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GENERALIZED MAGNETO-THERMOELASTICITY FOR AN INFINITE PERFECT CONDUCTING BODY WITH A CYLINDRICAL CAVITY

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Abstract. A model of the equations of generalized magneto-thermoelasticity for a perfect conducting isotropic thermoelastic media developed in [1] is given. This model is applied to solve a problem of an infinite body with a cylindrical cavity is considered in the presence of an axial uniform magnetic field. The boundary of the cavity is subjected to a combination of thermal and mechanical shock acting for a finite period of time. The solution is obtained by a direct approach by using the thermoelastic potential function. Laplace transform techniques are used to derive the solution in the Laplace transform domain. The inversion process is carried out using a numerical method based on Fourier series expansions. Numerical computations for the temperature, the displacement and the stress distributions as well as for the induced magnetic and electric fields are carried out and represented graphically. Comparisons are made with the results predicted by the generalizations, Lord-Shulman theory, and Green-Lindsay theory as well as to the coupled theory.

Nomenclature

ρ density;
t time;
λ, μ Lamé's constants ;
T absolute temperature;
T_0 reference temperature chosen so that $\frac{|T - T_0|}{T_0} << 1$;
u_i components of displacement vector;
σ_{ij} components of stress tensor;
e_{ij} components of strain deviator tensor;
ε_{ij} components of strain tensor;
e $= \varepsilon_{kk}$, dilatation;
δ_{ij} Delta Kronecker;
α_T coefficient of linear thermal expansion;
γ $= (3\lambda+2\mu)\,\alpha_T$;
k thermal conductivity;
C_E specific heat at constant strain;
τ_o, ν relaxation times;

short time effects are considered, the full-generalized system of equations has to be used a great deal of accuracy is lost.

Among the authors who considered the generalized magneto-thermoelasticity equations are Nayfeh and Nasser [9] who studied the propagation of plane waves in a solid under the influence of an electromagnetic field. Choudhuri [10] extend these results to rotating media. Sherief [11] solved a problem for a solid cylinder, while Sherief and Ezzat [12] solved a thermal shock half-space problem using asymptotic expansions. Lately, Sherief and Ezzat [13] solved a problem for an infinitely long annular cylinder, while Ezzat [14-16] and Ezzat et al. [17-22] solved some problems for a perfect conducting media.

In this work we introduced a new model of the equations of generalized thermoelasticity for isotropic perfect conducting media in the presence a constant magnetic field. This model is applied to solve a problem of an infinite perfect conducting isotropic body with cylindrical cavity. The solution is obtained using a direct approach. The resulting formulation together with the Laplace transform technique is applied to a problem considered. The inversion of the Laplace transform is carried out using a numerical technique [23]. The results obtained are represented graphically.

2. Formulation of the problem

We shall consider a thermoelastic medium of perfect conductivity permeated by an initial magnetic field $\mathbf{H_o}$. Due to the effect of this magnetic field there arises in the conducting medium an induced magnetic field **h** and induced electric field **E**. Also, there arises a force **F** (the Lorentz force). Due to the effect of the force, points of the medium undergo a displacement **u**, which gives rise to a temperature. The linearized equations of electromagnetism for slowly moving media [14]

$$\text{Curl}\ \mathbf{h} = \mathbf{J} + \varepsilon_o \frac{\partial \mathbf{E}}{\partial t}, \tag{1}$$

$$\text{Curl}\ \mathbf{E} = -\mu_o \frac{\partial \mathbf{h}}{\partial t}, \tag{2}$$

$$\mathbf{E} = -\mu_o \frac{\partial \mathbf{u}}{\partial t} \wedge \mathbf{H_o}, \tag{3}$$

$$\text{div}\ \mathbf{h} = 0. \tag{4}$$

The above equations are supplemented by the displacement equations of the theory of generalized thermoelasticity, taking into account the external body force, which is here equal to the Lorentz force [15]

$$\rho \frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu) u_{j,ij} + \mu u_{i,jj} - \gamma (T + \nu \frac{\partial T}{\partial t})_{,i} + \mu_o (J \wedge H_o)_i, \tag{5}$$

and the generalized heat conduction equation [1]:

$$k T_{,ii} = \rho C_E \left(\frac{\partial T}{\partial t} + \tau_o \frac{\partial^2 T}{\partial t^2} \right) + \gamma T_o \left(\frac{\partial u_{i,i}}{\partial t} + n_o \tau_o \frac{\partial^2 u_{i,i}}{\partial t^2} \right) - \left(Q + n_o \tau_o \frac{\partial Q}{\partial t} \right), \tag{6}$$

where n_o is a constant.
The constitutive equation:

$$E = \mu_o H_o \frac{\partial u}{\partial t}. \tag{16}$$

Expressing the components of the vector **J** in terms of displacement, by eliminating from Eq. (14) the quantities h and E and introducing them into the displacement Eq. (5), we arrived at

$$\rho\,(1+\frac{\alpha_o^2}{c^2})\frac{\partial^2 u}{\partial t^2} = \rho(c_1^2+\alpha_o^2)\frac{\partial e}{\partial r} - \gamma\frac{\partial}{\partial r}\left(T + \nu\frac{\partial T}{\partial t}\right), \tag{17}$$

Equation (6) is to be supplemented by the constitutive Eqs. (10) and the heat conduction equation

$$k\nabla^2 T = \rho C_E\left(\frac{\partial}{\partial t} + \tau_o\frac{\partial^2}{\partial t^2}\right)T + \gamma T_o\left(\frac{\partial}{\partial t} + n_o\tau_o\frac{\partial^2}{\partial t^2}\right)e - \left(1 + n_o\tau_o\frac{\partial}{\partial t}\right)Q, \tag{18}$$

where ∇^2 is Laplace's operator in cylindrical coordinates, given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \Psi^2} + \frac{\partial^2}{\partial z^2}.$$

In case of dependence on r only, this reduces to

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}.$$

We shall use the following non-dimensional variables

$$r' = c_1\eta r, \quad u' = c_1\eta r, \quad t' = c_1^2\eta t, \quad \nu' = c_1^2\eta\nu, \quad \tau_o' = c_1^2\eta\tau_o, \quad \sigma_{ij}' = \sigma_{ij}/\mu, \quad e' = e,$$

$$\theta = (T - T_o)/T_o, \quad Q' = Q/kT_o\eta^2 c_1^2, \quad h' = \frac{h}{H_o}, \quad E' = \frac{E}{\mu_o H_o c_1}, \quad J' = \frac{J}{H_o c_1 \eta}.$$

Equations (11)-(18) take the following form (dropping the primes for convenience)

$$J = -\left(\frac{\partial h}{\partial r} + \frac{c_1^2}{c^2}\frac{\partial^2 u}{\partial t^2}\right), \tag{19}$$

$$h = -\,e, \tag{20}$$

$$E = \frac{\partial u}{\partial t}, \tag{21}$$

$$\alpha_1\frac{\partial^2 u}{\partial t^2} = \alpha_2\frac{\partial e}{\partial r} - b\frac{\partial}{\partial r}\left(\theta + \nu\frac{\partial \theta}{\partial t}\right), \tag{22}$$

$$\nabla^2\theta = \left(\frac{\partial}{\partial t} + \tau_o\frac{\partial^2}{\partial t^2}\right)\theta + g\left(\frac{\partial}{\partial t} + n_o\tau_o\frac{\partial^2}{\partial t^2}\right)e - \left(1 + n_o\tau_o\frac{\partial}{\partial t}\right)Q, \tag{23}$$

$$\bar{J} = \frac{\partial}{\partial r}\left(\nabla^2 \overline{\Phi} - p^2 V^2 \overline{\Phi}\right), \tag{36}$$

$$\bar{h} = -\nabla^2 \overline{\Phi}, \tag{37}$$

$$\bar{E} = p\frac{\partial \overline{\Phi}}{\partial r}, \tag{38}$$

$$(\nabla^2 - \alpha p^2)\overline{\Phi} = a(1 + \nu p)\bar{\theta}, \tag{39}$$

$$\nabla^2 \bar{\theta} = p(1 + \tau_o p)\bar{\theta} + g p(1 + n_o \tau_o p)\nabla^2 \overline{\Phi}, \tag{40}$$

$$\bar{\sigma}_{rr} = \beta^2 \nabla^2 \bar{\phi} - \frac{2}{r}\frac{\partial \bar{\phi}}{\partial r} - b(1 + \nu p)\bar{\theta}, \tag{41}$$

$$\bar{\sigma}_{\psi\psi} = (\beta^2 - 2)\nabla^2 \bar{\phi} + \frac{2}{r}\frac{\partial \bar{\phi}}{\partial r} - b(1 + \nu p)\theta, \tag{42}$$

Eliminating $\bar{\theta}$ between Eqs (35) and (36), we obtain the following fourth-order partial differential equation satisfied by $\overline{\Phi}$

$$\left\{\nabla^4 - [p(1+\varepsilon) + p^2(\alpha + \tau_o + \varepsilon\nu + \varepsilon n_o \tau_o (1 + \nu p))]\nabla^2 + \alpha p^2 (p + \tau_o p^2)\right\}\overline{\Phi} = 0 \,, \tag{43a}$$

where $\varepsilon = ga$.
This equation can be written in the form

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)\overline{\Phi} = 0, \tag{43b}$$

where k_1, k_2 are the positive roots of the characteristic equation

$$k^4 - p[1 + \varepsilon + p(\alpha + \tau_o + \varepsilon\nu + \varepsilon n_o \tau_o (1 + \nu p)]k^2 + \alpha p^3 (1 + \tau_o p) = 0. \tag{44}$$

The solution of the equation is given by

$$\bar{\theta} = \sum_{i=1}^{2} A_i K_o(k_i r), \tag{45}$$

$$\overline{\Phi} = \sum_{i=1}^{2} B_i K_o(k_i r) \tag{46}$$

where $K_0(*)$ is the modified Bessel function of the second kind of order zero from equation (39), we can get the parameters B_i

$$B_i = \frac{a(1 + \upsilon p)}{k_i^2 - \alpha p^2} A_i \quad i = 1,2 \tag{47}$$

Finally, we get the following equations

$$\bar{\theta} = \sum_{i=1}^{2} A_i K_o(k_i r), \tag{48}$$

$$a_{11} = -K_0(k_2R)(k_2^2-\alpha p^2)(k_1^2-\alpha p^2)p\,,$$

$$a_{12} = (k_1^2-\alpha p^2)(\alpha\beta^2p^3K_0(k_2R)+2K_1(k_2R)k_2),$$

$$a_{21} = K_0(k_1R)(k_1^2-\alpha p^2)(k_2^2-\alpha p^2)p\,,$$

$$a_{22} = -(k_2^2-\alpha p^2)(\alpha\beta^2p^3K_0(k_1R)+2K_1(k_1R)k_1),$$

$$\omega = \alpha\beta^2K_0(k_1R)K_0(k_2R)p^4(k_1^2-k_2^2)+2K_0(k_1R)K_1(k_2R)k_2p(k_1^2-\alpha p^2)$$

$$-2K_0(k_2R)K_1(k_1R)k_1p(k_2^2-\alpha p^2).$$

Finally, we have the solution in the Laplace transform domain in the following forms

$$\bar{\theta} = \frac{1}{\omega}\sum_{i=1}^{2}[a_{i1}\sigma_o + a_{i2}\theta_o]K_o(k_i r), \tag{58}$$

$$\bar{u} = \frac{-a(1+\upsilon p)}{\omega}\sum_{i=1}^{2}\frac{k_i[a_{i1}\sigma_o + a_{i2}\theta_o]}{k_i^2-\alpha p^2}K_1(k_i r), \tag{59}$$

$$\bar{h} = -\frac{a(1+\upsilon p)}{\omega}\sum_{i=1}^{2}\frac{k_i^2[a_{i1}\sigma_o + a_{i2}\theta_o]}{k_i^2-\alpha p^2}K_o(k_i r), \tag{60}$$

$$\bar{E} = \frac{-ap(1+\upsilon p)}{\omega}\sum_{i=1}^{2}\frac{k_i[a_{i1}\sigma_o + a_{i2}\theta_o]}{k_i^2-\alpha p^2}K_1(k_i r), \tag{61}$$

$$\bar{\sigma}_{rr} = \frac{a(1+\upsilon p)}{\omega}\sum_{i=1}^{2}\frac{[a_{i1}\sigma_o + a_{i2}\theta_o]}{k_i^2-\alpha p^2}\left[\alpha\beta^2p^2K_o(k_i r)+\frac{2k_i}{r}K_1(k_i r)\right], \tag{62}$$

$$\bar{\sigma}_{\psi\psi} = \frac{a(1+\upsilon p)}{\omega}\sum_{i=1}^{2}\frac{[a_{i1}\sigma_o + a_{i2}\theta_o]}{k_i^2-\alpha p^2}\left[(\alpha\beta^2p^2-2)K_o(k_i r)-\frac{2k_i}{r}K_1(k_i r)\right], \tag{63}$$

5. Inversion of the Laplace transform

We shall now outline the method used to invert the Laplace transform in the above equations.

Let $\bar{f}(s)$ be the Laplace transform of a function f(t). The inversion formula for Laplace transform can be written as

$$f(t) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{st}\,\bar{f}(s)\,ds\,,$$

where d is an arbitrary real number greater than all the real parts of the singularities of $\bar{f}(s)$. Taking s = d + i y , the above integral takes the form

$$f(t) = \frac{e^{dt}}{2\pi}\int_{-\infty}^{\infty} e^{ity}\,\bar{f}(d+iy)dy\,.$$

Expanding the function h(t) = exp(-dt) f(t) in a Fourier series in the interval [0, 2L], we obtain the approximate formula [23]:

s_m , m =1,2,3, ...

All constants are given in SI units.
The actual procedure used to invert the Laplace transforms consists of using equation (67) together with the ε-algorithm. The values of d and L are chosen according the criteria outlined in [23].

6. Numerical results

The copper material was chosen for purposes of numerical evaluations. The computations were carried out for one value of time, namely for t=0.15 and we have considered $\sigma_o = \theta_o = 1$. The Figures 1, 2 and 3 represent the solutions for thermo-mechanical problem while, Figs 4, 5 and 6 represent the solution for thermal shock problem. The temperature distributions are shown in Figs. 1 and 4, the displacement distributions are shown in Figs. 2 and 5 and the radial stress components distributions are shown in Figs. 3 and 6. In all these figures dotted lines represent the values predicted by the coupled theory, solid lines represent the values predicted by the L-S theory and dashed lines represent the values predicted by the G-L theory. The constants of the problem are given in the Table 1.

Table 1. Values of the constants.

k= 386 N/K sec	α_T= 1.78 10^{-5} K^{-1}	C_E= 383.1 $m^2/K\ sec^2$	η= 8886.73 m/sec^2
μ= 3.86 10^{10} N/m^2	λ= 7.76 10^{10} N/m^2	c_1= 4.158 10^3 m/sec	ρ= 8954 kg/m^3
ε_o= $10^{-9}/(36\pi)$ $C^2/N\ m^2$	μ_o= $4\pi\ 10^{-7}$ N m sec^2 C^{-2}	H_o= 1 C/m sec	τ_o=0.02
υ= 0.03	b= 0.042	g= 1.61	β^2= 4
V= 1.39 10^{-5}	T_o= 293 K	α_1= 1.005	α_2= 1.001

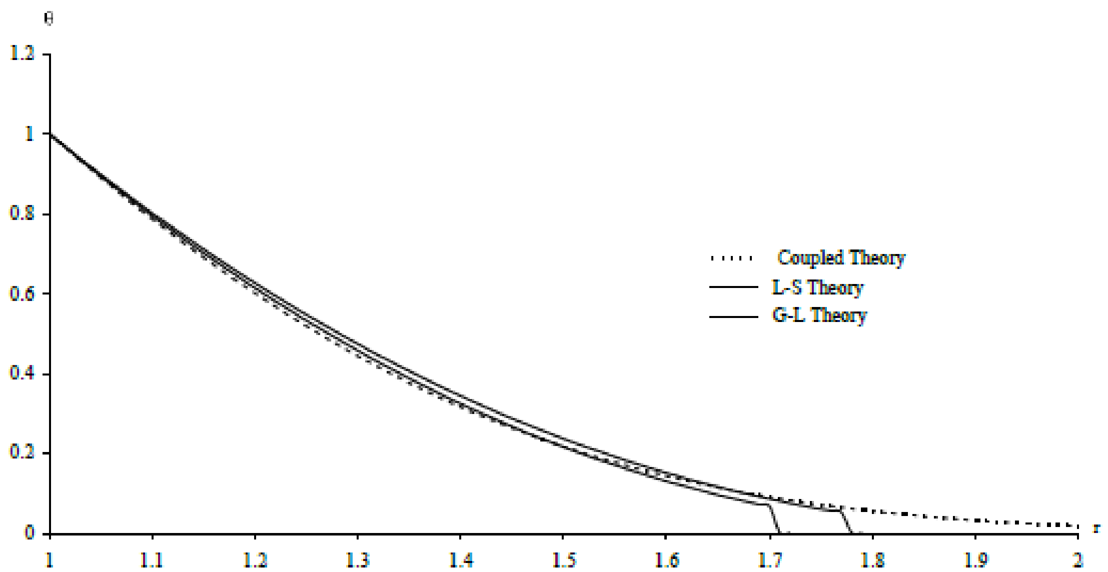

Fig. 1. Temperature distribution due to thermo-mechanical shock.

It is clear from all figures that the results for generalized thermoelasticity are distinctly different from those of coupled thermoelasticity. The solution of any of the considered

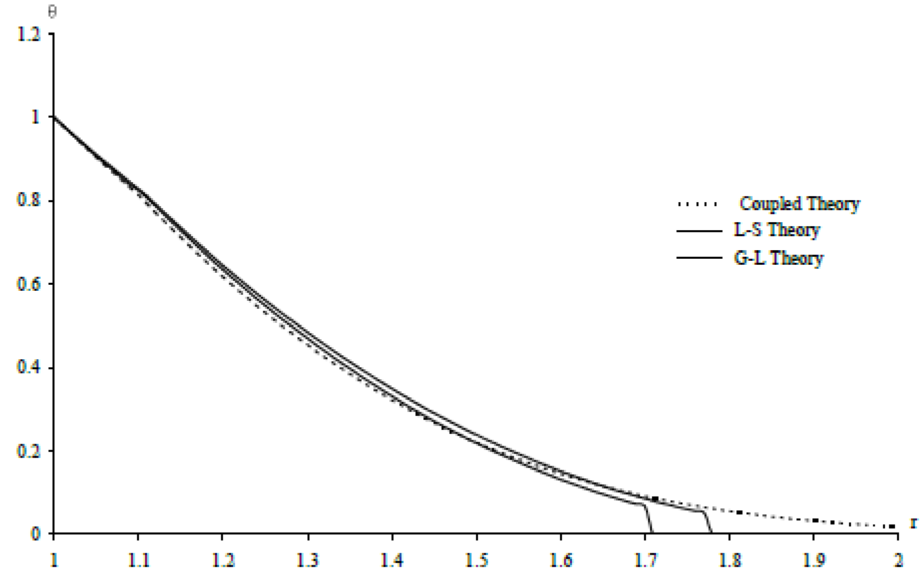

Fig. 4. Temperature distribution due to thermal shock.

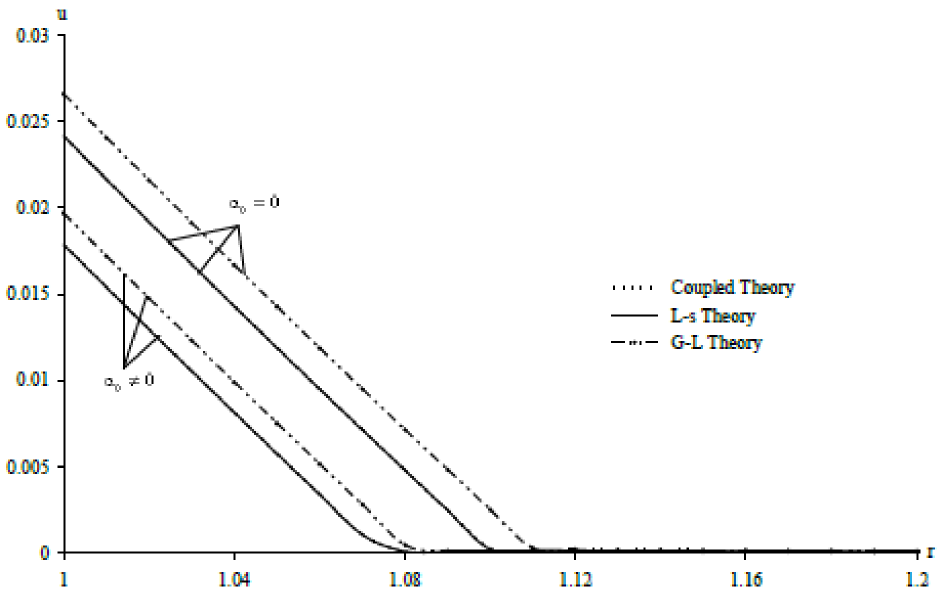

Fig. 5. Displacement distribution due to thermal shock.

7. Appendix

For the thermal shock problem, we can get the solution by substitution $\sigma_o = 0$ in the equations (58)-(63), we get

$$\bar{\theta} = \frac{\theta_o}{\omega} \sum_{i=1}^{2} a_{i2} K_o(k_i\, r)\,, \tag{68}$$

$$\bar{u} = \frac{-a\theta_o(1+\upsilon p)}{\omega} \sum_{i=1}^{2} \frac{k_i\, a_{i2}}{k_i^2 - \alpha p^2} K_1(k_i\, r), \tag{69}$$

$$\bar{h} = -\frac{a\,\theta_o(1+\upsilon p)}{\omega} \sum_{i=1}^{2} \frac{k_i^2 a_{i2}}{k_i^2 - \alpha p^2} K_o(k_i\, r), \tag{70}$$

$$\overline{E} = \frac{-a\,\theta_o\, p(1+\upsilon p)}{\omega}\sum_{i=1}^{2}\frac{k_i a_{i2}}{k_i^2-\alpha p^2}K_1(k_i\, r), \tag{71}$$

$$\overline{\sigma}_{rr} = \frac{a\,\theta_o\,(1+\upsilon p)}{\omega}\sum_{i=1}^{2}\frac{a_{i2}}{k_i^2-\alpha p^2}\left[\alpha\beta^2 p^2 K_o(k_i\, r)+\frac{2k_i}{r}K_1(k_i\, r)\right], \tag{72}$$

$$\overline{\sigma}_{\psi\psi} = \frac{a\,\theta_o(1+\upsilon p)}{\omega}\sum_{i=1}^{2}\frac{a_{i2}}{k_i^2-\alpha p^2}\left[\left(\alpha\beta^2 p^2-2\right)K_o(k_i\, r)-\frac{2k_i}{r}K_1(k_i\, r)\right], \tag{73}$$

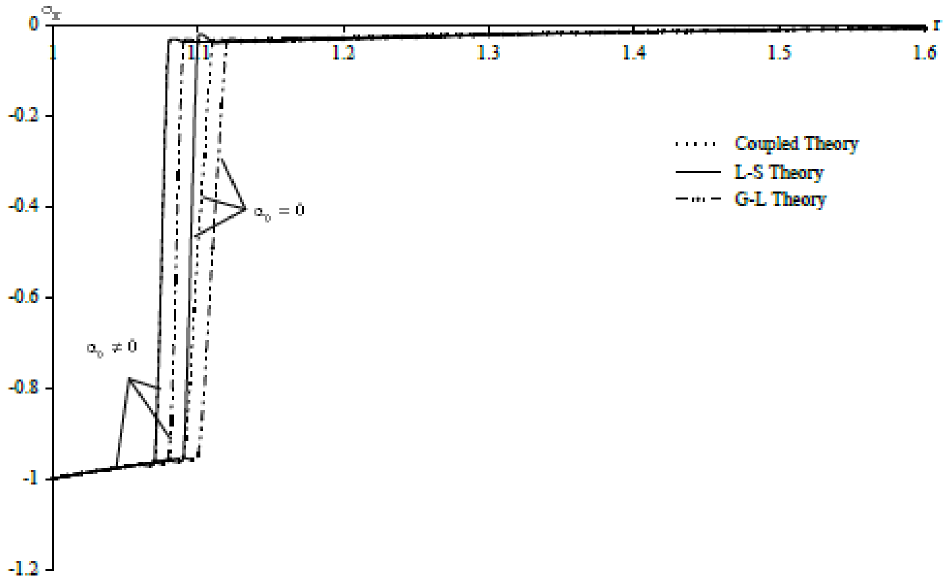

Fig. 6. Radial stress distribution due to thermal shock.

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