The influence of contact angle's hysteresis on the cylindrical drop's dynamics

Aleksey Alabuzhev
alabuzhev@mail.ru

Abstract

The forced oscillations of a cylindrical drop are considered in the present work. The drop is suspended in the different fluid and confined by two parallel rigid plates, subjected to vibrations. The vibration axis is perpendicular to the symmetry axis. The amplitude of vibrations is small in comparison with the drop radius. The equilibrium contact angle is right. The specific boundary conditions, assumed by Hocking (1987), is applied to take into account: the contact line starts to slide only when the deviation of the contact angle exceeds a certain critical value. As a result, the stick-slip dynamics can be observed.

1 Problem statement

This investigation assumes that a drop with a density ρ_i^* is suspended in a fluid of different density ρ_e^*. In the absence of external forces the drop has cylindrical shape with radius R^*. The system is confined between two parallel rigid plates, subjected to vibrations perpendicular to the drop axis (fig.1). In the absence of vibrations a contact angle between a lateral surface of a drop and bounding plate equals $\pi/2$. The thickness of the layer is h. The equilibrium contact angle between the lateral surface of drop and the rigid plate is θ_0.

The vessel is closed at the infinity and undergoes high-frequency oscillations according to the law $\vec{r}=\vec{R}+A\vec{j}\cos(\omega t)$ ($\vec{r}$, $\vec{R}$ are the radius - vectors of an arbitrary point of a drop surface in the plane parallel to the rigid plates (horizontal plane) in the presence and absence of vibrations, respectively, A is the amplitude of vibrations, $\vec{j}$ is the unit vector in a horizontal plane). The vibration amplitude A is small as compared to R. It is assumed, that the lateral surface of the drop is $r = R + \zeta(\alpha, z, t)$ where α is the polar angle.

Velocity of motion of a contact line is assumed to be proportional to a deviation of a contact angle from equilibrium value [1]:

$$\zeta_t \quad = \quad \begin{cases} \Lambda\left(\gamma - \gamma_0\right), \ \gamma > \gamma_0 \\ \quad 0, \qquad\quad |\gamma| < \gamma_0 \\ \Lambda\left(\gamma + \gamma_0\right), \ \gamma < -\gamma_0 \end{cases} \tag{1}$$

The following quantities are chosen as the scales: time - $\sqrt{(\rho_e^* + \rho_i^*)h^2 R/\sigma}$, length - R, height - h, potential of velocity - A$\sqrt{\sigma R/((\rho_e^* + \rho_i^*)h^2)}$, density -$\rho_e^* + \rho_i^*$, pressure - $A\sigma/h^2$, deviations of the drop surface - A.

$$p = -\rho\left(\varphi_t + \frac{1}{2}\varepsilon(\nabla\varphi)^2\right), \Delta\varphi = 0, \tag{2}$$

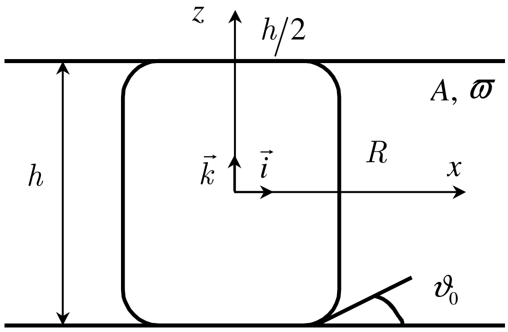

Figure 1: Geometry of a problem

$$r = 1 + \varepsilon\zeta \, [\vec{n} \cdot \nabla\varphi] = 0, F_t + \varepsilon\nabla\varphi \cdot \nabla F = 0, [p] = -\operatorname{div}\vec{n}, \tag{3}$$

$$z = \pm\frac{1}{2} : veck \cdot \nabla\varphi = 0 \tag{4}$$

$$z = \pm\frac{1}{2}, r = 1 + \varepsilon\zeta : \zeta_t \quad = \quad \begin{cases} \lambda\left(\gamma - \gamma_0\right), \ \gamma > \gamma_0 \\ \quad 0, \qquad |\gamma| < \gamma_0 \\ \lambda\left(\gamma + \gamma_0\right), \ \gamma < -\gamma_0 \end{cases} \tag{5}$$

Here $\vec{v} = \nabla\varphi$ - velocity potential, $F = r - R - \epsilon\zeta(\alpha, z, t)$ - drop surface, p - pressure, $\vec{n} = \frac{\nabla F}{|\nabla F|}$ - normal vector, $\vec{k}$ - unit vector of z-axis, $\gamma\left(t\right) = \mp\zeta_z|_{z=\pm 1/2}$ - contact line's deviation from equilibrium position, ζ - surface deviation of equilibrium shape. The effective boundary condition (5) shows that the contact line is fixed, if the absolute value of the deviation of the surface is less than some characteristic value of γ_0. The square brackets denote the jump in value at the interface between the external fluid and the drop in the index of the unknown functions are denoted derivatives with respect to relevant variables. The boundary problem (2)-(5)contains the following dimensionless parameters: small vibration amplitude - $\epsilon = A/R$, capillary parameter $\lambda{=}\Lambda\sqrt{h^2 R(\rho_e^* + \rho_i^*)/\sigma}$, the ratio of radius to height - $b = R/h$, density of internal fluid - $\rho = \rho_i^*/(\rho_e^* + \rho_i^*)$, density of external fluid - $\rho = \rho_e^*/(\rho_e^* + \rho_i^*)$.

2 Method of solution

Despite the fact that the boundary condition (5) makes the problem (2)-(5) is nonlinear, the solutions for the functions φ, ζ, p can be represented as a series (see [2]-[4]):

$$\varphi_i = \sum_{k=0}^{\infty} a_k\left(t\right) \mathrm{I}_0\left(\left(2k+1\right)\pi b r\right) \sin\left(2k+1\right)\pi z, \tag{6}$$

$$\varphi_e = \sum_{k=0}^{\infty} b_k\left(t\right) \mathrm{K}_0\left(\left(2k+1\right)\pi b r\right) \sin\left(2k+1\right)\pi z, \tag{7}$$

$$\zeta = \sum_{k=0}^{\infty} c_k\left(t\right) \sin\left(2k+1\right)\pi z, \tag{8}$$

$$p_i = -\rho_i\left(\varphi_{it} + \omega^2 z\cos\omega t\right), p_e = -\rho_e\left(\varphi_{et} + \omega^2 z\cos\omega t\right), \tag{9}$$

$$b_k\left(t\right) \ = a_k\left(t\right)\frac{\mathrm{I'}_0\left(\left(2k+1\right)\pi b\right)}{\mathrm{K'}_0\left(\left(2k+1\right)\pi b\right)}, a_k\left(t\right) = \frac{c_{kt}\left(t\right)}{\mathrm{I}_0\left(\left(2k+1\right)\pi b\right)}. \tag{10}$$

On the other hand, using the normal stress balance condition (3) , the solution of the surface deviation ζ can be written as:

$$\zeta = \frac{b\gamma}{\cos\left(\frac{1}{2b}\right)} \sin\left(\frac{z}{b}\right) + \sum_{k=0}^{\infty} \left(\frac{c_{ktt}}{\Omega_k^2} + \frac{(\rho_i - \rho_e)\,\omega^2 g_k \cos\omega t}{1 - (2k+1)^2 \pi^2 b^2} \sin\left((2k+1)\,\pi z\right) \right) \tag{11}$$

Here $g_k = \left(2(-1)^k\right) \Big/ \left((2k+1)^2\pi^2\right)$ is Furie expansion coefficient of function z on basic functions $\sin\left((2k+1)\,\pi z\right)$, Ω_k - eigen frequencies of cylindrical drop with free contact line:

$$\Omega_{mk}^2 = \frac{m^2 - 1 + 4\pi^2 b^2 k^2}{F_{mk}} R_{mk\,r}^i\,(1)\,, \tag{12}$$

whereL_k $=m = 0, 1, 2, ...$ - azimuthal number, $k = 0, 1, 2, ...$ - wave number, $R_{m0}^i(r) = r^m$, $R_{m0}^e(r) = r^{-m}$,

$$R_{mk}^i(r) = \mathrm{I}_m(2\pi bkr)$$

for $k \geqslant 1$, $R_{mk}^e(r) = \mathrm{K}_m(2\pi bkr)$ for $k \geqslant 1$, I_m, K_m - modified Bessel functions , $F_{mk} = \rho R_{mk}^i\,(1) - R_{mk\,r}^i\,(1)\, R_{mk}^e\,(1)/R_{mk\,r}^e\,(1)$, $R_{mk\,r}^{i,e}\,(r) = d\,R_{mk}^{i,e}\,(r)\Big/dr$ - subscript r is derivative on radius r. In our solution $m = 0$.

Comparing the solutions 8 and 11 for ζ, we obtain a system of ordinary differential equations for unknown amplitudes $c_k\,(t)$:

$$c_{ktt} + \Omega_k^2 c_k = \Omega_k^2 S_k \gamma - \Omega_k^2 L_k \cos\omega t, \tag{13}$$

where $S_k = bf_k \operatorname{ces}\left(\frac{1}{2b}\right)$, $L_k = \frac{(\rho_i - \rho_e)\omega^2 g_k}{(2k+1)^2\pi^2 b^2 - 1}$, $f_k = \frac{2b(-1)^k}{(2k+1)^2\pi^2 b^2 - 1} \cos\left(\frac{1}{2b}\right)$, f_k - Furie expansion coefficient of function $\sin\left(z/b\right)$.

Equation (13) must be solved together with 95). Note that the series for ζ converges very slowly. However, the eigen frequencies Ω_k are growing rapidly, therefore, from some k, can neglect the first term on the left side of equation (5) as compared with the rest, ie . $c_k \approx S_k\gamma - L_k \cos\omega t$. Thus, using a finite sum and the solution of (8), we obtain

$$\begin{aligned} \zeta_t &= \sum_{k=0}^{N} c_{kt} \sin\left((2k+1)\,\pi z\right) + \sum_{k=N+1}^{\infty} \left(S_k \gamma_t + L_k \omega \sin\left(\omega t\right)\right) \sin\left((2k+1)\,\pi z\right) = \\ &= \sum_{k=0}^{N} D_{kt} \sin\left((2k+1)\,\pi z\right), \end{aligned} \tag{14}$$

where $D_k = c_k + S_k\gamma - L_k \cos\left(\omega t\right)$. Substituting (14) into (5), we obtain the equation for λ:

$$\gamma_t = \frac{\left(-\sum\limits_{k=0}^{N} (-1)^k c_{kt} + \sum\limits_{k=0}^{N} (-1)^k L_k \omega \sin\left(\omega t\right) - \begin{cases} \lambda\left(\gamma - \gamma_0\right), & \gamma > \gamma_0 \\ 0, & |\gamma| < \gamma_0 \\ \lambda\left(\gamma + \gamma_0\right), & \gamma < -\gamma_0 \end{cases} \right)}{\sum\limits_{k=0}^{N} (-1)^k S_k}. \tag{15}$$

The system of differential equations (13), (15) was solved using the method of Gear.

3 Results

The boundary condition of Hocking (5) shows that the contact line is fixed, if the contact angle does not exceed a certain critical value. Otherwise, the contact line moves. Figures (2-3) shows the areas in which the contact line is moving or at rest.

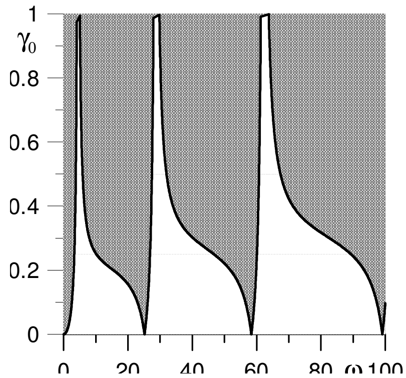

Figure 2: The diagram of contact line motion on the plane (ω, γ_0), $\rho_i = 0.3$, $b = 1$. The solid lines are determined by the condition and separate the domains of oscillations with the fixed contact line, in gray and with the contact line moving in the stick-slip regime

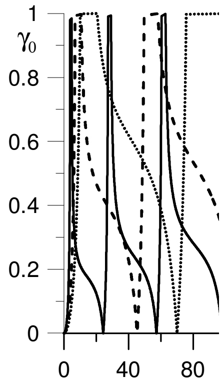

Figure 3: The diagram of contact line motion on the plane (ω, γ_0), ($\rho_i = 0.7$). $b = 1$ - solid, $b = 1.5$ - dashed, $b = 2$ - dotted.

Figures (4-5) show the dependence of the oscillation amplitude of the frequency of the external influence of the capillary for different values of the parameter and the critical value of contact angle. Previously, it was found that in the absence of hysteresis are forcing frequencies at which the contact line is not moving. In the presence of hysteresis, the contact line is not moving in a certain range of frequencies. By increasing the contact angle values characteristic time during which the contact line is not moving, growing, and the frequency range in which it is in motion, is shrinking.

4 Conclusion

The effect of contact angle hysteresis on the dynamics of liquid drops in equilibrium has the form of a cylinder and axially bounded by two parallel solid surfaces under the action of axial vibration. The equilibrium contact angle between the side surface of the droplets and solid surfaces is assumed to be straight. Considered their own and forced vibrations of the drop. The influence of the dynamics of the contact line was taken into account by an effective boundary condition, allowing the contact angle hysteresis. Due to the dissipative nature of the effective boundary condition there is a stable regime of nonlinear oscillations. There is evidence to reject the surface and the frequency characteristics depending on the constant Hawking, and the characteristic value of the contact angle. Previously, it was found that in the absence of hysteresis are forcing frequencies at which the contact line is not moving. In the presence of hysteresis, the contact line is not moving in a certain range of frequencies. By increasing the contact angle values characteristic time during which the contact line is not moving, growing, and the frequency range in which it is in motion, is shrinking. For large values of the constant Hawking, when the contact line is

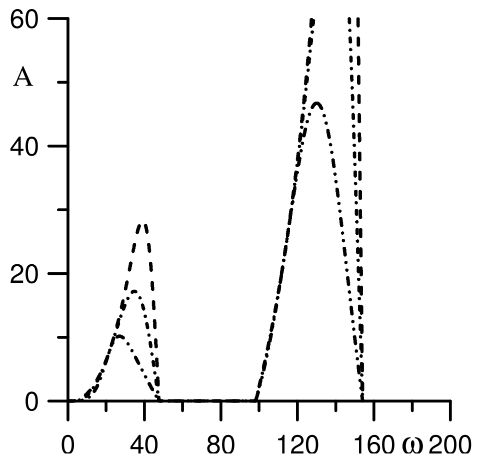

Figure 4: Amplitude-frequency response ($\gamma_0 = 1$).$\lambda = 3$ - dotted line, $\lambda = 5$ - dash-dotted, $\lambda = 10$ - dashed-2-dotted

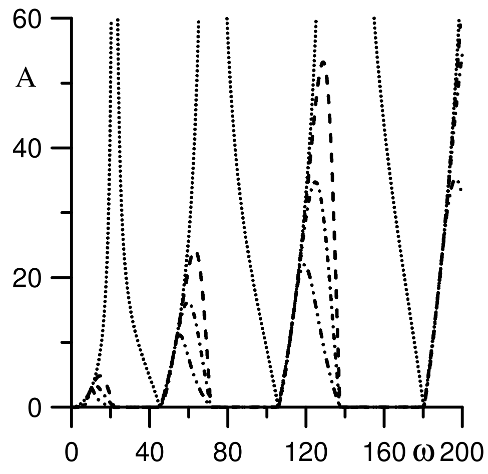

Figure 5: Amplitude-frequency response ($\gamma_0 = 10$).$\lambda = 0$ - dotted line, $\lambda = 3$ - dashed, $\lambda = 5$ - dash-dotted, $\lambda = 10$ - dash-2-dotted

weakly interacts with the substrate and the dissipation is small, the possible existence of resonances.

Acknowledgements

The work was supported by the President RF grant MK-2368.2011.1

References

[1] Hocking L.M. The damping of capillary-gravity waves at a rigid boundary. J. Fluid Mech., 1987, v.179, pp.253-266.

[2] Alabuzhev A.A., Lyubimov D.V. Effect of the contact-line dynamics on the natural oscillations of a cylindrical droplet. Journal of Applied Mechanics and Technical Physics. 2007. V. 48. No 5. P. 686-693.

[3] Fayzrakhmanova I., Straube A. Stick-slip dynamics of an oscillated sessile drop // Phys. Fluids 2009. V. 21, 072104.

[4] Irina S. Fayzrakhmanova, Arthur V. Straube, and Sergey Shklyaev. Bubble dynamics atop an oscillating substrate: Interplay of compressibility and contact angle hysteresis. Phys. Fluids 23, 102105 (2011).

Aleksey A. Alabuzhev, ICMM UB RAS, Perm, Russia