PLANE WAVE AND FUNDAMENTAL SOLUTION IN THERMOPOROELASTIC MEDIUM

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Abstract. The present article deals with the study of propagation of plane wave and fundamental solution in the thermoporoelastic medium. It is found that for two dimensional model, their exist three longitudinal waves, namely $P_1$-wave, $P_2$-wave and T-wave in addition to transverse wave. Characteristics of waves like phase velocity, attenuation coefficient, specific loss and penetration depth are computed numerically and depicted graphically. The representation of the fundamental solution of the system of equations in the thermoporoelastic medium in case of steady oscillations is considered in term of elementary functions. Some basic properties of the fundamental solution are established. Some special cases are also deduced.

Keywords: plane wave; fundamental solution; thermoporoelastic medium; steady oscillations.

1. Introduction
Poroelasticity is the mechanics of poroelastic solids with pores filled with fluid. Mathematical theory of poroelasticity deals with the mechanical behaviour of fluid saturated porous medium. Pore fluid generally includes gas, water and oil. Due to different motions of solid and fluid phases and complicated geometry of pore structures, it is very difficult to study the mechanical behaviour of a fluid saturated porous medium. The discovery of fundamental mechanical effects in saturated porous solids and the formulation of the first porous media theories are mainly due to Fillunger [1], Terzaghi [2,3,4] and their successors.

Based on the work of Von Terzaghi [2,3], Biot [5] proposed a general theory of three dimensional consolidation. Taking the compressibility of the soil into consideration, the water contained in the pores was taken to be incompressible. Biot [6,7] developed the theory for the propagation of stress waves in porous elastic solids containing a compressible viscous fluid and demonstrated the existence of two types of compressional waves (a fast and a slow wave) along with one share wave. Biot’s model was broadly accepted and some of his results have been taken as standard references and the basis for subsequent analysis in acoustic, geophysics and other such fields.

For the thermoporoelasticity problems, coupled thermal and poro-mechanical processes play an important role in a number of problems of interest in the geomechanics such as stability of boreholes and permeability enhancement in geothermal reservoirs. A thermoporoelastic approach combines the theory of heat conduction with poroelastic constitutive equations and coupling the temperature fields with the stresses and pore pressure.

Rice and Cleary [8] presented some basic stress-diffusion solutions for fluid saturated elastic porous media with compressible constituents. There exists a substantial literature treating the extension of the well known isothermal theory to account for the effects of thermal
expansion of both the pore fluid and the elastic matrix [eg. Schiffman [9], Bowen[10], Noorishad[11]].

McTigue [12] developed a linear theory of fluid saturated porous thermoelastic material and this theory allows compressibility and thermal expansion of both the fluid and solid constituents. He presented a general solution scheme in which a diffusion equation with temperature dependent source term governs a combination of the mean total stress and the fluid pore pressure.

Kurashige [13] extends the Rice and Cleary [8] theory to incorporate the heat transportation by a pore fluid flow in addition to the effect of difference in expansibility between the pore fluid and the skeletal solid and presented a thermoelastic theory of fluid-filled porous materials. This theory shows that the displacement field is completely coupled with the pore pressure and temperature field in general, however, for irrotational displacement, the first field is decoupled from the last two, which are still coupled to each other. This pore pressure-temperature coupling involves nonlinearity.


Jabbari and Dehbani [17] considered the classical coupled thermoporoelastic model of hollow and solid cylinders under radial symmetric loading conditions and presented a unique solution. Ganbin et al. [18] obtained the solution in saturated porous thermoviscoelastic medium, with cylindrical cavity that is subjected to time dependent thermal load by using Laplace transform technique. Gatmiri et al. [19] presented the two-dimensional fundamental solutions for non-isothermal unsaturated deformable porous medium subjected to quasistatic loading in time and frequency domain. Li et al. [20] presented the study state solutions for transversely isotropic thermoporoelastic media in three dimensions.

Jabbari and Dehbani [21] considered the quasi-static porothermoelasticity model of hollow and solid sphere and obtained the displacement, temperature distribution and pressure distribution due mechanical, thermal and pressure source. Liu et al. [22] studied the relaxation effect of a saturated porous media using the two dimensional generalized thermoelastic theory. Belotserkovets and Prevost [23] obtained an analytical solution of thermoporoelastic response of fluid-saturated porous sphere.

Bai [24] derived an analytical method for the thermal consolidation of layered saturated porous material subjected to exponential decaying thermal loading. Mixed variation principal for dynamic response of thermoelastic and poroelastic continua was discussed by Apostolakis and Dargus [25]. Hou et al. [26] discussed the three dimensional Green’s function for transversely isotropic thermoporoelastic biomaterial. Jabbari et.al. [27] presented the thermal buckling analysis of functionally graded thin circular plate made of saturated porous material and obtained the closed form solutions for circular plates subjected to temperature load.


Svanadze [32, 33, 34] constructed the fundamental solutions in thermoelasticity with microtemperature and micromorphic elastic solid with microtemperature. Svanadze and his co-workers [35, 36, 37, 38] also constructed the fundamental solution and basic properties in thermomicrostretch, micropolar thermoelasticity without energy dissipation and full coupled theory of elasticity for solids with double porosity.

Kumar and Gupta [44] studied the plane wave propagation in an anisotropic thermoelastic medium with fractional order derivative and void. Sharma and Kumar [45] studied the propagation of Plane waves and fundamental solution in thermoviscoelastic medium with voids. Plane wave propagation in microstretch thermoelastic medium with microtemperature was studied by Kumar and Kaur [46]. Fundamental and plane wave solution in swelling porous medium was studied by Kumar et al. [47].

Scarpetta et al. [48] constructed the fundamental solution in the theory of thermoelasticity for solids with double porosity. Kumar and Gupta [49] studied the Plane wave propagation and domain of influence in fractional order thermoelastic material with three phase lag heat transfer.

The present study deals with the study of propagation of plane waves and fundamental solution in the thermoporoelastic medium. Characteristics of waves like phase velocity and attenuation coefficient, specific loss and penetration depth are computed numerically and depicted graphically. The representation of the fundamental solution of the system of equations in the thermoporoelastic medium in the case of steady oscillations is considered in term of elementary functions.

2. Basic Equations

Following Jabbari and Dehbani [50], the field equations are given by

\[(\lambda + \mu)\nabla \nabla \cdot \mathbf{u} + \mu \nabla^2 \mathbf{u} - \alpha \nabla p - \beta \nabla T = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}, \quad (1)\]

\[\frac{k}{\gamma_w} \nabla^2 p - \alpha_p \nabla \dot{p} - Y \dot{T} = \alpha c \nabla \dot{u} = 0, \quad (2)\]

\[K \nabla^2 T - Z (\nabla T \cdot \dot{T} + \gamma_w \nabla \cdot \dot{u}) - \beta \nabla T_0 \nabla \cdot \dot{u} = 0, \quad (3)\]

where \(\mathbf{u}\) is the displacement component, \(p\) is the pore pressure, \(\rho\) is the bulk mass density, \(\alpha = 1 - \frac{C_s}{C}\) is the Biot’s coefficient, \(C_s = 3(1 - 2\nu_s) / E_s\) is the coefficient of volumetric compression of solid grain, with \(E_s\) and \(\nu_s\) being the elastic modulus and Poisson’s ratio of solid grain, \(C = 3(1 - 2\nu) / E\) is the coefficient of volumetric compression of solid skeleton, with \(E\) and \(\nu\) being the elastic modulus and Poisson’s ratio of solid skeleton, \(T_0\) is initial reference temperature, \(\beta = \frac{3\alpha_s}{C}\) is the thermal expansion factor, \(\alpha_s\) is the coefficient of linear thermal expansion of solid grain, \(Y = 3(n\alpha_w + (\alpha - n)\alpha_s)\) and \(\alpha_p = n(C_w - C_s) + \alpha C_s\) are coupling parameters, \(\alpha_w\) and \(C_w\) are the coefficients of linear thermal expansion and volumetric compression of pore water, \(n\) is the porosity, \(k\) is the hydraulic conductivity, \(\gamma_w\) is the unit of pore water and \(Z = \frac{(1-n)p_s c_s + n \rho_w c_w}{T_0}\) is coupling parameter, \(\rho_w\) and \(\rho_s\) are densities of pore water and solid grain and \(c_w\) and \(c_s\) are heat capacities of pore water and solid grain and \(K\) is the coefficient of heat conductivity.

3. Formulation of the problem

We consider a homogeneous thermoporoelastic medium. For two dimensional problems, we take displacement vector \(\mathbf{u}\) as:

\[\mathbf{u} = (u_1, 0, u_2)\]

We define the dimensionless quantities as:

\[x' = \frac{\omega^* x}{c_1}, \quad u' = \frac{\omega^* p c_1}{\beta T_0}, \quad p' = \frac{p}{\beta T_0}, \quad c_s^2 = \frac{\lambda + 2\mu}{\rho}, \quad \frac{t'}{T_0}, \quad \omega^* = \frac{Z T_0 c_1^2}{K}, \quad (5)\]
\( i = 1, 2, 3 \).

The displacement components \( u_1 \) and \( u_3 \) are related to the potential functions \( \Phi(x_1, x_3, t) \), \( \Psi(x_1, x_3, t) \) as:

\[
u_i = \frac{\partial \Phi}{\partial x_1} - \frac{\partial \Psi}{\partial x_3}, \quad u_3 = \frac{\partial \Phi}{\partial x_3} + \frac{\partial \Psi}{\partial x_1}
\]  

(6)

Using equation (4) on (1)-(3) and applying the dimensionless quantities defined by (5), with the aid of (6), after suppressing the prime, yield

\[
\nabla^2 \Phi - \alpha p - T - \frac{\partial^2 \Phi}{\partial t^2} = 0,
\]

(7)

\[
\delta^2 \nabla^2 \Psi - \frac{\partial^2 \Psi}{\partial t^2} = 0,
\]

(8)

\[
b_1 \nabla^2 p - b_2 \frac{\partial p}{\partial t} - b_3 \frac{\partial T}{\partial t} - \frac{\partial}{\partial t} [\nabla^2 \Phi] = 0,
\]

(9)

\[
b_4 \nabla^2 T - b_5 \frac{\partial T}{\partial t} + b_6 \frac{\partial \Phi}{\partial t} - \frac{\partial}{\partial t} [\nabla^2 \Psi] = 0,
\]

(10)

where

\[
\delta^2 = \frac{\mu}{\kappa + 2 \mu}, \quad b_1 = \frac{k \omega^2 p}{\gamma \omega^2}, \quad b_2 = \frac{\alpha \rho c^2}{\kappa}, \quad b_3 = \frac{\gamma p c^2}{\kappa}, \quad b_4 = \frac{k \omega^2 p}{\rho^2 T_0}, \quad b_5 = \frac{z p c^2}{\beta^2}, \quad b_6 = \frac{\gamma p c^2}{\beta^2}
\]

and \( e = \frac{\delta u_3}{\delta x_1} + \frac{\delta u_3}{\delta x_3}, \quad \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \).

### 4. Solution of Plane waves

For Plane harmonic waves, we assume the solution of the form:

\[
(\Phi, \Psi, p, T) = (\Phi, \Psi, \tilde{p}, \tilde{T}) e^{i[(l_1 x_1 + l_3 x_3) - \omega t]},
\]

(11)

where \( \omega (= \xi c) \) is the frequency and \( \xi \) is the wave number and \( c \) is the phase velocity. \( \Phi, \Psi, \tilde{p}, \tilde{T} \) are undetermined constants that are independent of time \( t \) and coordinates \( x_1, x_3 \), \( l_1 \) and \( l_3 \) are the direction cosines of the wave normal to the \( x_1 x_3 \)-plane with the property \( l_1^2 + l_3^2 = 1 \).

Using (11) in (7), (9) and (10), we obtain a system of three homogeneous equations in three unknowns and these equations has non-trivial solution if the determinant of the coefficients of the system vanish, which yields the following characteristic equation in \( c \) as:

\[
H_1 c^6 + H_2 c^4 + H_3 c^2 + H_4 = 0,
\]

(12)

where

\[
H_1 = \frac{F_4}{\omega^6}, \quad H_2 = \frac{F_3}{\omega^4}, \quad H_3 = \frac{F_2}{\omega^2}, \quad H_4 = F_1,
\]

\[
F_1 = -b_1 b_4, \quad F_2 = b_1 b_4 \omega^2 + i \omega (b_1 b_5 + b_2 b_4 + \alpha b_4 + b_1),
\]

\[
F_3 = -i \omega^3 (b_1 b_5 + b_2 b_4) + \omega^2 (b_2 b_5 + b_3 b_6 - \alpha b_3 + \alpha b_5 + b_5 + b_2),
\]

\[
F_4 = -\omega^4 (b_2 b_5 + b_3 b_6).
\]

The complex coefficient implies that three roots of this equation may be complex. The complex phase velocities of the longitudinal waves, given by \( c_i, \ i = 1, 2, 3 \), will be varying with the direction of phase propagation. The complex velocity of a longitudinal wave, i.e. \( c_i = c_R + ic_I \), defines the phase propagation velocity \( V_i = (c_R^2 + c_I^2) / c_R \), attenuation quality factor \( Q_i^{-1} = -2 c_I / c_R \), Specific loss \( R_i = 4\pi |\frac{c_i}{c_R}| \), Penetration depth \( S_i = |(c_R^2 + c_I^2) / \omega c_I| \) for the corresponding wave. Therefore, three waves propagating in such a medium are attenuating. The same directions of wave propagation and attenuation vector of these waves make them homogeneous wave. The waves with phase velocity, attenuation quality factor, specific loss and penetration depth i.e. \( V_i, Q_i^{-1}, R_i \) and \( S_i (i = 1, 2, 3) \) may be named as \( P_1 \)-wave, \( P_2 \)-wave and \( T \)-wave that are propagating with the descending order of their velocities, respectively.

Substituting the value of \( \Psi \) from (11) in (8), we obtain:

\[
(\omega^2 - \delta^2 \xi^2) \bar{\Psi} = 0,
\]

(13)

which yield
\[ c = \pm \delta \]  

The equation (13) corresponds to transverse wave which travel with phase velocity \( \delta \).

### 5. Special Case

In the absence of porosity effect, characteristic equation reduces to:

\[ H_5c^4 + H_6c^2 + b_4 = 0, \]  

where

\[ H_5 = \frac{F_6}{\omega^4}, \quad H_6 = \frac{F_5}{\omega^2}, \quad F_5 = -b_4\omega^2 - i\omega(b_{50} + 1), \quad F_6 = i\omega^3b_{50}, \]

and the equation (13) remains the same because it is not effected by porous effect.

### 6. Steady Oscillations

For steady oscillations, we assume the displacement vector, pressure and temperature change, of the form:

\[ (u(x, t), p(x, t), T(x, t)) = (\tilde{u}, \tilde{p}, \tilde{T})e^{-i\omega t} \]  

Making use of dimensionless quantities defined by (5), in (1)-(3) and with the aid of (16) yields:

\[ \begin{align*}
(1 - \delta^2)\nabla \cdot u + (\delta^2\nabla^2 + \omega^2)u - \alpha \nabla p - \nabla T &= 0, \\
\alpha_2 \nabla^2 p + i\omega a_2 p + i\omega T + i\omega a_3 \text{div} u &= 0, \\
\alpha_4 \nabla^2 T + i\omega a_4 T + i\omega a_5 p + i\omega a_7 \text{div} u &= 0,
\end{align*} \]

where

\[ \begin{align*}
\alpha_1 &= \frac{k\omega^2}{\gamma\rho Fc_t^2}, \quad \alpha_2 = \frac{a_p\beta}{\gamma}, \quad \alpha_3 = \frac{\alpha\beta}{\gamma\rho c_t}, \quad \alpha_4 = \frac{k\omega^2}{c_t^2}, \quad \alpha_5 = ZT_0, \quad \alpha_6 = \gamma\beta T_0, \quad \alpha_7 = \frac{\beta^2\tau_0}{\rho c_t^2}.
\end{align*} \]

We introduce the matrix differential operator:

\[ F(D_x) = \|F_{gh}(D_x)\|_{5 \times 5}, \]  

where

\[ \begin{align*}
F_{mm}(D_x) &= (\delta^2\nabla^2 + \omega^2)\delta_{mm} + (1 - \delta^2)\frac{\partial^2}{\partial x_m \partial x_n}, \\
F_{m5}(D_x) &= -\frac{\partial}{\partial x_m}, \\
F_{4n}(D_x) &= i\omega a_3 \frac{\partial}{\partial x_n}, \\
F_{5n}(D_x) &= i\omega a_7 \frac{\partial}{\partial x_n}, \\
F_{44}(D_x) &= i\omega a_2 + a_4 \nabla^2, \\
F_{45}(D_x) &= i\omega, \\
F_{54}(D_x) &= i\omega a_5, \\
F_{55}(D_x) &= i\omega a_5 + a_4 \nabla^2, \\
\end{align*} \]

and \( \delta_{mn} \) is the Kronecker delta.

The system of equations (17)-(19) can be written as:

\[ F(D_x)U(x) = 0, \]  

where \( U = (u, p, T) \) is a five component vector function on \( E^3 \).

We assume that

\[ \delta^2 a_4 a_5 \neq 0 \]  

If the condition (23) is satisfied, then \( F \) is an elliptic differential operator, Hörmander [55].

**Definition:** The fundamental solution of the system of equations (17)-(19) (the fundamental matrix of operator \( F \)) is the matrix \( G(x) = \|G_{gh}(x)\|_{5 \times 5} \), satisfying condition, Hörmander [51]:

\[ F(D_x)U(x) = \delta(x)I(x), \]

where \( \delta() \) is the Dirac delta function, \( I = \|\delta_{gh}\|_{5 \times 5} \) is the unit matrix and \( x = E^3 \).

Now we construct \( G(x) \) in terms of elementary functions.

#### 6.1 Fundamental solution of system of equations of steady oscillations

We consider the system of equations:
\[\delta^2 \nabla^2 \mathbf{u} + (1 - \delta^2) \nabla \cdot \mathbf{u} + i\omega_3 \nabla p + i\omega_7 \nabla T + \omega^2 \mathbf{u} = \mathbf{H};\]  
(25)

\[-div \mathbf{u} + (i\omega_2 + a_1 \nabla^2) p + i\omega_4 T = L;\]  
(26)

\[-div \mathbf{u} + i\omega p + (i\omega_5 + a_4 \nabla^2) T = M,\]  
(27)

where \( \mathbf{H} \) is three components of vector function on \( E^3 \) and \( L, M \) are scalar functions of \( E^3 \). The system of equations (25)-(27) may be written in the form:

\[\mathbf{F}^{tr}(\mathbf{D}_2)U(\mathbf{x}) = \mathbf{Q}(\mathbf{x}),\]  
(28)

where \( \mathbf{F}^{tr} \) is the transpose of \( \mathbf{F}, \mathbf{Q} = (\mathbf{H}, L, M) \) and \( \mathbf{x} = E^3 \).

Applying the operator \( div \) to (25), we obtain:

\[(\nabla^2 + \omega^2) div \mathbf{u} + i\omega_3 \nabla^2 p + i\omega_7 \nabla^2 T + \omega^2 div \mathbf{u} = div \mathbf{H}\]  
(29)

The equations (26), (27) and (29) can be written in the form:

\[\mathbf{N}(\Delta) \mathbf{S} = \mathbf{Q},\]  
(30)

where \( \mathbf{S} = (div \mathbf{u}, p, T), \mathbf{Q} = (d_1, d_2, d_3) = (div \mathbf{H}, L, M) \), and

\[\mathbf{N}(\Delta) = \|N_{mn}(\Delta)\|_{3 \times 3} = \begin{bmatrix} \mathcal{V}^2 + \omega^2 & i\omega_3 \nabla^2 & i\omega_7 \nabla^2 \\ -\alpha & i\omega_2 + a_1 \nabla^2 & i\omega_6 \\ -1 & i\omega & i\omega_5 + a_4 \nabla^2 \end{bmatrix}\]  
(31)

The equation (30) can also be written as:

\[\Gamma_1(\Delta) \mathbf{S} = \mathbf{\Psi},\]  
(32)

where \( \mathbf{\Psi} = (\Psi_1, \Psi_2, \Psi_3) \), \( \Psi_n = e^\ast \sum_{m=1}^3 N_{mn} d_m \),

\[\Gamma_1(\Delta) = e^\ast \det \mathbf{N}(\Delta), e^\ast = \frac{1}{a_1a_4}, n = 1,2,3,\]  
(33)

and \( N_{mn} \) is the cofactor of the elements \( N_{mn} \) of the matrix \( \mathbf{N} \).

From (31) and (33), we see that

\[\Gamma_1(\Delta) = \prod_{m=1}^3 (\Delta + \lambda^2_m),\]  
(34)

where \( \lambda^2_m, m = 1,2,3 \) are the root of the equation \( \Gamma_1(-\kappa) = 0 \) (with respect to \( \kappa \)).

Applying operator \( \Gamma_1(\Delta) \) on (25), we have:

\[\Gamma_1(\Delta)[\delta^2 \nabla^2 \mathbf{u} + (1 - \delta^2) \nabla \cdot \mathbf{u} + i\omega_3 \nabla p + i\omega_7 \nabla T + \omega^2 \mathbf{u}] = \Gamma_1(\Delta) \mathbf{H}\]  
(35)

\[\Gamma_1(\Delta)[\mathcal{V}^2 + \omega^2 \mathbf{u}] = \frac{1}{\delta^2} \left[ \Gamma_1(\Delta) \mathbf{H} - \text{grad} \{(1 - \delta^2) \Psi_1 + i\omega_3 \Psi_2 + i\omega_7 \Psi_3 \}\right]\]  
(36)

\[\Gamma_1(\Delta)[\mathcal{V}^2 + \lambda^2_m \mathbf{u}] = \frac{1}{\delta^2} \left[ \Gamma_1(\Delta) \mathbf{H} - \text{grad} \{(1 - \delta^2) \Psi_1 + i\omega_3 \Psi_2 + i\omega_7 \Psi_3 \}\right]\]  
(37)

\[\mathbf{\Psi}'' = \frac{1}{\delta^2} \left[ \Gamma_1(\Delta) \mathbf{H} - \text{grad} \{(1 - \delta^2) \Psi_1 + i\omega_3 \Psi_2 + i\omega_7 \Psi_3 \}\right]\]  
(38)

From equations (32) and (35), we obtain:

\[\mathbf{\Theta}(\Delta) \mathbf{U}(\mathbf{x}) = \mathbf{\Psi}(\mathbf{x}),\]  
(39)

where \( \mathbf{\Psi}(\mathbf{x}) = (\mathbf{\Psi}'', \Psi_2, \Psi_3) \),

and \( \mathbf{\Theta}(\Delta) = \|\Theta_{gh}(\Delta)\|_{5 \times 5}; \Theta_{mm}(\Delta) = \Gamma_1(\Delta)[\mathcal{V}^2 + \lambda^2_4], m = 1,2,3; \Theta_{44}(\Delta) = \Theta_{55}(\Delta) = \Gamma_1(\Delta), \Theta_{gh}(\Delta) = 0, g,h = 1,2,3,4,5, g \neq h.

The equations (33) and (36) can be written in the form:

\[\mathbf{\Psi}'' = \frac{1}{\delta^2} \left[ \Gamma_1(\Delta) \mathbf{H} - \text{grad} \{(1 - \delta^2) e^\ast \sum_{m=1}^3 N_{m1} d_m + i\omega_3 \text{grad} e^\ast \sum_{m=1}^3 N_{m2} d_m + i\omega_7 \text{grad} e^\ast \sum_{m=1}^3 N_{m3} d_m \}\right].\]  
(40)

\[\mathbf{\Psi}'' = \frac{1}{\delta^2} \Gamma_1(\Delta) \mathbf{H} + \frac{1}{\delta^2} e^\ast \text{grad} \text{div} \mathbf{H} \{(1 - \delta^2) N_{11} + i\omega_3 N_{12} + i\omega_7 N_{13}\} + \frac{1}{\delta^2} e^\ast \text{grad} \mathbf{L} \{(1 - \delta^2) N_{21} + i\omega_3 N_{22} + i\omega_7 N_{23}\} + \frac{1}{\delta^2} e^\ast \text{grad} \mathbf{M} \{(1 - \delta^2) N_{31} + i\omega_3 N_{32} + i\omega_7 N_{33}\};\]  
(41)

\[\mathbf{\Psi}'' = \frac{1}{\delta^2} \Gamma_1(\Delta) \mathbf{J} + q_{11}(\Delta) \text{grad} \text{div} \mathbf{H} + q_{21}(\Delta) \text{grad} \mathbf{L} + q_{31}(\Delta) \text{grad} \mathbf{M},\]  
(42)

\[\mathbf{\Psi}_2 = q_{12}(\Delta) \text{div} \mathbf{H} + q_{22}(\Delta) \mathbf{L} + q_{32}(\Delta) \mathbf{M},\]  
(43)
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\[ \Psi_3 = q_{13} \Delta \, d \psi H + q_{23} \Delta \, L + q_{33} \Delta \, M, \]

where \( J = \| \delta_{gh} \|_{3 \times 3} \) is the unit matrix and

\[
\begin{align*}
q_{11} \Delta &= \frac{1}{\delta^2} e^{*} \{ -(1 - \delta^2) N_{11}^* + i \omega a_3 N_{12}^* + i \omega a_7 N_{13}^* \}, \\
q_{21} \Delta &= \frac{1}{\delta^2} e^{*} \{ -(1 - \delta^2) N_{21}^* + i \omega a_3 N_{22}^* + i \omega a_7 N_{23}^* \}, \\
q_{31} \Delta &= \frac{1}{\delta^2} e^{*} \{ -(1 - \delta^2) N_{31}^* + i \omega a_3 N_{32}^* + i \omega a_7 N_{33}^* \}, \\
q_{m2} \Delta &= e^{*} N_{m2}^*, \quad q_{m3} \Delta = e^{*} N_{m3}^*, \quad m = 1, 2, 3.
\end{align*}
\]

Now from equations (38), we have:

\[ \Psi(x) = R^{tr}(D(x))Q(x), \quad (39) \]

where \( R^{tr} \) is the transpose of \( R \), and

\[ R = \| R_{mn} \|_{5 \times 5}, \]

\[
\begin{align*}
R_{mn}(D(x)) &= \frac{1}{\delta^2} \frac{\partial^2 \xi_n}{\partial x_m \partial x_n}, \quad R_{m4}(D(x)) = q_{21} \frac{\partial}{\partial x_m}, \quad R_{m5}(D(x)) = q_{31} \frac{\partial}{\partial x_m}, \\
R_{4n}(D(x)) &= q_{12} \frac{\partial}{\partial x_n}, \quad R_{5n}(D(x)) = q_{13} \frac{\partial}{\partial x_n}, \quad m, n = 1, 2, 3, \\
R_{45}(D(x)) &= q_{22}, \quad R_{46}(D(x)) = q_{32}, \quad R_{54}(D(x)) = q_{23}, \quad R_{55}(D(x)) = q_{33}. \quad (40)
\end{align*}
\]

From equations (28), (37) and (39), we have:

\[ \Theta U = R^{tr}F^{tr}U. \quad (41) \]

It implies that \( R^{tr}F^{tr} = \Theta \), and hence

\[ F(D(x))R(D(x)) = \Theta(D). \quad (42) \]

We assume that \( \lambda_m^2 \neq \lambda_n^2 \neq 0 \), \( m, n = 1, 2, 3, 4 \), \( m \neq n \).

Let \( Y(x) = ||Y_{\alpha}(x)||_{5 \times 5} \), \( Y_{mn}(x) = \sum_{n=1}^{4} r_{1n} e_n(x) \), \( m, n = 1, 2, 3, 4 \).

\[
Y_{44}(x) = Y_{55}(x) = \sum_{n=1}^{3} r_{2n} e_n(x), \\
Y_{uv}(x) = 0, \quad u, v = 1, 2, 3, 4, 5, \quad u \neq v,
\]

where

\[
\xi_n(x) = -\frac{1}{4\pi|x|} \exp(i\lambda_n |x|), \quad n = 1, 2, 3, 4,
\]

\[
r_{2l} = \prod_{m=1, m \neq l}^{4} (\lambda_m^2 - \lambda_l^2)^{-1}, \quad l = 1, 2, 3, 4,
\]

\[
r_{2v} = \prod_{m=1, m \neq v}^{4} (\lambda_m^2 - \lambda_v^2)^{-1}, \quad v = 1, 2, 3, 4. \quad (43)
\]

We will prove the following Lemma:

The matrix \( Y \) defined above is the fundamental matrix of operator \( \Theta(D) \), that is

\[ \Theta(D)Y(x) = \delta(x)I(x). \quad (44) \]

Proof: To prove the lemma, it is sufficient to prove that:

\[ l_1(D) = (\Delta + \lambda_4^2)Y_{11}(x) = \delta(x), \quad l_1(D)Y_{44}(x) = \delta(x). \quad (45) \]

We find that:

\[ r_{11} + r_{12} + r_{13} + r_{14} = \frac{t_{1} + t_{2} + t_{3} + t_{4}}{t_{5}}, \quad (46) \]

\[
\begin{align*}
t_1 &= (\lambda_4^2 - \lambda_3^2)(\lambda_3^2 - \lambda_2^2)(\lambda_2^2 - \lambda_1^2), \quad t_2 = (\lambda_4^2 - \lambda_3^2)(\lambda_4^2 - \lambda_2^2)(\lambda_3^2 - \lambda_1^2), \\
t_3 &= (\lambda_4^2 - \lambda_3^2)(\lambda_4^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2), \quad t_4 = (\lambda_4^2 - \lambda_3^2)(\lambda_4^2 - \lambda_3^2)(\lambda_3^2 - \lambda_2^2), \\
t_5 &= (\lambda_4^2 - \lambda_3^2)(\lambda_3^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2). \quad (47)
\end{align*}
\]

Using (47) in (46), yield

\[ r_{11} + r_{12} + r_{13} + r_{14} = 0, \quad (48) \]

\[ r_{12}(\lambda_1^2 - \lambda_2^2) + r_{13}(\lambda_1^2 - \lambda_3^2) + r_{14}(\lambda_1^2 - \lambda_4^2) = 0, \quad (49) \]

\[ r_{13}(\lambda_2^2 - \lambda_3^2) + r_{14}(\lambda_2^2 - \lambda_4^2) = 0, \quad (50) \]

\[ r_{14}(\lambda_3^2 - \lambda_4^2) = 1, \quad (51) \]

\[ (\Delta + \lambda_4^2)\xi_m(x) = \delta(x) + (\lambda_m^2 - \lambda_4^2)\xi_m(x), \quad m, n = 1, 2, 3, 4. \quad (52) \]

Now consider:

\[ l_1(D) = (\Delta + \lambda_4^2)Y_{11}(x) = (\Delta + \lambda_4^2)(\Delta + \lambda_2^2)(\Delta + \lambda_1^2) \sum_{n=1}^{4} r_{1n} \xi_n(x) = \]

\[ = (\Delta + \lambda_4^2)(\Delta + \lambda_3^2)(\Delta + \lambda_2^2) \sum_{n=1}^{4} r_{1n} [\delta(x) + (\lambda_2^2 - \lambda_4^2)\xi_n(x)] = \]

\[ = (\Delta + \lambda_4^2)(\Delta + \lambda_3^2)(\Delta + \lambda_2^2) [\delta(x) \sum_{n=1}^{4} r_{1n} + \sum_{n=2}^{4} r_{1n}(\lambda_2^2 - \lambda_4^2)\xi_n(x)]. \]
whereas in case of TE, the value of $c_{ww}$

hence the attenuation coefficient, specific loss and penetration depth of plane waves, i.e. $T_E = 6 \times 10^5 \mu m^{-1}$ corresponds to thermoelastic medium (TE).

Figures, the solid line corresponds to thermoporoelastic medium (PTE) and dotted line and penetration depth with respect to frequency are shown in Figs. 1-12 respectively. In all the figures, the value of $\omega$ increases.

Due to the effect of porosity the value of $V_2$ for PTE and TE increases gradually as $\omega$ increases. The value of $V_2$ for PTE and TE increases gradually as $\omega$ increases.

The software Matlab 7.0.4 has been used to determine the values of phase velocity, attenuation coefficient, specific loss and penetration depth of plane waves, i.e. $P_1$-wave, $P_2$-wave and $T$-wave. The variations of phase velocity, attenuation coefficients, specific loss and penetration depth with respect to frequency are shown in Figs. 1-12 respectively. In all the figures, the solid line corresponds to thermoporoelastic medium (PTE) and dotted line corresponds to thermoelastic medium (TE).

**7. Numerical results and discussion**

With the view of illustrating the theoretical results and for numerical discussion we take a model for which the values of the various physical parameters are taken from Jabbari and Dehbani [54]:

\[
E = 6 \times 10^5 \mu m, \quad \nu = 0.3, \quad T_0 = 293 ^\circ K, \quad K_s = 2 \times 10^{10} \mu m, \quad K = 5 \times 10^9 \mu m, \quad K = 0.5W/m ^\circ C, \quad \alpha_s = 1.5 \times 10^{-5}1/ ^\circ C, \quad \alpha_w = 2 \times 10^{-4}1/ ^\circ C, \quad c_s = 0.8 J/g ^\circ C, \quad c_w = 4.2 J/g ^\circ C, \quad \rho_s = 2.6 \times 10^6 g/m^3, \quad \rho_w = 1 \times 10^6 g/m^3, \quad n = 0.4, \quad \alpha = 1
\]

The software Matlab 7.0.4 has been used to determine the values of phase velocity, attenuation coefficient, specific loss and penetration depth of plane waves, i.e. $P_1$-wave, $P_2$-wave and $T$-wave. The variations of phase velocity, attenuation coefficients, specific loss and penetration depth with respect to frequency are shown in Figs. 1-12 respectively. In all the figures, the solid line corresponds to thermoporoelastic medium (PTE) and dotted line corresponds to thermoelastic medium (TE).

**7.1. Phase velocity.** Fig. 1 depicts the variations of phase velocity of $V_1$ with frequency $\omega$ for PTE and TE. In case of PTE, the value of $V_1$ increases exponentially as $\omega$ increases, whereas in case of TE, the value of $V_1$ oscillates in the range $0 \leq \omega \leq 2.5$ and then increase as $\omega$ increases.

Fig. 2 shows the variations of $V_2$ with $\omega$ for PTE and TE. The value of $V_2$ for PTE and TE increases gradually but due to the effect of porosity the value of $V_2$ is less for PTE as compared to TE as $\omega$ increases.

Fig. 3 shows that the value of $V_3$ for PTE when $\alpha = 0.5$ increases gradually as $\omega$ increases whereas there is sharp increase in the value of $V_3$ when $\alpha = 0.75$ as $\omega$ increases. Due to the effect of porosity the value of $V_3$, when $\alpha = 0.5$ is more as compared to $\alpha = 0.75$ in the range $0 \leq \omega \leq 0.4$ and then show the opposite behaviour for higher values of $\omega$.
7.2. Attenuation quality factor. Fig. 4 shows the variation of $Q_1$ with $\omega$ for PTE and TE. The value of $Q_1$ for PTE and TE increases gradually as $\omega$ increases but due to the effect of porosity the value of $Q_1$ for PTE is more as compared to TE as $\omega$ increases.

The variation of $Q_2$ with $\omega$ for PTE and TE is shown in Fig. 5. The value of $Q_2$ is consistent for PTE whereas for TE, there is a small increase in the value of $Q_2$ as $\omega$ increases. The value of $Q_2$ for TE is more as compared to PTE due to the effect of porosity as $\omega$ increases.
Fig. 6 depicts the variations of $Q_3$ with frequency $\omega$ for PTE. When $\alpha = 0.5$, there is a small increase in the value of $Q_3$ as $\omega$ increases, whereas for $\alpha = 0.75$, the value of $Q_3$ decreases exponentially for higher values of $\omega$.

![Graph of Fig. 6](image)

**Fig. 4.** Variation of attenuation quality factor w.r.t frequency.

![Graph of Fig. 5](image)

**Fig. 5.** Variation of attenuation quality factor w.r.t frequency.

![Graph of Fig. 6](image)

**Fig. 6.** Variation of attenuation quality factor w.r.t. frequency.

**7.3. Specific loss.** The variation of specific loss $R_1$ with frequency $\omega$ is shown in Fig. 7. There is a small increase in the value of $R_1$ for PTE as $\omega$ increases whereas for TE, the value of $R_1$ increases exponentially as $\omega$ increases. Due to the effect of porosity the value of $R_1$ for PTE is more as compared to TE as $\omega$ increases.

Fig. 8 shows the variations of $R_2$ with $\omega$ for PTE and TE. The value of $R_2$ for PTE increases in the range $0 \leq \omega \leq 3.5$ then starts decreasing as $\omega$ increases whereas for TE, the
value of $R_2$ decreases exponentially for higher values of $\omega$. The value of $R_2$ for PTE is less as compared to TE due to the effect of porosity as $\omega$ increases.

Fig. 9 shows the variation of $R_3$ with frequency $\omega$ for PTE. There is small increase in the value of $R_3$ for PTE when $\alpha = 0.5$ as $\omega$ increases whereas for $\alpha = 0.75$, the value of $R_3$ decreases sharply for higher values of $\omega$.

7.4. Penetration depth. The variation of $S_1$ with $\omega$ for PTE and TE is shown in Fig.10. The value of $S_1$, for PTE, decreases exponentially whereas for TE, with small initial decrease, the value of $S_1$ becomes consistent for higher values of $\omega$. Due to the effect of porosity the value of $S_1$ for PTE is more as compared to TE as $\omega$ increases.
Fig. 11 depicts the variation of $S_2$ with frequency $\omega$. The value of $S_2$ decreases gradually for all values of $\omega$ for PTE and TE but the value of PTE is less as compared to TE in the range $0 \leq \omega \leq 3$ and then show the opposite behaviour for higher values of $\omega$ due to the effect of porosity.

Fig. 12 shows the variations of $S_3$ with $\omega$ for PTE. The value of $S_3$ for PTE decreases gradually for both values of $\alpha$, but due to the effect of porosity, the value of $S_3$ is less for $\alpha = 0.5$ as compared to the value of $\alpha = 0.75$. 

![Graph of S2 vs Frequency](image1.png)

**Fig. 10.** Variation of penetration depth w.r.t frequency.

![Graph of S3 vs Frequency](image2.png)

**Fig. 11.** Variation of penetration depth w.r.t frequency.

![Graph of S3 vs Frequency with Alpha](image3.png)

**Fig. 12.** Variation of penetration depth w.r.t frequency.
8. Conclusion

The present study deals with the propagation of plane wave and fundamental solution in the thermoporoelastic medium. It is found that for two dimensional model, there exist three longitudinal waves, namely $P_1$-wave, $P_2$-wave and T-wave in addition to transverse wave. The phase velocity, attenuation coefficient, specific loss and penetration depth are computed numerically and depicted graphically. The fundamental solution of the system of equations in the thermoporoelastic medium in the case of steady oscillations is considered in term of elementary functions.

Due to the presence of porous effect, the phase velocities of $P_1$-wave for PTE remains more in comparison to TE, whereas the phase velocity of T-wave is less for PTE in comparison to TE for all values of frequency.

Attenuation quality factor, penetration depth and specific loss of $P_1$-wave for PTE is more in comparison to TE and reverse behaviour is shown in case of T-wave. The value of attenuation quality factor, penetration depth and specific loss of $P_2$-wave for $\alpha = 0.5$ is less as compared to $\alpha = 0.75$ and opposite behaviour is shown for penetration depth whereas the value of phase velocity of $P_2$-wave is more for $\alpha = 0.5$ as compared to $\alpha = 0.75$ for initial values of $\omega$ and shows the opposite behaviour for higher values of $\omega$.

References