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ON CALCULATION OF EFFECTIVE ELASTIC PROPERTIES OF MATERIALS WITH CRACKS

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Abstract. A simple approach for calculation of anisotropic effective elastic properties of cracked materials is presented. Square computational domain containing randomly distributed cracks under plane strain conditions is considered. Effective elastic properties are expressed in terms of average displacement discontinuities on cracks in three test problems: uniaxial loading in two orthogonal directions and pure shear. These problems are solved using the displacement discontinuity method. Resulting effective compliances are averaged over realizations with different crack distributions. This approach is employed for calculation of effective elastic properties for two particular crack configurations: (i) one family of parallel cracks and (ii) two families of parallel cracks inclined at angle 30^0. Crack densities up to 0.8 are considered. It is shown that for both configurations the effective elastic properties are orthotropic even at large crack densities. Dependencies of Young's moduli on crack density are obtained. At crack densities up to 0.1, the effective properties can be estimated analytically using the non-interaction approximation (NIA). At higher crack densities, the NIA strongly overestimates effective stiffnesses. Quantitative agreement with results obtained in the literature using more sophisticated methods is demonstrated.

Keywords: effective elastic properties; cracked materials; crack interactions; orthotropy; non-interaction approximation; boundary element method; displacement discontinuity method.

1. Introduction

Calculation of effective elastic properties of materials with cracks is a long standing problem in mechanics of materials. Changes in effective elastic properties due to cracks can be very significant [1]. Therefore accurate prediction of these properties is crucial in many fields, including mechanical engineering [2, 3], geomechanics [4, 5], material science [6] etc.

In literature, the effective elastic properties are computed both analytically and numerically. In the framework of analytical methods, effective elastic properties are represented as a function of crack density [1] (or crack density tensor [3]). At relatively low crack densities, the effective properties can be estimated using the non-interaction approximation (NIA) [3]. In the NIA, it is assumed that the effect of many cracks is equal to a sum of independent effects from individual cracks. The calculation of effective properties is reduced to calculation of average displacement discontinuity for a single crack subjected to a given loading at infinity. With increasing crack density, the effect of mutual influence of cracks become significant. Then more accurate approximate methods, such as differential scheme [7], self-consistent scheme [2], Mori-Tanaka scheme [8] should be used. However, these schemes yield different dependencies of effective properties on crack density and the choice of a proper scheme is not always straightforward. Moreover, approximate schemes [9], except for the NIA, are usually

limited to the case of uniform distribution of crack orientations (isotropic effective properties). Therefore, at high crack densities, anisotropic effective elastic properties are usually calculated numerically.

Numerical calculation of effective properties of cracked materials is also a challenge. To calculate the effective properties, deformation of the computational domain containing large number of cracks under the given loads should be considered. In the two-dimensional case, reasonable accuracy can be reached either if the number of cracks in the computational domain is of order of 10^4 or if averaging over realizations with different crack distributions is used. In the latter case, number of cracks in the computational domain can be of order of 10^2. Periodic boundary conditions also allow to minimize finite size effects [1]. Deformation of the computational domain can be described using, for example, the finite element method (FEM) or the boundary element method (BEM). Finite element solution requires very fine mesh, especially at high crack densities. In BEM, only the boundaries are discretized. This allows to decrease the number of degrees of freedom, compared to FEM. At the same time, the matrix corresponding to a system of linear equations of BEM is dense. Therefore, the choice between two methods is not straightforward.

In the present paper, we present a simple approach for calculation of effective elastic properties in the two-dimensional case (plane strain). The square computational domain containing randomly distributed cracks is considered. For each crack distribution, deformation of the computational domain under three different loads is simulated using the displacement discontinuity method [10] (the simplest version of the BEM). Then effective compliances are calculated using average displacement discontinuities on cracks [3]. Additionally, the compliances are averaged over realizations in order to reduce finite size effects. This simple approach is employed for calculation of anisotropic effective elastic properties for two crack configurations: (i) one family of parallel cracks and (ii) two families of parallel cracks inclined at angle 30^0. Numerical results allow to estimate the range of applicability of the NIA and to study anisotropic elastic properties of the cracked material. Comparison with results obtained in the literature using more sophisticated methods [1] is carried out.

2. Calculation of effective elastic properties of cracked materials

In the present section, we describe the approach for calculation of effective elastic properties of cracked materials. Two-dimensional statement (plane strain) is considered. The matrix material is isotropic, linearly elastic.

To calculate elastic properties, we consider deformation of a square computational domain containing cracks under three different loads: uniaxial load in two orthogonal directions and pure shear. In these cases, the mean strain, $\boldsymbol{\varepsilon}$, and applied stresses, $\boldsymbol{\sigma^0}$, are related to components of the effective compliance tensor M by Hooke's law as follows.

- Uniaxial load in the "horizontal" direction ($\sigma^{\mathbf{0}} = \sigma^0 e_1 e_1$):

$$\begin{pmatrix} M_{1111} \\ M_{2211} \\ M_{1211} \end{pmatrix} = \frac{1}{\sigma^0} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{pmatrix}. \tag{1}$$

- Uniaxial load in the "vertical" direction ($\sigma^0 = \sigma^0 e_2 e_2$):

$$\begin{pmatrix} M_{1122} \\ M_{2222} \\ M_{1222} \end{pmatrix} = \frac{1}{\sigma^0} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{pmatrix}. \tag{2}$$

- Shear load $\sigma^0 = \sigma^0 (e_1 e_2 + e_2 e_1)$:

$$\begin{pmatrix} M_{1112} \\ M_{2212} \\ M_{1212} \end{pmatrix} = \frac{1}{\sigma^0} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{pmatrix}. \tag{3}$$

Tractions corresponding to loads (1)-(3) are applied at boundaries of the computational domain. The mean strain is calculated. Given known the mean strain, the effective properties are calculated using formulas (1)-(3).

The mean strain, ε, is calculated using the following decomposition [3]:

$$\varepsilon = \varepsilon_0 + \Delta\varepsilon, \tag{4}$$

where ε_0 is the strain in material without cracks (known), $\Delta\varepsilon$ is the unknown extra strain due to cracks. The extra strain due to cracks, $\Delta\varepsilon$, is expressed in terms of the average displacement discontinuities (jumps of displacement) on cracks as follows [3]:

$$\Delta\varepsilon = \frac{1}{2A}\sum_k (bn + nb)^{(k)}, \tag{5}$$

where $b^{(k)}$ is the average displacement discontinuity on the k-th crack; $n^{(k)}$ is unit normal to k-th crack, A is the area of the computational domain. Crack openings, $b^{(k)}$, under given loads (1)-(3) are calculated using the displacement discontinuity method [10], described in the next section.

Thus, the effective elastic constants are computed as follows. For each crack distribution, cracks and sides of the computational domain are divided into boundary elements. The system of linear algebraic equations for displacement discontinuities in all boundary elements is solved under loading conditions, corresponding to (1)-(3). The extra strains due to cracks are calculated using formula (5). Then components of the effective compliance tensor are computed using formulas (1)-(3). Since the distribution of cracks is random, the resulting compliances are averaged over realizations with different crack distributions.

In the following sections, this simple procedure is employed for calculation of effective properties for two particular crack configurations.

3. The displacement discontinuity method (DDM)

In the present section, we describe numerical method used for calculation of displacement discontinuities on cracks.

The displacement discontinuity method (DDM), introduced in the paper [10], is the simplest version of the boundary element method [11]. Boundaries of the computational domain and cracks are divided into elements, each having the normal and the shear displacement discontinuities, D_y and D_x (Fig. 1). The displacement discontinuity along a crack is a peace-wise constant function (e.g. zero order approximation is used).

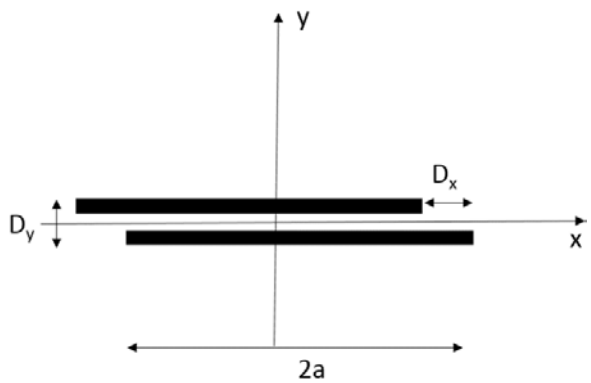

Fig. 1. Displacement discontinuities associated with one boundary element.

Each element can be subjected to the shear and normal tractions. The displacement discontinuities, D, and the tractions, T, at all elements form columns:

$$D = \begin{pmatrix} D_x^1 \\ D_y^1 \\ \cdots \\ D_x^N \\ D_y^N \end{pmatrix}, \; T = \begin{pmatrix} T_x^1 \\ T_y^1 \\ \cdots \\ T_x^N \\ T_y^N \end{pmatrix}, \tag{6}$$

where N is the total number of elements. Displacement discontinuities and tractions are interrelated by a system of linear equations:

$$\begin{aligned} T_x^i &= \sum_{j=1}^N A_{xx}^{ij} D_x^j + \sum_{j=1}^N A_{xy}^{ij} D_y^j, \; i = 1,..N, \\ T_y^i &= \sum_{j=1}^N A_{yx}^{ij} D_x^j + \sum_{j=1}^N A_{yy}^{ij} D_y^j, \; i = 1,..N. \end{aligned} \tag{7}$$

Here:

$$\begin{aligned} A_{xx}^{ij} &= -2G\left(-\sin(2\gamma) f_{xy} + \cos(2\gamma) f_{xx} + y\big(\sin(2\gamma) f_{xyy} - \cos(2\gamma) f_{yyy}\big)\right), \\ A_{xy}^{ij} &= -2Gy\big(\cos(2\gamma) f_{xyy} + \sin(2\gamma) f_{yyy}\big), \\ A_{yx}^{ij} &= 2G\big(2\sin^2\gamma f_{xy} + \sin(2\gamma) f_{xx} - y\big(\cos(2\gamma) f_{xyy} + \sin(2\gamma) f_{yyy}\big)\big), \\ A_{yy}^{ij} &= -2G\left(f_{xx} - y\big(\sin(2\gamma) f_{xyy} - \cos(2\gamma) f_{yyy}\big)\right), \end{aligned} \tag{8}$$

$$f = -\frac{1}{4\pi(1-\nu)}\left(y\left(arctg\frac{y}{x-a} - arctg\frac{y}{x+a}\right) - (x-a)ln\sqrt{(x-a)^2+y^2} + (x+a)ln\sqrt{(x+a)^2+y^2}\right),$$

where γ is the angle between elements i и j; G is the shear modulus; x, y are local coordinates of the j-*th* element in the coordinate system of the i-th element. Note that the matrix, corresponding to system (7), is dense.

Thus, displacement discontinuities on cracks, D, under given loads, are calculated using the system of linear equations (7). The resulting average discontinuities on cracks are used for calculation of effective compliances of the material as described in the previous section.

4. Test problem: opening of a single crack

To verify numerical implementation of the DDM, the problem of a single crack in a square domain under uniaxial tension is solved (see Fig. 2). This problem also allows to estimate the minimum number of elements required for discretization of cracks in calculations of effective properties.

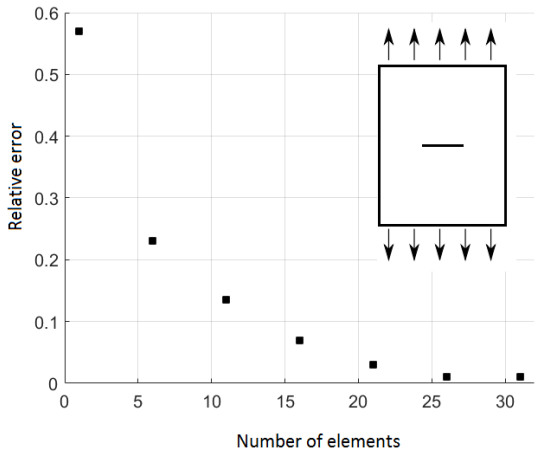

Fig. 2. Relative error in average crack opening as a function of number of elements per crack.

The crack and sides of the computational domain are divided into elements. The ratio of domain size to crack length is equal to 20. Trial and error approach shows that it is sufficient to

use 40 elements per side of the computational domain. Average opening of the crack is calculated. Results are compared with the exact analytical solution for a crack in an infinite domain [3]. The relative error (Δ) in average crack opening as a function of number of elements per crack is shown in Fig. 2.

Fig. 2 shows that the DDM overestimates the crack opening. Therefore, the compliance of cracked material is also overestimated. Acceptable accuracy (relative error about 3 %) is reached, when 20 elements per crack are used. Therefore, this number of elements is used in further calculations.

5. Calculation of anisotropic effective elastic properties

In the present section, we compute effective elastic properties for two crack configurations: (i) one family of parallel cracks and (ii) two families of parallel cracks inclined at 30°.

In our simulations, all cracks have the same length. To operate with dimensionless quantities, effective compliances are normalized by the shear modulus of the matrix. The 3D Poisson's ratio equal to 1/4 is used, implying 1/3 in the case of plane strain. Size of the computational domain is normalized by crack length.

The crack density ρ is calculated as [3]:

$$\rho = \frac{Nl^2}{4A}, \tag{9}$$

where N is the number of cracks in computation domain, l is a crack length, A is the area of the computation domain. Examples of crack arrays with different crack densities are shown in Fig. 3.

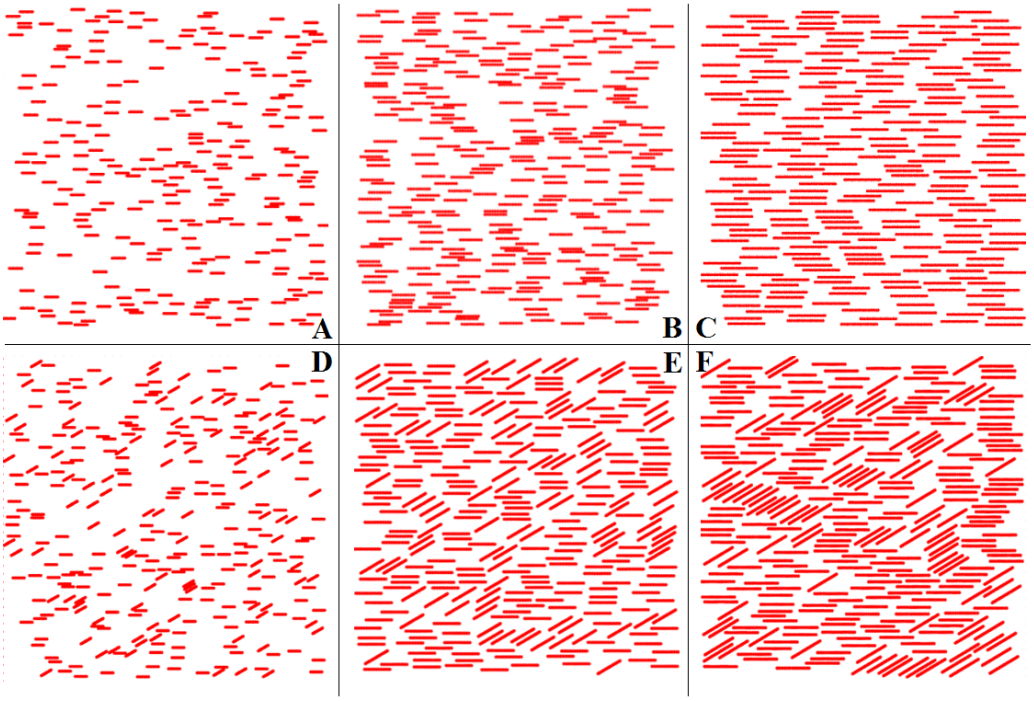

Fig. 3. Examples of crack arrays with different crack densities ($\rho = 0.1, \rho = 0.5, \rho = 0.8$). One family of parallel cracks (A)-(C) and two families of parallel cracks inclined at angle 30° (D)-(F).

In all simulations, crack densities were in the interval $0.01 \leq \rho \leq 0.8$. For each value of crack density, 450 to 650 crack arrays were generated, with locations of crack centers determined by random number generator and subject to the restriction of the minimal distance

between cracks being larger than 0.1 crack length. Details of numerical simulations are summarized in Table 1.

Table 1. Parameters of numerical simulations.

Parameter	Value
Number of cracks in one array	300-350
Number of boundary elements per one crack	20
Number of boundary elements per one side of the computational domain	40
Number of realizations for each value of crack density	450-650

5.1. One family of parallel cracks. Consider effective elastic properties of a material containing randomly distributed parallel cracks (see subfigures (A)-(C) in Fig. 3). The effective properties are calculated using the approach described in section 2.

It is shown that the effective elastic properties are orthotropic. One of the orthotropic axes is collinear with cracks. Non-zero components of the effective stiffness tensor (averaged over realizations) are shown in Fig. 4. Predictions of the non-interaction approximation and results obtained in the paper [3] are also shown in Fig. 4.

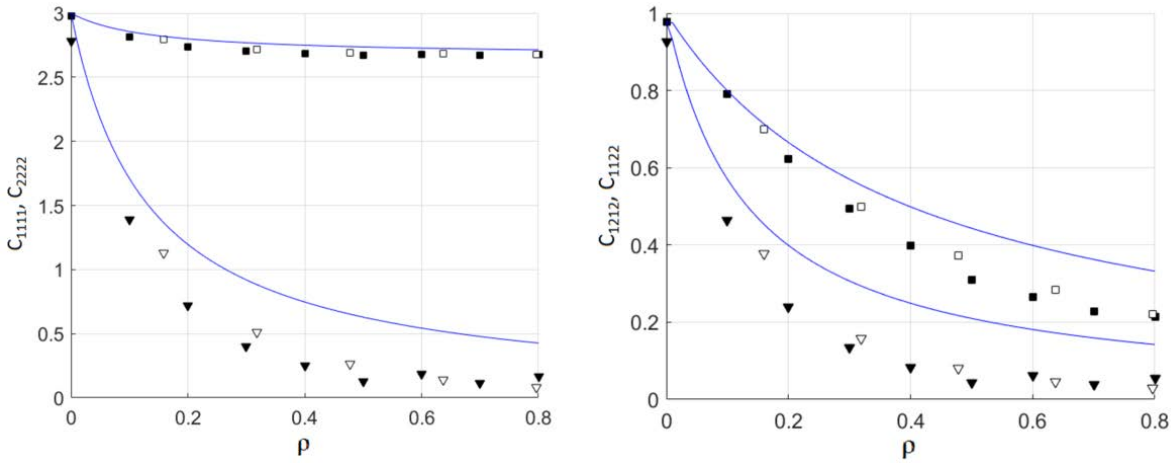

Fig. 4. Effective stiffnesses for a material with parallel cracks. Our results (filled squares and triangles) and results obtained in the paper [1] (empty squares and triangles). Solid lines show predictions of the non-interaction approximation.

It is seen from Fig. 4 that our results are in a good agreement with ones obtained in paper [1] by more accurate method. Stiffnesses, predicted by our method, are below the values obtained in the paper [1]. The reason for this difference is that the DDM slightly overestimates crack openings (see section 4). Therefore, compliances are also overestimated and stiffnesses are underestimated. At the same time, the approach described in the present paper is significantly simpler.

Note that the NIA has reasonable accuracy at crack densities up to 0.1. Therefore, in this case, the effective elastic properties can be estimated numerically.

From engineering point of view, it may be more informative to plot effective Young's moduli rather than stiffnesses. Young's moduli are related to components of the compliance tensor, M, as follows

$$E_x = \frac{1}{M_{1111}}, E_y = \frac{1}{M_{2222}}, \tag{10}$$

where the x-axis is directed along cracks. Effective Young's moduli of a material with parallel cracks are shown in Fig. 5.

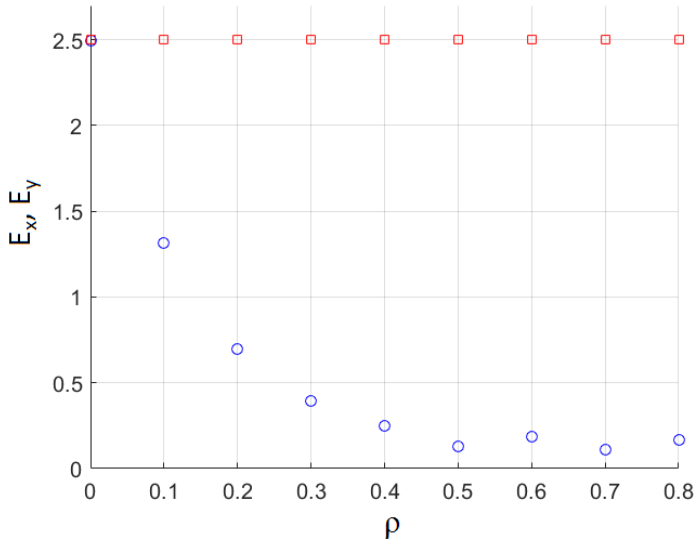

Fig. 5. Young's moduli of a material with parallel cracks, in the case of parallel family of cracks (squares – E_x, circles – E_y).

As expected, Young's modulus in the direction of cracks, E_x, does not depend on crack density. In contrast, Young's modulus E_y is significantly affected by cracks.

5.2. Two families of parallel cracks inclined at angle 30°. Consider effective elastic properties of a material containing two families of parallel cracks inclined at angle 30° (see subfigures (D)-(F) in Fig. 3). Partial crack densities of the two families differ by the factor of two: $\rho_1 = 2\rho_2$, $\rho_1 + \rho_2 \equiv \rho$. This problem was originally analyzed in the paper [3] using the non-interaction approximation. In the paper [3], it was shown that the effective elastic properties are orthotropic. The angle between one of the orthotropic axes and *x*-axis (horizontal in Fig. 3) is approximately equal to 9.8° (counter-clockwise rotation). In the present section, we show that at high crack densities the effective properties are still orthotropic.

The effective compliance tensor is computed as described in section 2. We assume that the effective properties are orthotropic. Principal axes of the compliance tensor are found as follows. Coordinate axes are rotated by angle α. Components of the compliance tensor in new (rotated axes) are calculated using formulas [12]:

$$M^{rot} = F^T M F, F(\alpha) = \begin{pmatrix} cos^2\alpha & sin^2\alpha & 2\,sin\,\alpha\,cos\,\alpha \\ sin^2\alpha & cos^2\alpha & -2\,sin\,\alpha\,cos\,\alpha \\ -\,sin\,\alpha\,cos\,\alpha & sin\,\alpha\,cos\,\alpha & cos^2\alpha - sin^2\alpha \end{pmatrix}, \tag{11}$$

where α is a rotation angle, F^T is transposed rotation matrix. Since the orthotropy of effective properties is approximate, we find the principal axes such that deviation from orthotropy, δ, has minimum. The deviation is measured by the Euclidean norm [13]:

$$\delta = \frac{\sqrt{M_{1112}^2 + M_{2212}^2 + M_{1222}^2 + M_{1211}^2}}{\|M\|} \tag{12}$$

Minimization of δ with respect to rotation angle α yields the orientation of principal axes. Dependencies of deviation from orthotropy, δ, on rotation angle, α, for several values of crack density are shown in Fig. 6. It is seen that for all crack densities δ has minimum at $\alpha \approx 9.5^0$. This value is in a good agreement with prediction of the non-interaction approximation ($\alpha_{\mathrm{NIA}} \approx 9.8^0$). Therefore, the effective elastic properties are orthotropic even at high crack densities.

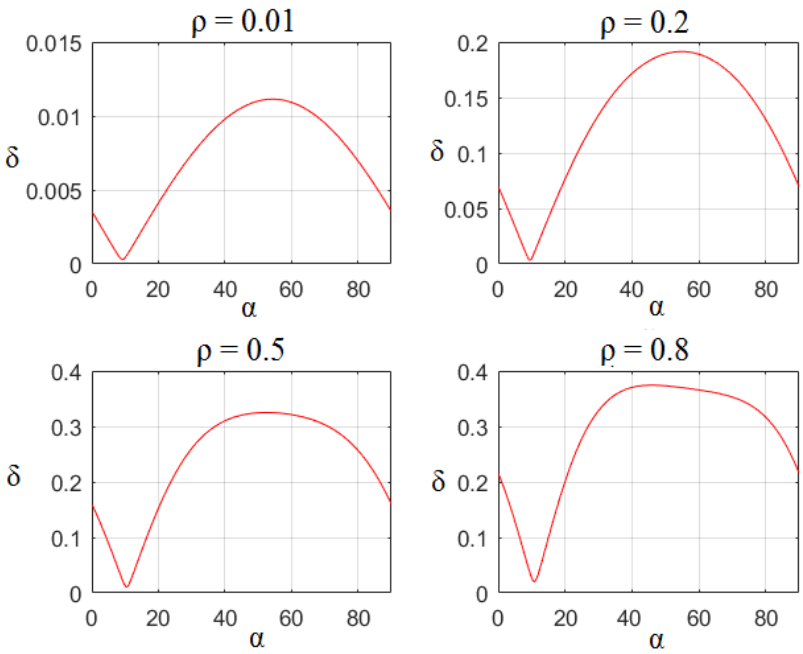

Fig. 6. Deviation from orthotropy, measured by norm (12), as a function of the rotation angle. Note that minimum value of deviation, δ, is less than several percent.

Effective Young's moduli in principal axes of the compliance tensor are calculated (see Fig. 7). In contrast to the previous problem, both moduli decrease with increasing crack density.

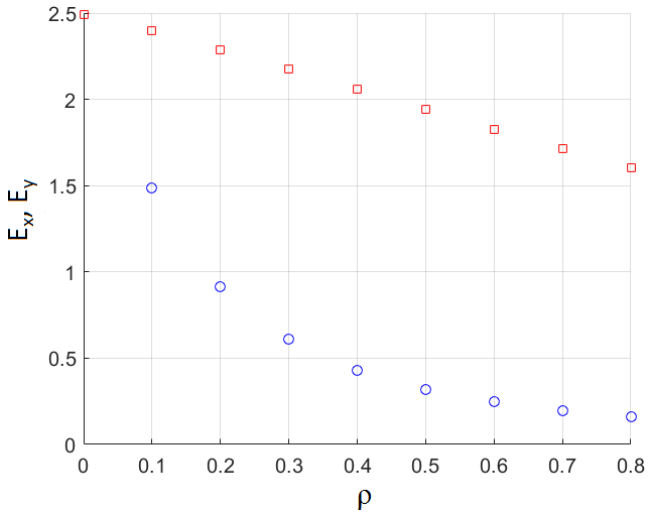

Fig. 7. Young's moduli for a material with two families of parallel cracks inclined at angle 30° (squares – E_x, circles – E_y). Here x and y are the principal axes of the compliance tensor.

6. Conclusions

A simple approach for calculation of effective elastic properties for cracked materials was presented. The approach has three main ingredients: (i) relation between extra strain due to cracks and average displacement discontinuities on cracks [3], (ii) the displacement discontinuity method [10] and (iii) averaging over realizations with different crack distributions. The approach was employed for calculation of effective elastic properties for two crack configurations: one family of parallel cracks and two families of parallel cracks inclined at angle 30°. Comparison with predictions of the non-interaction approximation was carried out. It was shown that the non-interaction approximation has acceptable accuracy for crack densities up to 0.1. Therefore, for these densities, the effective properties can be estimated

analytically. At higher crack densities, the non-interaction approximation strongly overestimates the effective stiffness. For both crack configurations, elastic properties are orthotropic even at large crack densities up to 0.8. Quantitative agreement with results reported in literature was demonstrated. The approach described above can be used in a variety of applications, including geomechanical problems, e.g. simulation of hydraulic fracturing in naturally fractured reservoirs [14].

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