A simple statement of the essential spectrum with some applications : thin shells, plates with sharp edges

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Abstract

We first recall some properties of classical and non classical vibrating problems in continuous media. If the classical properties of compactness are not satisfied, we can obtain an essential spectrum of the self-adjoint operator which defines the spectral problem. In these cases, for very small data, we can obtain a large response and so these sequences can be physically interpreted as some kind of resonance. This local phenomena are quick oscillations in some directions which are called weakness directions. We will see that the singularities will propagate along directions which are orthogonal to these weakness ones.

An example is given by the spectral problem of thin elastic shells in membrane approximation. We propose to determine this spectrum and the corresponding weakness directions in every point of the shell. For an anisotropic shell, we obtain exactly the relation between the elements of the essential spectrum and the components of the weakness direction by using the second fundamental form and the compliance coefficients. Then we study the particular case of homogeneous and isotropic shells. We obtain a new relation and it appears that the essential spectrum depends only on the quotient of the fundamental forms and on the Young modulus, but is independent of the Poisson coefficient. We note that a geometrical interpretation of the quotient of the fundamental forms is the normal curvature of the surface in the directions of propagation. Conversely,if a spectral value is given, then we can find the directions of propagation of singularities. Their number can be zero, two or four. We illustrate with some concrete examples. In the last part about shells, we study the problem of propagation and reflection of singularities for an isotropic cylindrical shell and we show that the equation of propagation does not depend on the Poisson coefficient.

Another example is given by the two dimensional models for thin plates with sharp edges. The spectrum will contain a non empty essential part if the edge is sharp enough. Let us consider some point P near the edge of the plate. We define the thickness of the plate in this point, $h(P)$, and $d(P)$ the distance to the edge. These functions are very small and the degree of sharpness is given by α so that $h(P) = O(d(P)^{\alpha})$. Let us consider the longitudinal oscillations of the plate ; It can be shown that the spectrum is discrete. If $\alpha < 1$ all eigenvalues are positive and if α?1 zero is a triple eigenvalue with rigid modes. For the transversal oscillations we have the same results only if $\alpha < 2$. But if α?2 the essential spectrum is not empty.

1 Introduction

1.1 Classical and non classical vibrating problems

The classical form of a vibrating problem of structures can be writen in the form

$$\rho\frac{d^2u}{dt^2} + Au = 0 \qquad (1)$$

and a solution $u(x)e^{i\omega t}$ satisfies to a spectral problem

$$Au = \rho\omega^2 u = \lambda u \tag{2}$$

or in the variational form,

$$\forall v \in H,\ a(u,v) = \lambda(u,v) \tag{3}$$

where H is a good Hilbert space and λ the spectral parameter. In the classical case, A is a selfadjoint operator and with a compact resolvent and there exists a sequence of eigenvalues

$$O < \lambda_0 I \lambda_1 I ... I \lambda_k I \longrightarrow +\infty \tag{4}$$

with orthogonal modes.

1.2 Essential spectrum

Let us recall that the resolvent set is defined by,

$$\rho(A) = \{\ \zeta\ /\ (A - \zeta Id)^{-1} \in \mathcal{L}(H)\ \} \tag{5}$$

Its complement, the spectrum $\Sigma(A)$, is constituted of isolated eigenvalues of finite multiplicity - for these ζ, $(A-\zeta Id)^{-1}$ does not exist - and of other values for which $(A-\zeta Id)^{-1}$ exists but does not belong to $\mathcal{L}(H)$. They are eigenvalues of infinite multiplicity, accumulation points of eigenvalues and continuous spectrum.

The set of these ζ which are not isolated eigenvalues of finite multiplicity is the essential spectrum $\Sigma_{ess}(A)$. It can be characterized as the set of ζ for which there exists a sequence (u_k) called Weyl's sequence so that,

$$\begin{array}{rl} \|\ u_k\ \| & = 1 \\ u_k & \longrightarrow 0 \ \text{ in H weakly} \\ (A - \zeta Id)(u_k) & \longrightarrow 0 \ \text{ in H strongly} \end{array} \tag{6}$$

For very small data, we can obtain a large response and so these sequences can be physically interpreted as some kind of resonance. This local phenomena are quick oscillations in some directions which are called weakness directions. We will see that the singularities will propagate along directions which are orthogonal to these weakness ones.

2 The case of shells in membrane approximation

2.1 Essential spectrum

We consider a thin shell with a middle surface S. This surface is described by the map,

$$y = (y^1, y^2) \in \Omega \rightarrow r(y^1, y^2) \in \mathbb{R}^3 \tag{7}$$

where Ω is a domain of the plane. Let u be the displacement vector of the surface. We introduce the Hilbert space $H = (L(\Omega))^3$ and we denote by (u,v) the scalar product. The displacement u belongs to the subset $V_1 = H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega)$ which can be modified to take boundary conditions in account. The variational form of the problem of vibrations is,

We search for $u \in V_1$ so that,

$$\forall v \in V_1,\ a_m(u,v) + \epsilon^2 a_f(u,v) = \lambda(u,v) \tag{8}$$

The bilinear forms a_m and a_f correspond respectively to the membrane problem and the flexion problem. They are continuous on V_1. This problem is classical with a selfadjoint operator and compact resolvent and so there exists a sequence of eigenvalues but if the relative thickness of the shell, ϵ, is very small, then the membrane approximation is an appropriate representation. The formulation of this problem is different. In this case, u belongs to the space $V = H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$. The inclusion of V in H is dense and continuous but is not compact.

The problem is written as,

We search for $u \in V$ so that,

$$\forall v \in V, \quad a_m(u, v) = \lambda(u, v) \tag{9}$$

This spectral problem is an elliptic system with mixed order. The classical properties of compactness are not satisfied and the spectrum both contains a sequence of eigenvalues depending on the domain, and an essential spectrum.

We define the fundamental forms of the shell, $A(x^1, x^2)$ and $B(x^1, x^2)$.
The equations of the vibrating shell in the membrane approximation give an explicit spectral problem on the displacement u. There appears derivatives of second order in u_1 and u_2 and of first order in u_3. The classical properties of compactness are not satisfied and there exists an essential spectrum. We have weakness directions noted by (ξ_1, ξ_2) and the orthogonal ones (x_1, x_2) will be the directions of propagation of the singularities. The values of λ and (ξ_1, ξ_2) correspond to the non-ellipticity of the system in the Douglis and Nirenberg sense.

We can obtain the relation between λ and (ξ_1, ξ_2),

$$\lambda = \frac{[B(x_1, x_2)]^2}{s^{\alpha\beta\lambda\mu} x_\alpha x_\beta x_\lambda x_\mu} \tag{10}$$

where $s^{\alpha\beta\lambda\mu}$ are the coefficients of the compliance coefficients. If a point is given on S, then the spectral parameter λ belongs to a segment. In the case of an isotropic shell, we obtain the following outstanding form,

$$\lambda = E\Big[\ \frac{B(x_1, x_2)}{A(x_1, x_2)} \ \Big]^2 \tag{11}$$

and it appears that the essential spectrum depends only on the geometry and the Young modulus but is independent of the Poisson coefficient. A geometrical interpretation of the quotient of the fundamental forms is the normal curvature k_x of the surface in direction (x_1, x_2). The essential spectrum is then exactly the segment, $\Sigma_{ess} = [\ E.Inf k_x^2,\ E.Sup k_x^2\ ]$. So, in an elliptic point of S we have two directions (x_1, x_2) but in a hyperbolic point several cases are possible, four or two directions and sometimes double directions.

2.2 Propagation of singularities in a cylindrical thin shell

We consider the cylindrical shell defined by the map,

$$(y^1, y^2) \in [0,1] \times [0, 2\pi[\longrightarrow (x^1 = Ry^1, x^2 = Rcosy^2, x^3 = Rsiny^2) \tag{12}$$

The essential spectrum is the segment $[0, \frac{E}{R^2}]$. Denoting by λ a value in the essential spectrum and by θ the polar angle of the directions of propagation $\underline{x}$, we obtain

$$\lambda = \frac{E}{R^2}\Big[\frac{tan^2\theta}{1+tan^2\theta}\Big]^2 \tag{13}$$

For the vibrating shell in the membrane approximation, we obtain the following homogeneous spectral problem,

$$\begin{cases} u_{1,11} + \frac{1-\nu}{2}u_{1,22} + \frac{1+\nu}{2}u_{2,12} - \nu Ru_{3,1} &= -\frac{\lambda}{KR^2}u_1 \\ \frac{1+\nu}{2}u_{1,12} + \frac{1-\nu}{2}u_{2,11} + u_{2,22} - Ru_{3,2} &= -\frac{\lambda}{KR^2}u_2 \\ \nu u_{1,1} + u_{2,2} - Ru_3 &= -\frac{\lambda}{KR}u_3 \end{cases} \tag{14}$$

where u^1, u^2 and u^3 are the contravariant components of the displacement and $K = \frac{Eh}{(1-\nu^2)R^4}$.
To study the propagation of the singularities, we introduce the following right-hand side which represents a point normal force

$$\begin{cases} u_{1,11} + \frac{1-\nu}{2}u_{1,22} + \frac{1+\nu}{2}u_{2,12} - \nu Ru_{3,1} + \frac{\lambda}{KR^2}u_1 &= 0 \\ \frac{1+\nu}{2}u_{1,12} + \frac{1-\nu}{2}u_{2,11} + u_{2,22} - Ru_{3,2} + \frac{\lambda}{KR^2}u_2 &= 0 \\ \nu u_{1,1} + u_{2,2} - Ru_3 + \frac{\lambda}{KR}u_3 &= \frac{1}{KR}\delta(y^1)\delta(y^2) \end{cases} \tag{15}$$

and we suppose that λ belongs to the essential spectrum but is not an eigenvalue. We search for asymptotic expansions of the displacements on the form,

$$\begin{cases} u_\alpha(y^1,y^2) &= U^1_\alpha(y^1)\delta(y^2 - my^1) + U^2_\alpha(y^1)Y(y^2 - my^1) + ... \quad \alpha = 1,2 \\ u_3(y^1,y^2) &= U^1_3(y^1)\delta'(y^2 - my^1) + U^2_3(y^1)\delta(y^2 - my^1) + ... \end{cases} \tag{16}$$

where $m = tan\theta$ is the slope of the direction of propagation, and we substitute these expressions in the problem. By identifications of the leading terms, we then still obtain $\lambda = KR^2(1-\nu^2)\frac{m^4}{(1+m^2)^2} = \frac{Ehm^4}{R^2(1+m^2)^2}$. We note that it does not depend on the Poisson coefficient. It is easy to calculate $U^1_1(y^1)$ and $U^1_2(y^1)$ according to $U^1_3(y^1)$:

$$U^1_1(y^1) = -Rm\frac{\nu m^2 - 1}{(1+m^2)^2}U^1_3(y^1) \tag{17}$$

and

$$U^1_2(y^1) = R\frac{(\nu+2)m^2 + 1}{(1+m^2)^2}U^1_3(y^1) \tag{18}$$

By writing the identifications of the next order terms, we obtain a nonhomogeneous system and then we have to satisfy a compatibility condition. A straightforward computation gives the following compatibility condition,

$$\nu mR(\nu m^2 - 1)U^1_3{}'(y^1) + \frac{m}{2}\Big((\nu^2 + \nu - 4)m^2 + 3\nu - 1\Big)U^1_2{}'(y^1) +$$

$$\Big(\nu m^4 - \frac{(1+\nu)(2-\nu)}{2}m^2 + \frac{1-\nu}{2}\Big)U_1^1{}'(y^1) = -(m^2+1)^2\frac{1}{KR}\delta(y^1) \tag{19}$$

By replacing $U_2^1(y^1)$ and $U_1^1(y^1)$ by their values, we finally obtain,

$$U_3^1{}'(y^1) = \frac{R^3(1+m^2)^3}{4Ehm^3}\delta(y^1) \tag{20}$$

and we see that this equation does not depend on the Poisson coefficient.
The function U_3^1 which defines the propagation of the singularity is

$$U_3^1(y^1) = \frac{R^3(1+m^2)^3}{4Ehm^3}(Y(y^1)+C) \tag{21}$$

where C is arbitrary. The propagation of singularities is characterized by U_3^1 and is along the characteristic straight line which cuts the edges of the domain in two points. So we will have two conditions to determine only one constant C. It shows that the singularity will not disappear by reaching a point P on $\partial\Omega$. We will have a reflection on another characteristic the slope of which is $-tan\theta$.
The previous calculus are not valid if $m = tan\theta$ and then λ are equal to zero. In that case which is the static case, the propagation of singularities is rather different and there is no reflection.
That study of the propagation of singularities has been done in a particular case of isotropic shell. The equations of vibration of a cylindrical shell have constant coefficients and then the propagations are along straight lines. The general case is more complicated:
The segments which constitute the essential spectrum could be different in every point of the surface S. The characteristic curves are some pieces of curves and the propagations along them would depend on the reached point. For example, it is conceivable that the value of the spectral parameter which is given, will be go out of the essential spectrum in some point and that the propagation will stop.
Moreover, if some propagation reaches a hyperbolic point at the edge of a shell with four directions of propagation, then we will have several possibilities for the reflection and we do not know what will happen.

3 Spectra of two-dimensional models of thin plates with sharp edges

Considering a plate clamped at the edge, the spectral problems of thin plates with uniform thickness is a classical problem, but if the edge is sharp enough, we have special properties. We only consider the case of homogeneous isotropic elastic material and we suppose that we have a geometrical symmetry of the thickness. But we can also extend some results with more general cases, where the interaction of deflection and longitudinal deformation occurs.
Let $\omega \subset \mathbb{R}^2$ be the longitudinal cross-section of a thin isotropic plate with variable thickness. The thickness function h vanishes on the edge $\partial\omega$ while,

$$h(x) = n^\alpha(H(s)+\tilde{h}(x)), \quad x \in \ \omega \cap \mathcal{V}, \tag{22}$$

where $\alpha > 0$, (n,s) is the system of local curvilinear coordinates in a neighborhood of the $\partial\omega$, n is the orientated distance to $\partial\omega$, $n > 0$ inside ω and s is the arc length along $\partial\omega$ and H is a smooth positive function on $\partial\omega$ and $\tilde{h}$ is a smooth function which vanishes on $\partial\omega$.

3.1 Longitudinal oscillations of the plate

We work in a weighted Lebesgue space with a norm which depends on α.
The spectral problem where μ is the spectral parameter can be reduced to the abstract equation

$$T_\sharp v = \tau v \quad \text{in } \mathfrak{h}_\sharp \tag{23}$$

where τ is the new spectral parameter

$$\tau = (1+\mu)^{-1} \tag{24}$$

The Hilbert space $\mathfrak{h}_\sharp$ has a specific scalar product and $T_\sharp$ is a positive and continuous, symmetric, therefore, self-adjoint operator, which inherits some properties of compactness.
We obtain the following results,
For any α, the spectrum of the problem is discrete and form the eigenvalue sequence

$$\mu_1 I \mu_2 I \mu_3 ... I \mu_p I ... \longrightarrow +\infty \tag{25}$$

(multiplicity is taken into account). In the case $\alpha < 1$ all eigenvalues are positive while, for $\alpha i 1$, we have $\mu_4 > 0$ and $\mu_1 = \mu_2 = \mu_3 = 0$ with the eigenspace of rigid modes.

3.2 Transversal oscillations of the plate

In this case, we have a spectral problem where λ is the spectral parameter in some Hilbert space $\mathfrak{h}_3$.
We also reduce the problem to the abstract equation

$$T_3 v = \tau v \quad \text{in } \mathfrak{h}_3 \tag{26}$$

where τ is the new spectral parameter

$$\tau = (1+\mu)^{-1} \tag{27}$$

and we have the following results,
The spectrum of the problem is discrete if and only if $\alpha < 2$. The corresponding eigenvalue sequence

$$\lambda_1 I \lambda_2 I \lambda_3 ... I \lambda_p I ... \longrightarrow +\infty \tag{28}$$

is positive in the case $\alpha < 1$ but, for $\alpha \in [1, 2[$, we have $\lambda_4 > 0$ and $\lambda_1 = \lambda_2 = \lambda_3 = 0$ with the eigenspace of rigid modes.
Moreover, if $\alpha i 2$, the essential spectrum of the problem is not empty and it is possible to find a singular Weyl sequence for operator T_3 at the point $\tau = 1$.

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