# Functionally invariant solutions of nonlinear Klein-Gordon equation

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**Abstract**

New approach to the integration of nonlinear Klein-Gordon equation is given. Solutions $U(x, y, z, t)$ are searched in the form of a composite function $U = f(W)$. It is assumed that $W(x, y, z, t)$ simultaneously satisfies to two partial differential equations and $f(W)$ to the self-similar nonlinear ordinary differential equation. Functionally invariant solutions are constructed for $W$ which contain arbitrary function $F(\alpha)$. Ansatz $\alpha(x, y, z, t)$ may be found as a root of linear algebraic equation of variables $(x, y, z, t)$ with coefficients in the form of arbitrary functions of $\alpha$. Particular expressions of ansatz $\alpha$ are found. Proposed approach is illustrated by the solution of sine-Gordon equation.

## 1 Introduction

Nonlinear Klein-Gordon (NKG) equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 U}{\partial t^2} = V'(U) \tag{1}$$

plays the fundamental role in the modern natural sciences. Here $V'(U)$ is a nonlinear function of $U$, prime denotes differentiation with respect to the argument. Equation (1) with $V'(U) = \exp U$ pioneered in the theory of surfaces of constant curvature. It was solved by Liouville [1]. To the present time are studied cases when $V'(U)$ has a form of truncated exponential, Taylor, Fourier and $\operatorname{sh} nU$, $\operatorname{ch} nU$ $(n = 1, 2, \ldots)$ series. Equation (1) with $V'(U)$ in the form of sum of exponents describes the oscillations of the chain of nonlinear pendulums [2] (Toda's chain). NKG equation with cubic nonlinearity $V'(U) = U^3 - U$ is used in the field theory models [3]. Function $V'(U) = \sin U$ defines sine-Gordon equation (SG). It brings into existence one of the most beautiful and universal object of the modern scientific studies — soliton. SG equation has extremely wide area of applications: modeling of nonlinear lattice [4, 5], orientational structure of liquid crystals [6], orientation of spins in ferromagnetic materials [7], propagation of fluxons in Josephson transitions [8], propagation of perturbations in macromolecules [9], processes in the Earth's crust [10], surface metrics [11], and many others. A lot of studies in mathematics, applied and theoretical physics have been devoted to the equations with $V'(U) = p_1 \sin U + p_2 \sin 2U$ (double sine-Gordon (DSG)) and $V'(U) = p_1 \sin U + p_2 \sin 2U + p_3 \sin 3U$ (triple sine-Gordon (TSG)). In the theory of nonlinear lattice dynamics DSG describes cardinal transformation of the near atomic order, lattice fragmentation produced by large deformations, defect creation and propagation of dislocations [12, 13]. In the nonlinear optics DSG equation is modeling propagation of short light impulses in the fivefold degenerated media [14, 15]. TSG equation is used in the studies of magnetoelastic waves in ferromagnets such as garnet ferrites [16]. There are papers devoted to study of NKG equations with $V'(U)$ represented by the sum of $\operatorname{sh} nU$ and $\operatorname{ch} nU$ [17]. A lot of studies have been devoted to the development of mathematical methods for solving NKG equations. Fundamental results have been obtained both in integration of NKG equation of special forms and general forms of $V'(U)$. Robust methods of computational study of NKG equations are developed [7].

However new approach to the construction of NKG equations solutions is of great interest because new solutions allow to understand more deeply the nature of nonlinear equations' solutions and could find application to the description of physical phenomena and technological processes.

Method of functionally invariant NKG equations solutions construction is proposed hereafter. Basic concepts of the method have been stated in the papers [18, 19]. Authors' papers [20]–[25] are devoted to the construction of functionally invariant solutions of SG, DSG and TSG equations.

## 2 New approach to the construction of exact solutions for nonlinear Klein-Gordon equation

We seek the solution of NKG equation in the form of a composite function $U = f[W(x, y, z, t)]$. Then equation (1) takes a form

$$f'' \left[ \left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2 - \frac{1}{v^2}\left(\frac{\partial W}{\partial t}\right)^2 \right] + \\ + f' \left[ \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 W}{\partial t^2} \right] = V'(f). \tag{2}$$

Assume that function $W(x, y, z, t)$ simultaneously satisfies to two equations

$$\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2 - \frac{1}{v^2}\left(\frac{\partial W}{\partial t}\right)^2 = P(W), \tag{3}$$

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 W}{\partial t^2} = Q(W), \tag{4}$$

and $f(W)$ is the solution of nonlinear ordinary differential equation

$$P(W)\, f'' + Q(W)\, f' = V'(f). \tag{5}$$

Here $P(W)$ and $Q(W)$ are arbitrary functions.

Integration of equations (3)–(5) in the general form is not easier than solving the original equation (1). However for particular form of functions $P(W)$ and $Q(W)$ exact solutions of the equations (3)–(5) could be found and therefore of the equation (1).

**2.1.** Assume that

$$P(W) = W^2, \qquad Q(W) = W. \tag{6}$$

Then equations (3)–(5) take the form

$$\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2 - \frac{1}{v^2}\left(\frac{\partial W}{\partial t}\right)^2 = W^2, \tag{7}$$

$$\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2 W}{\partial z^2}-\frac{1}{v^2}\,\frac{\partial^2 W}{\partial t^2}=W, \tag{8}$$

$$W^2 f''+W f'=V'(f). \tag{9}$$

Equation (9) after change of variable

$$\zeta=\ln W \tag{10}$$

is reduced to the well known equation of nonlinear mathematical pendulum

$$\frac{d^2 f}{d\zeta^2}=V'(f), \tag{11}$$

which has first integral

$$\frac{df}{\sqrt{E+V(f)}}=\pm\sqrt{2}\,d\zeta. \tag{12}$$

Therefore for the case (6) equation (5) is integrated and function $f(W)$ is found at least in the form of quadrature.

Function $W(x,y,z,t)$ could be found using the method of construction of functionally invariant solutions of wave equation [18, 19]. Assume that

$$W=e^{\varphi}V(x,y,z,t), \qquad \varphi=a_1x+a_2y+a_3z-\sigma v^2 t. \tag{13}$$

Here $(a_1,a_2,a_3,\sigma)$ are arbitrary constants. Then if

$$a_1^2+a_2^2+a_3^2=1+\sigma^2v^2, \tag{14}$$

function $V(x,y,z,t)$ must simultaneously satisfy to three equations:

$$a_1\frac{\partial V}{\partial x}+a_2\frac{\partial V}{\partial y}+a_3\frac{\partial V}{\partial z}+\sigma\frac{\partial V}{\partial t}=0, \tag{15}$$

eikonal type equation

$$\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2-\frac{1}{v^2}\left(\frac{\partial V}{\partial t}\right)^2=0 \tag{16}$$

and wave equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}-\frac{1}{v^2}\,\frac{\partial^2 V}{\partial t^2}=0. \tag{17}$$

We seek function $V(x,y,z,t)$ in the form

$$V(x,y,z,t)=F(\alpha). \tag{18}$$

Here $F(\alpha)$ is an arbitrary function. Ansatz $\alpha(x,y,z,t)$ is the root of the equation

$$x\,l(\alpha)+y\,m(\alpha)+z\,n(\alpha)-v^2tp(\alpha)+g(\alpha)=0, \tag{19}$$

where $(l(\alpha),\ m(\alpha),\ n(\alpha),\ p(\alpha),\ g(\alpha))$ are arbitrary functions of $\alpha$.

Function $V(x,y,z,t)$ in the form (18) will be a solution of the equation (15), provided that algebraic equation

$$a_1l+a_2m+a_3n=v^2\sigma p, \tag{20}$$

is satisfied, and satisfy to equations (16), (17), if

$$l^2 + m^2 + n^2 = v^2 p^2. \tag{21}$$

**2.2.** Solution of NKG equation could be obtained also by other methods. Assume that

$$P(W) = 1, \qquad Q(W) = 0. \tag{22}$$

Then $f(W)$ will coincide with $f(\zeta)$, if to change argument $\zeta$ to $W$, and

$$W(x, y, z, t) = \varphi + F(\alpha). \tag{23}$$

Here $\varphi(x, y, z, t)$ is given by (13), and $F(\alpha)$ is an arbitrary function of $\alpha$. Ansatz $\alpha(x, y, z, t)$ is still the root of equation (19).

Also

$$W(x, y, z, t) = x\, a_0(\alpha) + y\, b_0(\alpha) + z\, c_0(\alpha) - t\, v^2 d_0(\alpha) + e_0(\alpha), \tag{24}$$

will be the solution of equations (3), (4), (22) with equation (19) for the ansatz if

$$\begin{aligned} a_0(\alpha) &= \int l(\alpha)\, d\alpha, \quad b_0(\alpha) = \int m(\alpha)\, d\alpha, \quad c_0(\alpha) = \int n(\alpha)\, d\alpha, \\ d_0(\alpha) &= \int p(\alpha)\, d\alpha, \quad e_0(\alpha) = \int g(\alpha)\, d\alpha, \end{aligned} \tag{25}$$

and conditions

$$a_0^2 + b_0^2 + c_0^2 = 1 + v^2 d_0^2, \tag{26}$$

$$(a_0')^2 + (b_0')^2 + (c_0')^2 = v^2 (d_0')^2 \tag{27}$$

are fulfilled.

# 3 Finding of ansatz $\alpha(x, y, z, t)$

For different solutions ansatz $\alpha(x, y, z, t)$ is found from the equation (19) which contains more unknown coefficients than number of algebraic equations to be satisfied by them. Therefore ansatz $\alpha$ is not defined uniquely. Not touching on the finding of general expression for $\alpha$, we note some simple particular cases.

$$\textbf{a)}\ \ l = vpx_1, \quad m = vpx_2, \quad n = vpx_3, \quad g = -vp\alpha, \tag{28}$$

$$\alpha = xx_1 + yx_2 + zx_3 - v^2 t. \tag{29}$$

Constants $x_1, x_2, x_3$ depend on $(v, \sigma, a_1, a_2, a_3)$ and new constant $f_0$ assuming that in general expressions for $(x_1, x_2, x_3)$ $\cos f(\alpha) = \cos f_0$:

$$\begin{aligned} x_1 &= \frac{v}{\sqrt{a_1^2 + a_2^2 + a_3^2}} \Big[\cos\delta \cos C \cos f_0 + \sin\delta \sin f_0 + v\sigma \cos A\Big], \\ x_2 &= \frac{v}{\sqrt{a_1^2 + a_2^2 + a_3^2}} \Big[\sin\delta \cos C \cos f_0 - \cos\delta \sin f_0 + v\sigma \cos B\Big], \\ x_3 &= \frac{v}{\sqrt{a_1^2 + a_2^2 + a_3^2}} \Big[-\sin C \cos f_0 + v\sigma \cos C\Big]. \end{aligned} \tag{30}$$

Here

$$\cos A = \frac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}}, \quad \cos B = \frac{a_2}{\sqrt{a_1^2+a_2^2+a_3^2}}, \quad \cos C = \frac{a_3}{\sqrt{a_1^2+a_2^2+a_3^2}}, \tag{31}$$

$$\cos\delta = \frac{a_1}{\sqrt{a_1^2+a_2^2}}, \qquad \sin\delta = \frac{a_2}{\sqrt{a_1^2+a_2^2}}. \tag{32}$$

$$\mathbf{b}) \;\; l = vpx_1, \quad m = vpx_2, \quad n = vpx_3, \quad f(\alpha) = \alpha, \quad g = 0. \tag{33}$$

In the second case $\alpha$ is the root of the trigonometrical equation

$$\eta\cos\alpha + \xi\sin\alpha + v\sigma\zeta = 0\,, \tag{34}$$

$$\begin{aligned} \eta &= (x-u_1t)\cos\delta\cos C + (y-u_2t)\sin\delta\cos C - (z-u_3t)\sin C\,, \\ \xi &= (x-u_1t)\sin\delta - (y-u_2t)\cos\delta\,, \\ \zeta &= (x-u_1t)\cos A + (y-u_2t)\cos B + (z-u_3t)\cos C, \end{aligned} \tag{35}$$

where

$$u_1 = \frac{1}{\sigma}\sqrt{1+v^2\sigma^2}\,\cos A, \quad u_2 = \frac{1}{\sigma}\sqrt{1+v^2\sigma^2}\,\cos B, \quad u_3 = \frac{1}{\sigma}\sqrt{1+v^2\sigma^2}\,\cos C.$$

From (34) we find that

$$\alpha = -(-1)^n\left[\arcsin\frac{\eta}{\sqrt{\xi^2+\eta^2}} + \arcsin\frac{v\sigma\zeta}{\sqrt{\xi^2+\eta^2}}\right] + n\pi\,, \tag{36}$$

$n = 0, \pm 1, \ldots$

Ansatz $\alpha$ as it follows from (36) is defined if

$$-1 \le \frac{v\sigma\zeta}{\sqrt{\xi^2+\eta^2}} \le 1\,. \tag{37}$$

Domain (37) is the exterior of cones with axis directed along the vector $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, and vertex of cone (at $t=0$) coincides with the origin of coordinates. With changing of time cones move along the vector $\mathbf{a}$ with the speed $\mathbf{u} = u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k}$. Cone opening is defined by the parameter $v\sigma$. Plane 1 is shown on the Fig. 1. It moves parallel to itself and perpendicular to the vector $\mathbf{a}$. It is described by the equation $a_1x+a_2y+a_3z-\sigma v^2t=0$. Plane 2 touches cone side surface. Its equation is $x_1x+x_2y+x_3z-v^2t=0$.

$$\mathbf{c}) \;\; l = vp\cos\theta\cos\phi, \quad m = vp\cos\theta\sin\phi, \quad n = vp\sin\theta, \quad g = 0. \tag{38}$$

Here $(p,\theta,\phi)$ are arbitrary functions of $\alpha$. Equation (21) is identically satisfied. Equation (20) relates $\theta(\alpha)$ with $\phi(\alpha)$. We seek function $\theta(\alpha)$ from (19) which takes the form

$$N\sin^2\theta - 2Q\sin\theta + M = 0. \tag{39}$$

From (39) one can find that

$$\sin\theta_1(\alpha) = \frac{Q-\sqrt{Q^2-MN}}{N}, \quad \sin\theta_2(\alpha) = \frac{Q+\sqrt{Q^2-MN}}{N}, \tag{40}$$

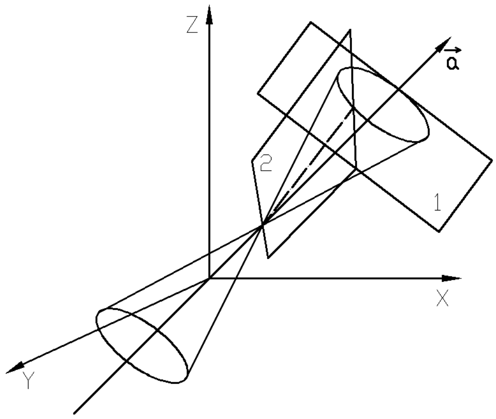


Figure 1: Domain of ansatz definition (37).

$$Q^2 - MN = (s^2 - \varphi^2)(xa_2 - ya_1)^2, \qquad s^2 = x^2 + y^2 + z^2 - v^2t^2,$$
$$N = (a_1^2 + a_2^2 + a_3^2)(x^2 + y^2 + z^2) - (a_1x + a_2y + a_3z)^2, \tag{41}$$
$$Q = v\left[a_3\sigma(x^2 + y^2 + z^2) - (a_3t + z\sigma)(a_1x + a_2y + a_3z) + (a_1^2 + a_2^2 + a_3^2)zt\right]$$

**d**) $l = \cos\phi, \quad m = \sin\phi, \quad n = \operatorname{sh}\psi, \quad \mathrm{vp} = \operatorname{ch}\psi, \quad \mathrm{g} = 0,$ (42)

where $\phi(\alpha), \psi(\alpha)$ are arbitrary functions of $\alpha$. Equation (21) is identically satisfied. Equation (20) relates $\phi(\alpha)$ with $\psi(\alpha)$. Equation (19) takes the form

$$M\cos^2\gamma + 2Q\sin\gamma\cos\gamma + N\sin^2\gamma = 0, \quad \gamma(\alpha) = \delta - \phi(\alpha). \tag{43}$$

Here

$$Q^2 - MN = \left(\frac{z\sigma v - vta_3}{\sigma^2v^2 - a_3^2}\right)^2 (s^2 - \varphi^2),$$
$$N = (x\sin\delta - y\cos\delta)^2 + \frac{(z\sigma v - vta_3)^2}{\sigma^2v^2 - a_3^2} \tag{44}$$
$$Q = \left(x\cos\delta + y\sin\delta + \frac{za_3 - \sigma v^2t}{\sigma^2v^2 - a_3^2}\sqrt{a_1^2 + a_2^2}\right)(x\sin\delta - y\cos\delta).$$

From (43) one can find that

$$\operatorname{tg}\gamma_1(\alpha) = \frac{\sqrt{\mathrm{Q}^2 - \mathrm{MN}} - \mathrm{Q}}{\mathrm{N}}, \quad \operatorname{tg}\gamma_2(\alpha) = -\frac{\sqrt{\mathrm{Q}^2 - \mathrm{MN}} + \mathrm{Q}}{\mathrm{N}}. \tag{45}$$

It should be kept in mind concerning the choice of given ansatzs $\alpha(x, y, z, t)$ that since we seek the solution $V(x, y, z, t)$ in the form of arbitrary function $F(\alpha)$ then $V(x, y, z, t)$ remains the solution if $\alpha$ is changed to the arbitrary function $\chi(\alpha)$.

For cases (22), (24)

**a**) $a_0 = \cos\psi_0, \quad b_0 = \sin\psi_0, \quad c_0 = \operatorname{tg}\alpha, \quad \mathrm{vd}_0 = \operatorname{tg}\alpha, \quad \psi_0 = \mathrm{const},$
$W = x\cos\psi_0 + y\sin\psi_0 + F(z - vt),$ (46)

$$\textbf{b)}\ \ a_0 = \frac{\operatorname{ch}\alpha}{\operatorname{sh}\alpha}\cos\psi_0,\quad b_0 = \frac{\operatorname{ch}\alpha}{\operatorname{sh}\alpha}\sin\psi_0,\quad c_0 = 1,\quad vd_0 = \frac{\operatorname{ch}\alpha}{\operatorname{sh}\alpha},$$
$$W = z + F(x\cos\psi_0 + y\sin\psi_0 - vt), \tag{47}$$

$$\textbf{c)}\ \ a_0 = 1,\quad b_0 = \operatorname{tg}\alpha,\quad \mathrm{c}_0 = \operatorname{tg}\alpha\operatorname{sh}\psi_0,\quad \mathrm{vd}_0 = \operatorname{tg}\alpha\operatorname{ch}\psi_0,$$
$$W = x + F(y + z\operatorname{sh}\psi_0 - \mathrm{vt}\operatorname{ch}\psi_0). \tag{48}$$

Here $F(\alpha)$ is arbitrary function of the given argument.

# 4 Solution of NKG equation for particular form of the function $V'(U)$

We illustrate new approach to the solving of NKG equation on the example of the integration of SG equation. If $V'(U) = \sin U$ then $F(\zeta)$ is given by the integral (12)

$$\int \frac{df}{\sqrt{E - \cos f}} = \pm\sqrt{2}(\zeta + \zeta_0). \tag{49}$$

Certain form of the solution $U = f(\zeta)$ is defined by the value of integration constant $E$:

$$U = 4\operatorname{arctg} e^{\pm(\zeta+\zeta_0)}, \qquad E = 1, \tag{50}$$

$$U = \pm 2\operatorname{arctg}\left[\sqrt{1-k^2}\,\frac{\operatorname{sn}\left[\frac{1}{k}(\zeta+\zeta_0), k\right]}{\operatorname{cn}\left[\frac{1}{k}(\zeta+\zeta_0), k\right]}\right], \qquad E > 1, \tag{51}$$

$$U = \pm 2\operatorname{arctg}\frac{\operatorname{dn}\left[(\zeta+\zeta_0), k\right]}{k\operatorname{sn}\left[(\zeta+\zeta_0), k\right]}, \qquad 0 < E < 1. \tag{52}$$

Solution (51) could be written in another form, namely

$$U = \pi - 2\operatorname{am}\left[K(k) - F(\psi, k),\, k\right], \qquad \psi = \pm\operatorname{am}\left[\frac{1}{k}(\zeta+\zeta_0), k\right]. \tag{53}$$

Here $\operatorname{sn}(\zeta, k)$, $\operatorname{cn}(\zeta, k)$, $\operatorname{dn}(\zeta, k)$ are elliptic sine, cosine and Jacobi delta-function correspondingly, $F(\psi, k)$ and $K(k)$ — incomplete and complete elliptic integrals, and $\operatorname{am}(\zeta, k)$ — Jacobi amplitude. Solution (53) is obtained from (51) using well known relation between incomplete elliptic integrals of the first kind

$$F(\phi, k) + F(\psi, k) = K(k), \tag{54}$$

with arguments related by the condition

$$\sqrt{1-k^2}\operatorname{tg}\phi\operatorname{tg}\psi = 1. \tag{55}$$

Solutions (51), (53) coincide in the period $0 < \zeta < 2K(k)$ and are different in the extension of the solution outside the period. Function (51) is extended periodically and (53) increases monotonically in the form of the "ladder". Functions (50)–(53) have different pattern of change. Solution (50) asymptotically tends to limits $(0, 2\pi)$. It is shown at the left of Fig. 2 where is shown graph of $U(x, z)$ for the solution (50) with

$$W = x\cos\psi_0 + y\sin\psi_0 + 2\sin(z - vt). \tag{56}$$

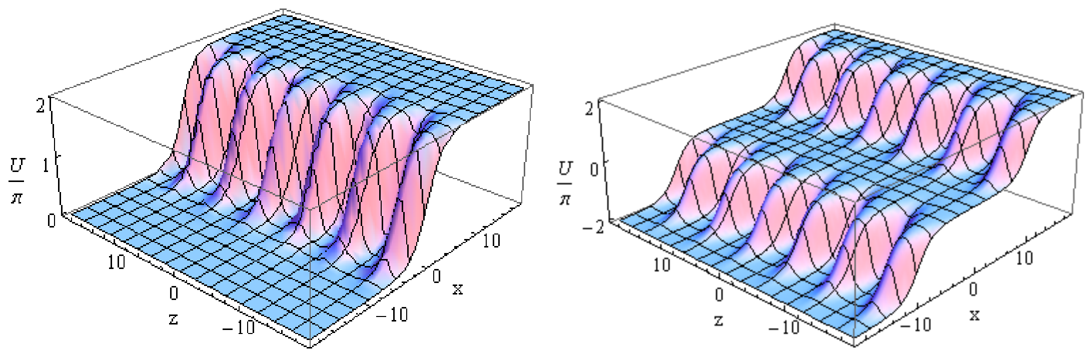


Figure 2: Graphs of solutions (50) at the left, and (53) at the right at $y = 0, t = 0$, $\psi_0 = \pi/4$, $k = 0.99999, \zeta_0 = 0$.

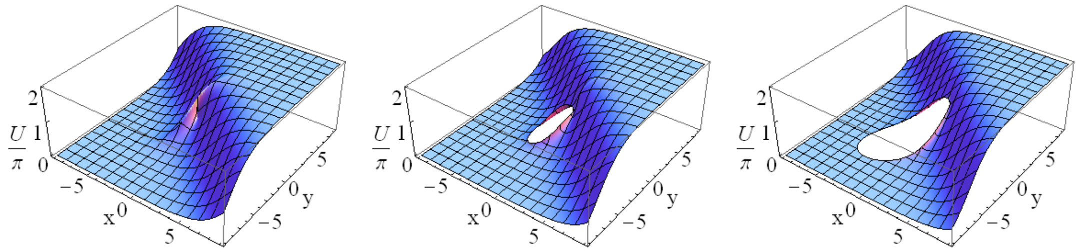


Figure 3: Graphs of solution (50) with ansatz (40) at $z = 0$ and $t = 0$ (at the left), $t = 1$ (at the centre), $t = 2$ (at the right); $v = 1$, $\sigma = 0.5$, $\zeta_0 = 0.1$, $a_1 = a_2 = 0.56, a_3 = 0.79$.

Graph has a form of kink with transition domain modulated by the periodic function.

Function (53) as mentioned above increases monotonically and its graph with the same function $W$ is shown at the right of Fig. 2. Solutions (51), (52) change periodically achieving maximum value $U = \arccos E$. On the Fig. 3 are shown graphs of solutions (50) with ansatz $\sin\theta_1(\alpha)$ (40). Note that for the given ansatz real solution exists not for all $(x, y, z, t)$. Domains where real solution does not exist correspond to the gaps on the graph. With increase of time such domains increase for the given solution.

## 5 Conclusion

We note finally:

1) New approach to the integration of NKG equation is proposed relating to the $(3+1)$ equation. However it could be easily extended on the spaces of any dimension $(n + 1)$. Moreover it has been shown in [26] that with increase of $n$ number of possible algebraic equations having ansatz $\alpha(x, y, z, \ldots, t)$ as root increases, also increase the number of arbitrary functions in the anzatz equation. For the three dimensional space equation (19) is unique [19].

2) New approach allows to find the solution at least in the form of quadrature for any nonlinear integrable function $V'(U)$. For the particular form of functions $V'(U)$ represented by the truncated exponential, ( sh nU, ch nU), Taylor, and Fourier series solution of NKG equation is reduced to the calculation and inversion of algebraic, elliptic, ultraelliptic and Abel integrals with genus defined by the number of summands.

## Acknowledgements

*This work was supported by RFBR, grant 10-01-00243-a.*

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