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"WANDERING" NATURAL FREQUENCIES OF AN ELASTIC CUSPIDAL PLATE WITH THE CLAMPED PEAK

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Abstract. Cuspidal irregularities of solids have been recognized as Vibrating Black Holes for elastic and acoustic waves. The corresponding absorption phenomenon is caused, in particular, by the appearance of the continuous spectrum $[\kappa_\dagger, +\infty)$ of the Lame system in a two-dimensional plate with the sharp cusp that provokes for wave processes in a finite volume. However, if the plate is clamped in the small h-neighborhood of the cusp top, the spectrum becomes discrete and consists of isolated natural frequencies κ_j^h of finite multiplicity. The asymptotics of κ_j^h as $h \to +0$ is constructed that describes the effect of the "wandering" of the natural frequencies above the threshold $\kappa_\dagger > 0$, namely the asymptotic formula $\kappa_j^h = K_j(\ln h) + O(h^\delta)$ with $\delta > 0$ is valid where K_j is a periodic function. In other words, some of frequencies flounce in the semi-axis $(\kappa_\dagger, +\infty)$ at a quite high rate $O(h^{-1})$. At the same time, natural frequencies below the threshold get the sustainable behaviour $\kappa_p^h = \kappa_p^0 + O(h^\delta)$, $\delta > 0$, as $h \to +0$.

Keywords: vibrating black holes, cuspidal plate, continuous spectrum, clamped peak, wandering eigenvalues, asymptotics

1. Formulation of the elasticity problems in a cuspidal solid

Let Ω be a two-dimensional isotropic and homogeneous elastic plate and let its edge $\Gamma = \partial\Omega$ be smooth everywhere with exception of the point $\mathcal{O}$, the origin of the Cartesian coordinate system $x = (x_1, x_2)$. In a neighborhood of $\mathcal{O}$, the domain gets the cuspidal shape, i.e.,

$$\{x \in \Omega: x_1 \leq d\} = \Pi_d,$$

where

$$\Pi_d = \{x \in \mathbb{R}^2: x_1 \in (0, d], |x_2| < H x_1^2\}, \qquad d > 0, H > 0. \tag{1}$$

First of all, we assume that the plate edge Γ is traction-free. Then the longitudinal oscillations of the plate Ω are described by the spectral boundary-value problem

$$-\partial_1 \sigma_{j1}(u; x) - \partial_2 \sigma_{j2}(u; x) = \rho\kappa^2 u(x), \ \ x \in \Omega, \ \ j = 1,2, \tag{2}$$

$$\sigma_j^{(n)}(u; x) := n_1(x)\, \sigma_{j1}(u; x) - n_2(x)\, \sigma_{j2}(u; x) = 0, \ \ x \in \Gamma \backslash \mathcal{O}, \ \ j = 1,2. \tag{3}$$

Here $\partial_j = \partial/\partial x_j$, $u = (u_1, u_2)$ is the displacement vector, $\sigma^{(n)} = \left(\sigma_1^{(n)}, \sigma_1^{(n)}\right)$ is the normal vector of stresses, and the Cartesian components of the stress tensor $\sigma(u)$ are given by

$$\sigma_{jk}(u) = \mu\left(\partial_j u_k + \partial_k u_j\right) + \lambda\, \delta_{j,k}(\partial_1 u_1 + \partial_2 u_2), \qquad j, k = 1,2. \tag{4}$$

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Moreover, $\lambda \geq 0$, $\mu > 0$ and $\rho > 0$ are the Lame constants and the density of the elastic material, respectively. Finally, $n = (n_1, n_2)$ is the unit vector of the outward normal, $\delta_{j,k}$ is the Kronecker symbol, and $\kappa \geq 0$ is the frequency of harmonic-in-time oscillations. We reduce the characteristic size of the domain Ω to unity and make dimensionless the coordinates x_1, x_2 and all geometric parameters.

It is known, see e.g. [1, 2], that the spectrum $\wp$ of the problem (2), (3) has the continuous component $\wp_{co} = [\kappa_\dagger, +\infty)$ with the positive cutoff value

$$\kappa_\dagger = \frac{5}{2} H \sqrt{3\frac{\rho}{M}}, \tag{5}$$

where H and ρ are taken from the relations (1) and (2), respectively, and

$$M = \frac{\mu(\lambda+\mu)}{\lambda+2\mu}. \tag{6}$$

The primitive formula $\wp_{co} \neq \emptyset$ follows from the fact observed in [3; Sect. 3.1], namely the traditional Korn inequality, cf., the review paper [4],

$$||u; H^1(\Omega)||^2 \leq c_\Omega(\mathcal{E}(u;\Omega) + ||u; L^2(\Omega)||^2) \tag{7}$$

does not hold true in the cuspidal domain (1). In (7), $L^2(\Omega)$ and $H^1(\Omega)$, respectively, are the Lebesgue and Sobolev spaces with the standard norms $\left||u; L^2(\Omega)|\right| = \left(\int_\Omega |u(x)|^2 dx\right)^{1/2}$ and $\left||u; H^1(\Omega)|\right| = \left(\int_\Omega (|\nabla u(x)|^2 + |u(x)|^2) dx\right)^{1/2}$, while $\mathcal{E}(u;\Omega)$ is the energy functional,

$$\mathcal{E}(u;\Omega) = \int_\Omega (|u_1(x)|^2 + |u_2(x)|^2)\, dx +$$
$$+\frac{1}{2\mu}\int_\Omega \left(\sum_{j,k=1,2}\left|\sigma_{jk}(u;x)\right|^2 - \frac{\lambda}{2(\lambda+\mu)}|\sigma_{11}(u;x) + \sigma_{22}(u;x)|^2\right) dx \tag{8}$$

Furthermore, the energy space $E(\Omega)$ obtained by completion of the linear set $C_c^\infty(\overline{\Omega}\backslash\mathcal{O})$ (infinitely differentiable vector functions vanishing near the point $\mathcal{O}$) with respect to the norm

$$||u; E(\Omega)||^2 = \mathcal{E}(u;\Omega)^{1/2}, \tag{9}$$

is much bigger that the Sobolev space $H^1(\Omega)$ and the embedding $E(\Omega) \subset L^2(\Omega)$ is not compact.

The latter assures that indeed the continuous spectrum $\wp_{co}$ of the operator of the problem (2), (3) is not empty according to general results in the operator theory, see, e.g., the monograph [5; Ch. 9, 10].

The continuous spectrum provokes for wave processes inside the cusp (1) which are known as Vibrating Black Holes, see the papers [6-8] where concrete engineering devices based on the effect under discussion, are described as well. We mention especially the pioneering paper [6] where, for the first time, the cutoff value (5) was computed on the basis of the Kirchhoff theory of thin elastic beams with variable thickness.

Let us assume now that the peak of the cusp (1) is clamped along the short arcs

$$\gamma_h^\pm = \{x: x_1 \in (0,h), |x_2| = \pm H x_1^2\};$$

here $h \ll 1$ is a small parameter. Then, for a vector field $u \in H^1(\Omega)$ satisfying the Dirichlet boundary conditions

$$u_j(x) = 0,\ x \in \gamma_h = \gamma_h^+ \cup \gamma_h^-,\quad j = 1,2, \tag{10}$$

the Korn inequality (9) becomes valid. Indeed, one may extend by zero the vector function u from Π_h on the rectangle $Q_h = \{x: x_1 \in (0,h), |x_2| < Hh^2\}$ and apply the standard Korn inequality, see e.g. [4], in the two domains $\Omega\backslash\Pi_h$ and Q_h with Lipschitz boundaries. Thus, the spectrum $\wp^h$ of the Lame system (2) with the boundary conditions (10) and

$$\sigma_j^{(n)}(u;x) = 0\,,\ \ x \in \Gamma \backslash \overline{\gamma_h},\ \ j = 1,2, \tag{11}$$

is fully discrete due to the compact embedding $H^1(\Omega) \subset L^2(\Omega)$. By [5; Theorems 10.1.5, 10.2.2], this spectrum composes the monotone unbounded eigenvalue sequence

$$0 < \kappa_1^h \le \kappa_2^h \le \kappa_3^h \le \cdots \le \kappa_p^h \le \cdots \ \to +\infty, \tag{12}$$

where multiplicity is counted in.

Our objective is to examine the asymptotic behavior of the eigenvalues κ_p^h as $h \to +0$. It is remarkable that asymptotic formulas become quite different below and above the threshold (5) and, for $\kappa_j^h > \kappa_\dagger$, they exhibit the new effect of "wandering" described in Section 5. For $\kappa_p^h < \kappa_\dagger$, we conclude with rather standard asymptotic expansions.

2. Waves inside the cusp

Since width $2Hx_1^2$ of the cusp (1) is much smaller than the distance $x_1 > 0$ to its top $\mathcal{O}$, the dimension reduction procedure, see e.g. [3; Ch. 4], can be applied in the same way as in the paper [1] in order to derive the asymptotic expansion of a solution $u(x)$ to the problem (2), (3) when $x_1 \to +0$. Thus, we accept the standard asymptotic ansatz

$$u(x) = U^{-2}(x) + U^{-1}(x) + U^0(x) + U^1(x) + U^2(x) + \cdots \tag{13}$$

with the first terms computed, for example, in the book [3; Sections 1.3 and 4.1] and the paper [1, Section 3],

$$U^{-2}(x) = e_{(2)}\, w(x_1), \qquad U^{-1}(x) = -x_2\, e_{(1)} \frac{dw}{dx_1}(x_1),$$

$$U^0(x) = e_{(2)} \frac{\lambda}{\lambda+2\mu}\frac{x_2^2}{2}\frac{d^2w}{dx_1^2}(x_1), \qquad U^1(x) = e_{(1)} \frac{3\lambda+4\mu}{\lambda+2\mu}\frac{x_2^3}{6}\frac{d^3w}{dx_1^3}(x_1) - \tag{14}$$

$$-2e_{(1)} \frac{\lambda+\mu}{\lambda+2\mu} H^2 x_1^4\, x_2 \frac{d^3w}{dx_1^3}(x_1) \ - 8e_{(1)} \frac{\lambda+\mu}{\lambda+2\mu} H^2 x_1^3\, x_2 \frac{d^2w}{dx_1^2}(x_1),$$

where $e_{(j)}$ stands for the unit vector of the x_j-axis. To construct the next term $U^2(x)$ in (13), we have to subject the unknown scalar function w of the longitudinal variable x_1 to the ordinary differential equation for the averaged bend of beams with variable thickness, cf., the original paper [6],

$$\frac{4}{3} M H^3 \frac{d^2}{dx_1^2} x_1^6 \ \frac{d^2w}{dx_1^2}(x_1) = 2Hx_1^2 \rho\kappa^2\, w(x_1), \quad\ x_1 \ > 0. \tag{15}$$

The coefficients H, ρ, κ and M are taken from the formulas (1), (2) and (6), respectively.

Clearly, the equation (15) with $\kappa = 0$ has the particular solution

$$w(x_1) = C_0 + C_1 x_1. \tag{16}$$

According to (14), the linear function (16) gives rise to the displacement field (13) which is nothing but a rigid motion. Solutions to the ordinary differential equation (15) of Euler type with $\kappa > 0$ have the form

$$w(x_1) = c x_1^{\tau-3/2}, \tag{17}$$

where the exponent τ is a root of the bi-quadratic equation

$$\left(\tau^2 - \frac{9}{4}\right)\left(\tau^2 - \frac{25}{4}\right) = \frac{3}{4}\frac{\rho\kappa^2}{MH^2}. \tag{18}$$

In the case

$$\kappa \le \kappa_\dagger,$$

i.e., below the threshold (5), the bi-quadratic equation (18) has four real rots but, for

$$\kappa > \kappa_{\dagger}, \tag{19}$$

i.e., above the threshold, the equation gets two real and two pure imaginary roots

$$\tau_{\pm}^{re} = \pm t_{+} \text{ and } \tau_{\pm}^{im} = \pm i t_{-} \tag{20}$$

where

$$t_{\pm}^{2} = T \pm \frac{17}{4} > 0, \qquad T = \frac{1}{2}\sqrt{16 + 3\frac{\rho\kappa^{2}}{MH^{2}}}. \tag{21}$$

The displacement field $u_{(\pm)}^{im}(x)$ constructed through the formulas (13), (14) and (20), (21) from the oscillatory solutions, see (17) and (20),

$$w_{(\pm)}^{im}(x_1) = c_{(\pm)}^{im} x_1^{\pm i t_{-} - 3/2} \tag{22}$$

are interpreted as one-dimensional bending elastic waves propagating along the cusp Π_h to its top $\mathcal{O}$ (minus) and from the point $\mathcal{O}$ (plus), see [6] and [1, 2]. The direction of propagation is found out by means of the Umov-Mandelstam (energy) radiation principle, see the monograph [9; Ch. 1] and, e.g., the paper [10] among many other publications.

Computing components of the stress tensor $\sigma(u^0)$ according to the relations (4) and (13), (14), we obtain

$$\sigma_{11}(u^0; x) = -4Mx_2 \frac{d^2 w}{dx_1^2}(x_1) + \cdots, \qquad \sigma_{22}(u^0; x) = \cdots \tag{23}$$

$$\sigma_{12}(u^0; x) = 4M\left(\frac{x_2^2}{2} - \frac{H^2}{2} x_1^4\right)\frac{d^3 w}{dx_1^3}(x_1) - 8MH^2 x_1^3 \frac{d^2 w}{dx_1^2}(x_1) + \cdots, \tag{24}$$

where dots stand for lower-order terms, namely for $O\left(x_1^{-1/2}\right)$ in (23) and $O\left(x_1^{+1/2}\right)$ in (24). In view of (14), (21) and (20) integrands on the right-hand side of (8) get order x_1^{-3} so that the integrals over the cusp (1) diverge at the rate $O(|\ln x_1|)$. The latter observation, in particular, indicates transportation of energy along the cusp.

The displacement fields $u_{(\pm)}^{re}(x)$ constructed from the solutions

$$w_{(\pm)}^{re}(x_1) = c_{(\pm)}^{re} x_1^{\pm t_{+} - 3/2} \tag{25}$$

with the real exponents in (20) possess quite different properties. Indeed, by the definitions (25), (21) and (14), (23), (24), the vector function $u_{(+)}^{re}$ gives rise to the finite energy functional $\mathcal{E}\left(u_{(+)}^{re}; \Pi_h\right)$. At the same time, the integrals on the right-hand side of (8) with $u = u_{(-)}^{re}$ diverge at the power rate $O\left(x_1^{-2t_{+}}\right)$.

In what follows we will deal with elastic fields involving the following linear combination of the constructed special displacement vectors:

$$u^0(x) = c_{(+)}^{im} u_{(+)}^{im}(x) + c_{(-)}^{im} u_{(-)}^{im}(x) + c_{(+)}^{re} u_{(+)}^{re}(x) + c_{(-)}^{re} u_{(-)}^{re}(x). \tag{26}$$

Notice that imposing various relationships between the coefficients $c_{(\pm)}^{im}$ and $c_{(\pm)}^{re}$ yields different operators of the problem (2), (3) with very distinct properties, cf., Section 5.

In the paper [2] it is proved that any solution of the problem (2), (3) verifying the estimate

$$|u(x)| \le c x_1^{-\theta - \frac{3}{2}}, \quad x \in \Omega, \tag{27}$$

with an exponent $\theta > t_{+}$, admits the asymptotic form

$$u(x) = \chi(x_1) u^0(x) + \tilde{u}(x), \tag{28}$$

where u^0 is a linear combination (26) with coefficients depending on the solution u, χ is a smooth cut-off function,

$\chi(x_1) = 1$ for $x_1 < \frac{d}{2}$ and $\chi(x_1) = 0$ for $x_1 > d$,

and the remainder $\tilde{u}$ enjoys the estimate

$$|\tilde{u}(x)| \leq c x_1^{\theta - \frac{3}{2}}, \quad x \in \Omega. \tag{29}$$

We emphasize that the displacement fields $u_{(\pm)}^{re}$ and $u_{(\pm)}^{im}$ do not satisfy the relation (29) so that detaching a linear combination (26) is necessary to achieve the appropriate decay of the remainder $\tilde{u}(x)$ as $x \to \mathcal{O}$.

3. The boundary layer phenomenon

In order to take into account the boundary conditions (10) we introduce the stretched coordinates

$$\xi = (\xi_1, \xi_2) = (h^{-2}(x_1 - h), h^{-2} x_2). \tag{30}$$

The coordinate dilation $x \mapsto \xi$ and formal setting $h = 0$ turn the cusp Π_h into the infinite strip

$\mathcal{S} = \{\xi \in \mathbb{R}^2 : \xi_1 \in \mathbb{R}, |\xi_2| < H\}$

of width $2H$. Since the endpoints $x = (h, \pm H h^2)$ of the arcs $\gamma_h^{\pm}$ are mapped into the points $\xi = (0, \pm H)$, the original problem (2), (11), (10) about the plate with the clamped peak converts into the following mixed boundary-value problem for the Lame system in the strip

$$\begin{aligned} &-\partial_1 \sigma_{j1}(w;\xi) - \partial_2 \sigma_{j2}(w;\xi) = 0, \quad \xi \in \mathcal{S}, \quad j = 1,2, \\ &\sigma_{j2}\left(w; \xi_1, \pm\tfrac{1}{2}\right) = 0, \quad \xi_1 > 0, \quad j = 1,2, \\ &w_j\left(\xi_1, \pm\tfrac{1}{2}\right) = G_j^{\pm}, \quad \xi_1 < 0, \quad j = 1,2. \end{aligned} \tag{31}$$

The boundary layer $w(\xi) = \big(w_1(\xi), w_2(\xi)\big)$ is intended to compensate for the main discrepancy of the vector function (28) in the Dirichlet conditions (10) and hence, the data of the problem (31) look as follows:

$$\begin{aligned} &G_2^{\pm}(\xi_1) = W_2^h := \textstyle\sum_{\pm}\Big(c_{(\pm)}^{im} h^{\pm i t_- - 3/2} + c_{(\pm)}^{re} x_1^{\pm t_+ - 3/2}\Big), \\ &G_1^{\pm}(\xi_1) = \pm W_1^h := \mp H \textstyle\sum_{\pm}\Big(\Big(\pm i t_- - \tfrac{3}{2}\Big) c_{(\pm)}^{im} h^{\pm i t_- - 3/2} + \Big(\pm t_+ - \tfrac{3}{2}\Big) c_{(\pm)}^{re} x_1^{\pm t_+ - 3/2}\Big). \end{aligned} \tag{32}$$

Thus, the solution of the problem (31) with the finite elastic energy is nothing but the bounded displacement field with the linear dependence on the transversal coordinate ξ_2, that is,

$w(\xi) = \big(-H\xi_2 W_1^h, W_2^h\big)$.

This boundary layer term gets the intrinsic property of the exponential decay as $\xi_1 \to \pm\infty$ only in the case

$$W_1^h = 0, \qquad W_2^h = 0. \tag{33}$$

This implies that the problem (31) must be homogeneous and the main term of the boundary layer must be absent in the asymptotics.

Let us find coefficients in the linear combination (26) which provides the relation (33). The first step looks quite unanticipated, namely we set

$$c_{(+)}^{re} = 0 \tag{34}$$

and, therefore, exclude from the linear combination the "decent" solution in the couple (25) but keep the "undeserving" one with objectionable behavior as $x_1 \to +0$. A reason for this procedure will be explained in the end of Section 5.

At the second step, we insert the formulas (32) into the equations (33) and exclude from the obtained system of linear algebraic equations

$$c_{(+)}^{im} h^{+it_- -3/2} + c_{(-)}^{im} h^{-it_- -3/2} + c_{(-)}^{re} h^{-t_+ -3/2} = 0,$$

$$c_{(+)}^{im}\left(it_- - \frac{3}{2}\right) h^{+it_- -5/2} - c_{(-)}^{im}\left(-it_- - \frac{3}{2}\right) h^{-it_- -5/2} - c_{(-)}^{re}\left(t_+ + \frac{3}{2}\right) h^{-t_+ -5/2} = 0 \qquad (35)$$

the unknown and irrelevant coefficient $c_{(-)}^{re}$. As a result, we come across the following relationship between the coefficients $c_{(-)}^{im}$ and $c_{(+)}^{im}$ of the oscillatory waves:

$$c_{(-)}^{im} = -h^{2it_-} \frac{t_+ + it_-}{t_+ - it_-} c_{(+)}^{im}. \qquad (36)$$

We emphasize that modulo of the coefficient on the right-hand side of (36) is equal to one. It should also be noted that the equalities (35) imply that the main asymptotic terms of the displacement field (26) with the coefficient (34) vanish at the cross-section $\{x: x_1 = h, |x_2| < Hh^2\}$ of the cusp Π_d and hence can be extended by zero onto the peak $\Pi_h$with the clamped sides $\gamma_h^{\pm}$ in order to satisfy the Dirirchlet conditions (10). The latter is the immediate consequence of the equalities (33) caused by our consideration of the boundary layer phenomenon.

4. Asymptotic conditions at the cusp top and the self-adjoint operator pencils

Let $\mathcal{W}(\Omega)$ be a space of vector functions in the form (9) where

$$u^0(x) = c_{(+)}^{im} u_{(+)}^{im}(x) + c_{(-)}^{im} u_{(-)}^{im}(x) + c_{(+)}^{re} u_{(+)}^{re}(x). \qquad (37)$$

Notice that in contrast to (34) we now have set

$$c_{(-)}^{re} = 0 \qquad (38)$$

and the linear combination (26) under the restriction (38) becomes nothing but (37). The norm in $\mathcal{W}(\Omega)$ is the sum of the coefficients moduli $\left|c_{(\pm)}^{im}\right| + \left|c_{(+)}^{im}\right|$ and an appropriate (see [2] and compare with (29)) weighted norm of the remainder $\tilde{u}$. Such spaces are called weighted spaces with detached asymptotics. Fixing some phase $\psi \in [0,2\pi)$, we impose the relationship

$$c_{(-)}^{im} = e^{i\psi}\, c_{(+)}^{im} \qquad (39)$$

and denote by $\mathcal{W}(\Omega;\psi)$ the subspace of vector functions $u \in \mathcal{W}(\Omega)$ subject to the restriction (39). Dealing with solutions of the problem (2), (3) in this space should be interpreted as imposing certain asymptotic conditions at the top $\mathcal{O}$ of the cusp.

To investigate general properties of the problem (2), (3), (39), we first of all insert any vector functions $u^{(m)} \in \mathcal{W}(\Omega;\psi), m = 1,2,$ into the Green formula on the domain $\Omega^\varepsilon = \Omega \setminus \Pi^\varepsilon$ and perform the limit passage $\varepsilon \to +0$. Denoting by $c_{(\pm)}^{im(m)}$, $c_{(+)}^{re(m)}$ and $\tilde{u}^{(m)}$ the introduced attributes of $u^{(m)}$ and setting

$$w^{(m)}(x_1) = c_{(+)}^{im(m)} x_1^{it_- -3/2} + c_{(-)}^{im(m)} x_1^{-it_- -3/2} + c_{(+)}^{re(m)} x_1^{t_+ -3/2}, \qquad (40)$$

we obtain

$$-\sum_{j=1}^{2}\left(\left(\partial_1\sigma_{j1}(u^{(1)}), u_j^{(2)}\right)_{\Omega^\varepsilon} + \left(\partial_2\sigma_{j2}(u^{(1)}), u_j^{(2)}\right)_{\Omega^\varepsilon} -\right.$$
$$\left. - \left(u_j^{(1)}, \partial_1\sigma_{j1}(u^{(2)})\right)_{\Omega^\varepsilon} + \left(u_j^{(1)}, \partial_2\sigma_{j2}(u^{(2)})\right)_{\Omega^\varepsilon}\right) +$$

$$+\left(\sigma^{(n)}\left(u^{(1)}\right),u^{(2)}\right)_{\Gamma\setminus\overline{\gamma_\varepsilon}}-\left(u^{(1)},\sigma^{(n)}\left(u^{(2)}\right)\right)_{\Gamma\setminus\overline{\gamma_\varepsilon}}=$$

$$=\frac{8}{3}\,MH^3\lim_{\varepsilon+0}\left(\frac{d}{dx_1}\left(x_1^6\frac{d^2w^{(1)}}{dx_1^2}(x_1)\right)\overline{w^{(2)}(x_1)}-x_1^6\frac{d^2w^{(1)}}{dx_1^2}(x_1)\overline{\frac{dw^{(2)}}{dx_1}(x_1)}+\right.$$

$$\left.\left.+x_1^6\frac{dw^{(1)}}{dx_1}\overline{\frac{d^2w^{(1)}}{dx_1^2}(x_1)}-w^{(1)}(x_1)\overline{\left(\frac{d}{dx_1}x_1^6\frac{d^2w^{(2)}}{dx_1^2}(x_1)\right)}\right)\right|_{x_1=\varepsilon}=$$

$$=4Tit_-\left(c_{(+)}^{im(1)}\overline{c_{(+)}^{\imath m(2)}}-c_{(-)}^{im(1)}\overline{c_{(-)}^{\imath m(2)}}\right)=0. \tag{41}$$

The numbers M, H and T are taken from (1), (6) and (21), respectively, and $(\ ,\ )_\Xi$ stands for the natural scalar product in the Lebesgue space $L^2(\Xi)$. In the middle of the calculation (41) we have used the formulas (23) and (24) while the last equality is due to the imposed relationship (39) which demonstrates that $c_{(+)}^{im(1)}\overline{c_{(+)}^{\imath m(2)}}=c_{(-)}^{im(1)}\overline{c_{(-)}^{\imath m(2)}}$.

Green's formula (41) means that the operator $\mathcal{A}_\gamma(\kappa)$ of the problem (2), (3) with the domain $\mathcal{W}(\Omega;\psi)$ is formally self-adjoint. It must be regarded as a holomorphic pencil, see the monographs [11; Ch. 1] and [12; Ch. 1]. Its spectrum in the half-plane $\mathbb{C}_\dagger=\{\kappa\in\mathbb{C}\colon \mathrm{Re}\,\kappa\geq\kappa_\dagger\}$ form the monotone unbounded sequence of normal real eigenvalues listed according to their multiplicity

$$\kappa_\dagger<k_N(\psi)\leq k_{N+1}(\psi)\leq\cdots\leq k_{N+q}(\psi)\leq\cdots+\infty. \tag{42}$$

By the relationship (39), the eigenvalues (42) depend 2π -periodically on the parameter ψ.

5. Asymptotics of eigenvalues of the problem with clamped peak

We now are in a position to formulate appropriate asymptotic conditions at the cusp top $\mathcal{O}$ in order to create an asymptotic model of the problem (2), (10), (11) with the help of asymptotic conditions. Comparing (39) with (36), we set

$$\psi(h)=\tau+2t_-\ln h, \tag{43}$$

where the phase $\tau\in[0,2\pi)$ is chosen such that

$$e^{i\tau}=-\frac{t_++it_-}{t_+-it_-}. \tag{44}$$

The operator $\mathcal{A}_{\psi(h)}(\kappa)$ of the problem (2), (3) defined in the space $\mathcal{W}(\Omega;\psi(h))$, that is, with the asymptotic conditions (28), (37), (39), (43), differs crucially from the operator $\mathcal{M}(\kappa)$ of the same problem but with the Umov-Mandelstam radiation conditions

$$u(x)=\chi(x_1)\left(c_{(-)}^{im}u_{(-)}^{im}(x)+c_{(+)}^{re}u_{(+)}^{re}(x)\right)+\tilde{u}(x), \tag{45}$$

which allow only the wave $u_{(-)}^{im}$ outgoing to the top $\mathcal{O}$, and of course, the field $c_{(+)}^{re}u_{(+)}^{re}$ with the finite elastic energy. Indeed, the operator $\mathcal{M}(\kappa)$ get a skew-symmetric component owing to Green's formula (41) whose right-hand side gets the form

$$-4Tit_-c_{(-)}^{im(1)}\overline{c_{(-)}^{\imath m(2)}}.$$

In contrast to $\mathcal{M}(\kappa)$, the operator $\mathcal{A}_{\psi(h)}(\kappa)$ is self-adjoint and possesses the discrete spectrum (42) with the parameter (43). The eigenvalues (42) of $\mathcal{A}_{\psi(h)}(\kappa)$ become (π/t_-)-periodic in the logarithmic scale, namely this period is attributed to the functions

$$\ln h\ \longmapsto k_N(\psi(h)). \tag{46}$$

In the case $\kappa \in (0, \kappa_\dagger)$ the operator of the problem (2), (3) in the energy space $E(\Omega)$ is Fredholm and self-adjoint too. Hence, it has a finite number of eigenvalues below the threshold (5)

$$0 = \kappa_1^0 = \kappa_2^0 = \kappa_3^0 < \kappa_4^0 \leq \cdots \leq \kappa_{N-1}^0 < \kappa_\dagger. \quad (47)$$

Here, the three null eigenvalues are generated by rigid motions, that is, two translations and one rotation, which are generated by the linear solutions (16) of the limit differential equation (15).

Asymptotic forms for the eigenvalues $\kappa_4^0, \dots, \kappa_{N-1}^0$ in (47) are well-known, see [13-16], and we only mention that the perturbation of the null eigenvalues by the Dirichlet conditions (10) is evaluated in [15, 16] as follows:

$$\kappa_p^h = O(h), \quad p = 1,2,3.$$

The justification procedure of the asymptotic forms

$$\kappa_j^h = k_j(\psi(h)) + O(h^\delta), \quad j = N, N+1, \dots \quad (48)$$

for the eigenvalues in the sequence (12) above the threshold (5) is much more complicated. To construct a proper approximation pattern of an eigenmode of the problem (2), (10), (11), we add to an eigenmode $u_{(j)}^0$ with a natural frequency (19) the displacement field

$$c_{(-)}^{re}(h)\, \chi(x_1)\, u_{(-)}^{re}(x) \quad (49)$$

with the very singular behavior but the small coefficient

$$c_{(-)}^{re}(h) = O(h^{t_+})$$

calculated through the system (35) of linear algebraic equations with the entries $c_{(\pm)}^{im}$ taken from the representation (28), (26), (39) of the chosen vector function $u_{(j)}^0$. The term (49) brings sufficiently small discrepancy into the problem (2), (19), (11) so that general results of the spectral theory of self-adjoint operators in Hilbert space, see [5, Ch. 6], provide the desiered asymptotic representation (48) with the exponent $\delta = t_+ > 0$.

It is worth to mention that the proposed structure of the approximate eigenmode with the singular term (49) explains our procedure in Section 3 to impose the "strange" requirement (34) and to derive the algebraic system (35) which leads to the key relationship (36).

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