1)

$$s_{ij} n_j = p_i \quad \text{on} \quad \partial\Omega \,, \tag{2.2}$$

where B_i is a prescribed function. The total strain is defined as

$$e_{ij} \equiv \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.3}$$

and it is composed of three parts

$$e_{ij} = e_{ij}^E + e_{ij}^P + e_{ij}^\Theta \,, \tag{2.4}$$

where e_{ij}^E, e_{ij}^P and e_{ij}^Θ denote the elastic and plastic strain fields.The stress s_{ij} is related to the strain e_{ij}^E and a temperature Θ by

$$s_{ij} = C_{ijnm} e_{mn}^E - \gamma_{ij}\Theta \,, \tag{2.5}$$

where $C_{ijmn} = C_{ijmn}(\boldsymbol{x})$ is the elastic tensor, and

$$e_{ij}^E \equiv e_{ij} - e_{ij}^P \,. \tag{2.6}$$

Moreover, γ_{ij} is related to a stress-temperature tensor M_{ij} by $\gamma_{ij} = -M_{ij}$.

From (2.1), (2.4), (2.5) and (2.2)

$$(C_{ijmn} e_{mn})_{,j} - \rho \ddot{u}_i = -f_i \,, \tag{2.7}$$

where

$$f_i = b_i - (C_{ijmn} e_{mn}^P + \gamma_{ij}\Theta)_{,j} \tag{2.8}$$

is a modified body force.

In further, we will assume that the medium is infinite $\Omega = \boldsymbol{E}^3$ and $\boldsymbol{b} = \boldsymbol{0}$. Moreover, the boundary condition is absent $\boldsymbol{p} = \boldsymbol{0}$ and the plastic deformation is absent $\boldsymbol{e}^P = \boldsymbol{0}$.

3. SPHERICAL THERMAL INCLUSION

We consider an infinite homogeneous thermoelastic body in which at the instant $t = 0$ certain spherical volume of radius a is suddenly heated to a temperature Θ^0 and the radius R of the heated volume expands linearly in time with a speed $v > 0$

$$R = a + v\,t, \tag{3.1}$$

Thus the thermal field is given by the relation

$$\Theta = \Theta(r,t) = \Theta^0 H(a + vt - r)\, H(t), \tag{3.2}$$

where Θ^0 is a constant and $H(t)$ denotes the Heaviside step function.

The plastic deformations are assumed to be absent and in view of spherical symmetry only one component of the displacement, the radial one in the spherical coordinates $u_r \equiv u$ does not vanish and only one equation of motion is to be solved, *cf.* Nowacki (1975) [11]

$$u_{,rr} + \frac{2}{r}u_{,r} + \frac{2}{r^2}u - \sigma^2 \ddot{u} = m\Theta_{,r} \,, \tag{3.3}$$

where

$$\sigma^2 = \frac{1}{c_1^2} = \frac{\rho}{\lambda + 2\mu},$$

$$m = \frac{\gamma}{\lambda + 2\mu},$$

$$\gamma = (3\lambda + 2\mu)\alpha$$

with λ and μ denoting the Lamé constants and α standing for the coefficient of linear thermal expansion. The stress $s \equiv s_{rr}$ is expressed by

$$s = 2\mu u_{,r} + \lambda\left(u_{,r} + \frac{2}{r}u\right) - \gamma\Theta \equiv$$

$$2\mu \frac{1-\nu}{1-2\nu}\left[u_{,r} + \frac{2\nu}{1-2\nu}\frac{u}{r}\right] - \gamma\Theta \,, \tag{3.4}$$

where ν denotes the Poisson ratio, and $\lambda = 2\mu(\nu / 1 - 2\nu)$.

If we choose the length and time units, $\breve{x}$ and $\breve{t}$, respectively, in such a way that $\breve{x}/\breve{t} = c_1$, then by letting

$$u = \Phi_{,r} \,, \tag{3.5}$$

where Φ stands for a displacement potential, and using (3.3) we obtain

$$\Phi_{,rr} + \frac{2}{r}\Phi_{,r} - \ddot{\Phi} = \Theta$$

or

$$(r\Phi)_{,rr} - r\ddot{\Phi} = r\,\Theta. \tag{3.6}$$

To Eq. (3.6) the initial conditions are adjoined

$$\Phi_{t=0} = 0, \quad \dot{\Phi}_{t=0} = 0. \tag{3.7}$$

The Laplace transform of Eq. (3.6) reads

$$\left(\frac{\partial^2}{\partial r^2} - p^2\right)(r\,\overline{\Phi}) = r\,\overline{\Theta}, \tag{3.8}$$

where

$$\overline{\Phi}(r,p) = \int_0^\infty e^{-pt}\Phi(r,t)\,dt$$

is the Laplace transform of $\Phi(r,t)$.

4. GREEN FUNCTION

We consider an auxiliary problem

$$\left(\frac{\partial}{\partial r^2} - p^2\right)(r\,\overline{\Phi}^*) = r\delta(r - r_0) \equiv r_0\delta(r - r_0). \tag{4.1}$$

The solution of this equation reads, *cf.* Nowacki [11]

$$r\,\overline{\Phi}^* = -\frac{2r_0}{\pi}\int_0^\infty \frac{\sin\alpha r_0 \sin\alpha r}{\alpha^2 + p^2}\,d\alpha \tag{4.2}$$

or

$$r\,\overline{\Phi}^* = \frac{r_0}{\pi}\left[\int_0^\infty \frac{\cos\alpha(r + r_0)}{\alpha^2 + p^2}\,d\alpha - \int_0^\infty \frac{\cos\alpha(r - r_0)}{\alpha^2 + p^2}\,d\alpha\right]$$

or

$$\overline{\Phi}^* = \frac{r_0}{r}\frac{1}{2p}\left[e^{-p(r + r_0)} - e^{-p|r - r_0|}\right]. \tag{4.3}$$

This is a unique solution of (4.1) for the infinite space.

5. SOLUTION OF THE PROBLEM

The appropriate displacement potential has the form

$$\overline{\Phi}(r,p) = \int_0^\infty \overline{\Phi}^*(r, r_0, p)\,\overline{\Theta}(r_0, p)\,dr_0\,, \tag{5.1}$$

where

$$\overline{\Theta}(r,p) = \Theta^0\,\frac{1}{p}\exp\left(-p\frac{r - a}{v}\right) \tag{5.2}$$

is the Laplace transform of (3.2). Hence

$$\overline{\Phi}(r,p) = \frac{\Theta^0}{r}\frac{v^2}{1 - v^2}\left\{\frac{2v}{1 - v^2}\left(-\frac{1}{p^4}\exp\left[-p\left(r - \frac{a}{v}\right)\right] + \frac{1}{p^4}\exp\left(-p\frac{r - a}{v}\right)\right) + r\frac{1}{p^3}\exp\left(-p\frac{r - a}{v}\right)\right\}. \tag{5.3}$$

Applying the inverse Laplace transformation we get

$$\Phi(r,t) = \frac{\Theta^0}{r}\frac{v^2}{1 - v^2}\times \left\{\frac{v}{1 - v^2}\frac{1}{3}\left[-\left(t - \frac{rv - a}{v}\right)^3 H\left(t - \frac{rv - a}{v}\right) + \left(t - \frac{r - a}{v}\right)^3 H\left(t - \frac{r - a}{v}\right)\right] + r\frac{1}{2}\left(t - \frac{r - a}{v}\right)^2 H\left(t - \frac{r - a}{v}\right)\right\}. \tag{5.4}$$

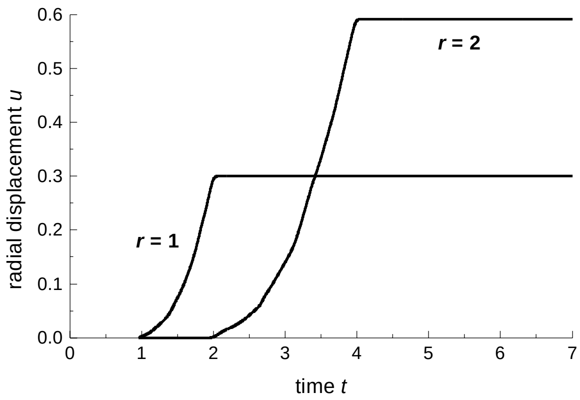

Fig. 1. The radial displacement u at $r = 1$ and $r = 2$ as a function of time t for $v = 1/2$ (for $\Theta^0 = 1$, $a = 0$ and $\nu = 1/3$).

The displacements are determined by (3.5) and they are plotted in Figs. 1-4 for $\Theta^0 = 1$ and $a = 0$.

To obtain the radial stress s, the dimensionless Duhamel-Neumann relation (3.4) is used

$$s = u_{,r} + \frac{2\nu}{1 - \nu}\frac{u}{r} - \Theta. \tag{5.5}$$

The temperature and stress units are selected as

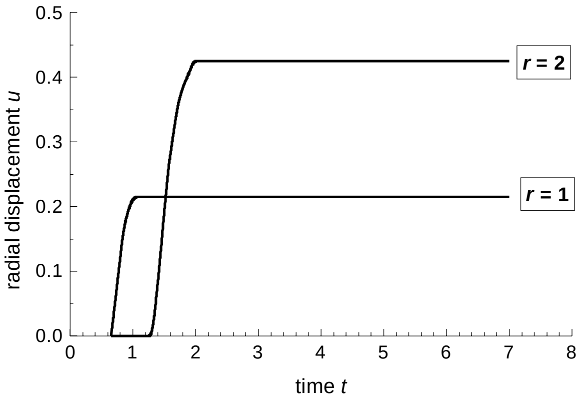

Fig. 2. The radial displacement u at r = 1 and r = 2 as a function of time t for v = 3/2 (for Θ^0 = 1, a = 0 and ν = 1/3).

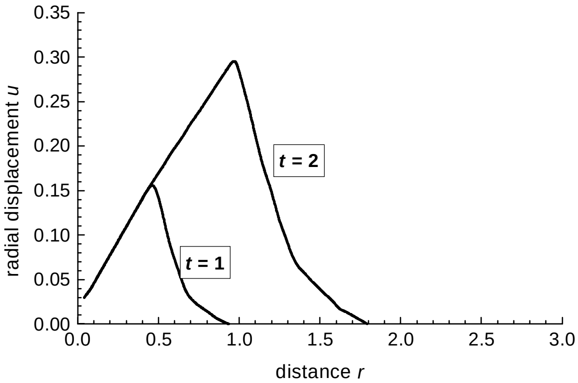

Fig. 3. The radial displacement u at t =1 and t = 2 as a function of distance r for v = 1/2 (for Θ^0 = 1, a = 0 and ν = 1/3).

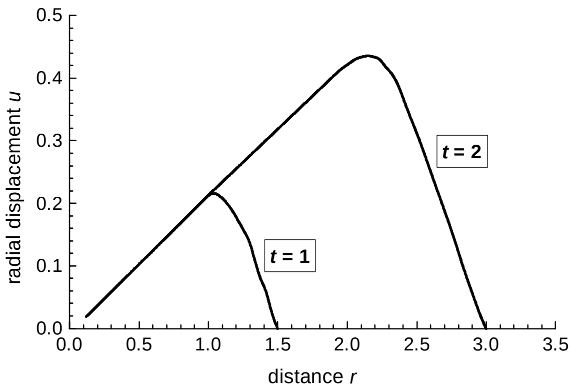

Fig. 4. The radial displacement u at t = 1 and t = 2 as a function of distance r for v = 3/2 (for Θ^0 = 1, a = 0 and ν = 1/3).

$$\breve{\Theta} = (\lambda + 2\mu)\gamma^{-1}$$

and

$$\breve{s} = \lambda + 2\mu \equiv 2\mu\frac{1-\nu}{1-2\nu},$$

respectively.

The stress behaviour is shown in Figs. 5 and 6.

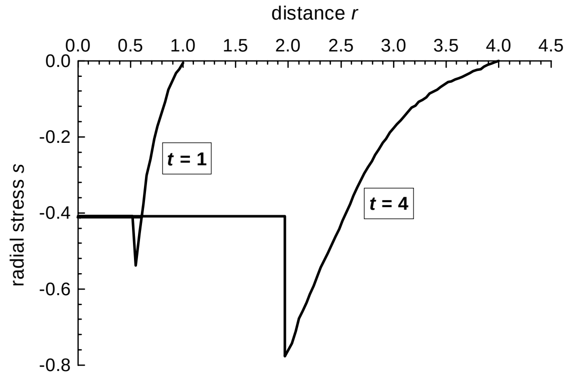

Fig. 5. The radial stress s at t = 1 and t = 4 as a function of distance r for v = 1/2 (for Θ^0 = 1, a = 0 and ν = 1/3).

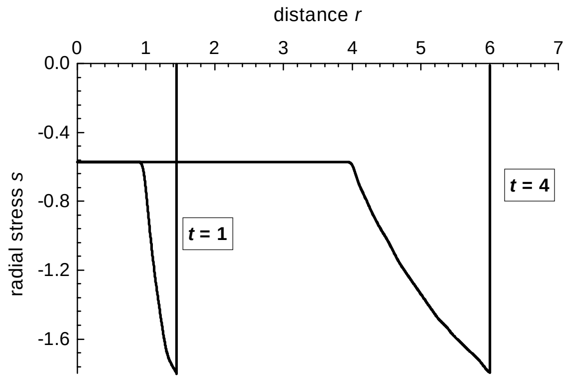

Fig. 6. The radial stress s at t = 1 and t = 4 as a function of distance r for v = 3/2 (for Θ^0 = 1, a = 0 and ν = 1/3).

6. DISCUSSION

We observe that the potential Φ from (5.4) generates the radial displacement $u = u(r,t)$ and the radial stress $s = s(r,t)$ which are finite for every $r > a$, $t > 0$.

The two wave fronts of the solution are clearly seen: one corresponding to the temperature inclusion boundary moving with the velocity v and the second moving with a unit velocity and corresponding to the longitudinal elastic waves.

The displacement u is a continuous function for every $r > a$, $t > 0$; while the stress s exhibits a finite jump on the r-axis corresponding to the spherical wave front $r = t\,v$.

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