Motion of the rigid body consisting of two disks

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Abstract

We present a kinematic analysis and dynamic simulation motion of a rigid body consisting of two equal elliptical disks whose symmetry planes are at right angle. The no-slip constraints of the body are integrable since the system is essentially holonomic. Trajectories of the ground contact points are found. All equilibrium positions of the body on the plane are found and their stability analysis is performed.

1 Introduction

Let us consider the rigid body of the following form: it comprises of two symmetric laminae whose planes of symmetry make a right angle between each other. The laminae are connected along the common axis of symmetry. When this body moves along the fixed horizontal plane it touches the plane in two points (Fig. 1).

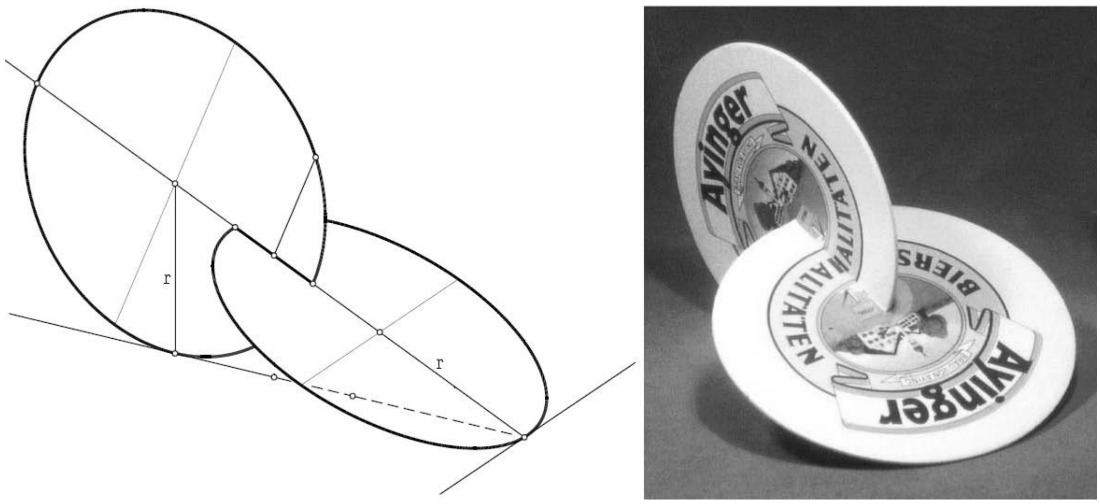

Figure 1: The Two-Circle-Roller and the Oloid.

The most known bodies of such a form are the Two-Circle-Roller [1, 2] and the Oloid [3, 4, 5]. The Two-Circle-Roller consists of two interlocked circular disks with the distance between their centers is $r\sqrt{2}$, where r is a common radius of the disks (Fig. 1). The Oloid is similar to Two-Circle-Roller: it consists of two interlocked circular disks but the distance between their centers equals to their radius r. For both of these bodies their motion on the fixed horizontal plane is studied in details [1]-[5]. However it is interesting to investigate the motion of the rigid body whose form differs from Two-Circle-Roller and Oloid.

The theory proposed in [4, 5] allows to investigate the motion of the rigid body when this body comprises of two symmetric laminae of the arbitrary form. In this paper we study the motion of the rigid body consisting of two interlocked elliptical disks whose planes of symmetry make a right angle between each other. The distance between their centers can be arbitrary value. Trajectories of the ground contact points are found. The equilibria of the body on the plane are found and their stability conditions are obtained.

2 Problem Formulation

Let two identical elliptical disks with semi-axes a and b, $(a > b)$ in perpendicular planes be given such that the distance between their centers C_1 and C_2 equals 2Δ and $\Delta < a$. Suppose that these elliptical disks are connected along their major axis of symmetry (Fig. 2). According to the theory discussed in [4, 5] let us introduce the moving coordinate frame $Gx_1x_2x_3$. The origin of this frame will be at the midpoint G of C_1C_2 (i.e. G is the center of mass of the system). The Gx_3 - axis is perpendicular to the plane Π_1 of the first disk, Gx_1 - axis is perpendicular to the plane Π_2 of the second disk and Gx_2 axis is directed along the common axis of symmetry of two elliptical disks. The unit vectors of this coordinate system will be $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$.

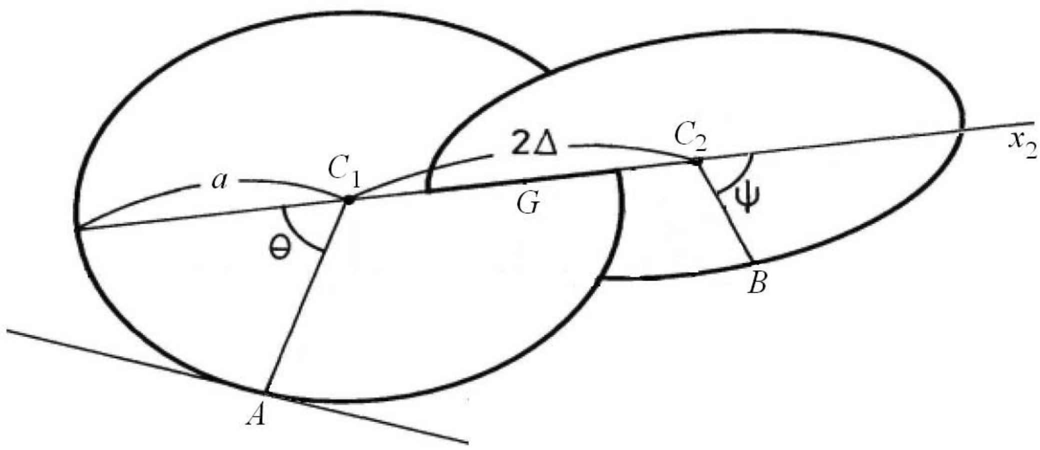

Figure 2: Rigid body consisting of two elliptical disks.

We will parametrize the first disk by the angle θ between the negative direction of Gx_2 axis and the direction to the point of contact A. Then the parametric equations for the bound of the first disk in terms of θ have the following form:

$$x_1 = \frac{ab\sin\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}, \quad x_2 = -\frac{ab\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}.$$

It is necessary to note that the natural arc-length parameter s is connected with the variable θ by the formula:

$$\frac{ds}{d\theta} = \frac{ab\sqrt{a^4\sin^2\theta + b^4\cos^2\theta}}{\left(a^2\sin^2\theta + b^2\cos^2\theta\right)^{\frac{3}{2}}}. \tag{1}$$

Let us introduce also the angle ψ for the parametrization of the second disk: let ψ be the angle between the positive direction of Gx_2 axis and the direction to the point of contact B. The parametric equations for the bound of the second disk have the form:

$$x_2 = \frac{ab\cos\psi}{\sqrt{a^2\sin^2\psi + b^2\cos^2\psi}}, \quad x_3 = -\frac{ab\sin\psi}{\sqrt{a^2\sin^2\psi + b^2\cos^2\psi}}.$$

The radius - vector of the point A can be written as follows:

$$\overrightarrow{GA} = \boldsymbol{r}_1 = \frac{ab\sin\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} e_1 - \left(\Delta + \frac{ab\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\right)\boldsymbol{e}_2.$$

The radius-vector of the point B has the form:

$$\overrightarrow{GB} = \boldsymbol{r}_2 = \left(\Delta + \frac{ab\cos\psi}{\sqrt{a^2\sin^2\psi + b^2\cos^2\psi}}\right)\boldsymbol{e}_2 - \frac{ab\sin\psi}{\sqrt{a^2\sin^2\psi + b^2\cos^2\psi}}\boldsymbol{e}_3.$$

When the considered rigid body rolls on a fixed horizontal plane the three vectors $\boldsymbol{r}_2 - \boldsymbol{r}_1$, $(\boldsymbol{r}_1)'_\theta$ and $(\boldsymbol{r}_2)'_\psi$ are always in this plane. We can write this condition as follows:

$$< \boldsymbol{r}_2 - \boldsymbol{r}_1,\, (\boldsymbol{r}_1)'_\theta,\, (\boldsymbol{r}_2)'_\psi >= 0,$$

where $< \cdot, \cdot, \cdot >$ is a triple scalar product of these vectors. From this condition we can find the following connection between θ and ψ:

$$\cos\psi = -\frac{a^2\cos\theta}{\sqrt{a^4 + 4ab\Delta\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + 4b^2\Delta^2\cos^2\theta}}$$

and obtain the radius – vector $\overrightarrow{GB} = \boldsymbol{r}_2$ in the θ – parametrization:

$$\boldsymbol{r}_2 = \left(\Delta - \frac{a^2b\cos\theta}{a\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + 2b\Delta\cos\theta}\right)\boldsymbol{e}_2 - \\ -\frac{b\sqrt{a^4\sin^2\theta + 4ab\Delta\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + 4b^2\Delta^2\cos^2\theta}}{a\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + 2b\Delta\cos\theta}\boldsymbol{e}_3. \qquad (2)$$

Expression (2) for the radius - vector $\boldsymbol{r}_2$ should make a sense therefore we should have

$$a^2\sin^2\theta + 4pb\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + 4p^2b^2\cos^2\theta \geq 0.$$

This means that $\cos\theta$ should satisfy to inequality:

$$\cos\theta > -\frac{a^2}{\sqrt{a^4 + 4a\Delta b^2 + 4\Delta^2 b^2}}.$$

Therefore we have the following restrictions for the variables θ and ψ:

$$-\arccos\left(-\frac{a^2}{\sqrt{a^4+4a\Delta b^2+4\Delta^2b^2}}\right) < \theta < \arccos\left(-\frac{a^2}{\sqrt{a^4+4a\Delta b^2+4\Delta^2b^2}}\right),$$

$$-\arccos\left(-\frac{a^2}{\sqrt{a^4+4a\Delta b^2+4\Delta^2b^2}}\right) < \psi < \arccos\left(-\frac{a^2}{\sqrt{a^4+4a\Delta b^2+4\Delta^2b^2}}\right),$$

3 Trajectories of the points of contact.

Let us derive now equation for the fixed plane in the $Gx_1x_2x_3$ coordinate system. This equation can be derived from the condition that points A, B and the tangent vector to the first disk at A are in the fixed plane. Therefore, after some simplifications we get:

$$-a^2\sin\theta X + b^2\cos\theta Y + b\left(a\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + b\Delta\cos\theta\right) +$$

$$+Z\sqrt{a^4\sin^2\theta + 4ab\Delta\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + 4b^2\Delta^2\cos^2\theta} = 0.$$

The unit vector

$$\boldsymbol{n} = -\frac{a^2\sin\theta}{\sqrt{2a^4\sin^2\theta + b^4\cos^2\theta + 4ab\Delta\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + 4b^2\Delta^2\cos^2\theta}}\boldsymbol{e}_1 +$$

$$+\frac{b^2\cos\theta}{\sqrt{2a^4\sin^2\theta + b^4\cos^2\theta + 4ab\Delta\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + 4b^2\Delta^2\cos^2\theta}}\boldsymbol{e}_2 +$$

$$+\frac{\sqrt{a^4\sin^2\theta + 4ab\Delta\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + 4b^2\Delta^2\cos^2\theta}}{\sqrt{2a^4\sin^2\theta + b^4\cos^2\theta + 4ab\Delta\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + 4b^2\Delta^2\cos^2\theta}}\boldsymbol{e}_3$$

is the normal vector to this plane. Therefore the angle between the plane of the first disk and the fixed plane is determined by the formula:

$$\cos\varphi = (\boldsymbol{n}\cdot\boldsymbol{e}_3) =$$

$$= \frac{\sqrt{a^4\sin^2\theta + 4ab\Delta\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + 4b^2\Delta^2\cos^2\theta}}{\sqrt{2a^4\sin^2\theta + b^4\cos^2\theta + 4ab\Delta\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + 4b^2\Delta^2\cos^2\theta}}.$$

The curvature of the bounding ellips of the first disk is:

$$k = \frac{ab\left(a^2\sin^2\theta + b^2\cos^2\theta\right)^{\frac{3}{2}}}{\left(a^4\sin^2\theta + b^4\cos^2\theta\right)^{\frac{3}{2}}}.$$

This curvature is connected with the curvature of the trajectory of the ground contact point of the first disk by the the following equation [4, 5]:

$$K = k\cos\varphi.$$

Having the expression for K we can easily find the parametric equations of the trajectory of point A on the fixed plane. For this purpose we introduce the fixed coordinate system $OXYZ$. The origin O of this system coincides with the point of contact of the first disk with the plane at $\theta = 0$. The OX - axis is tangent to the first disk at $\theta = 0$, the OZ - axis is directed upwards. The OY - axis forms the right triple with the OX and OZ axes. Let α be the angle between the tangent vector to the trajectory of the point A and the OX-axis. Then the functions $X_A = X_A(\theta)$, $Y_A = Y_A(\theta)$ which give the parametric representation of this trajectory in terms of the variable θ can be determined as a solution of the following system of differential equations [4, 5]:

$$\frac{dX_A}{d\theta} = \frac{ds}{d\theta}\cos\alpha, \quad \frac{dY_A}{d\theta} = \frac{ds}{d\theta}\sin\alpha, \quad \frac{d\alpha}{d\theta} = K(\theta)\frac{ds}{d\theta}. \tag{3}$$

Taking into account equation (1) we can rewrite equations (3) in the form:

$$\frac{dX_A}{d\theta} = \frac{ab\sqrt{a^4\sin^2\theta + b^4\cos^2\theta}}{\left(a^2\sin^2\theta + b^2\cos^2\theta\right)^{\frac{3}{2}}}\cos\alpha, \tag{4}$$

$$\frac{dY_A}{d\theta} = \frac{ab\sqrt{a^4\sin^2\theta + b^4\cos^2\theta}}{\left(a^2\sin^2\theta + b^2\cos^2\theta\right)^{\frac{3}{2}}}\sin\alpha, \tag{5}$$

$$\begin{aligned}\frac{d\alpha}{d\theta} = {} & \frac{a^2b^2}{\left(a^4\sin^2\theta + b^4\cos^2\theta\right)}\times \\ & \times\frac{\sqrt{a^4\sin^2\theta + 4ab\Delta\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + 4b^2\Delta^2\cos^2\theta}}{\sqrt{2a^4\sin^2\theta + b^4\cos^2\theta + 4ab\Delta\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + 4b^2\Delta^2\cos^2\theta}}.\end{aligned} \tag{6}$$

Equations (4)-(6) are very complicated. In general case it is impossible to find the explicit form of expressions for X_A, Y_A and α. Therefore we integrate these equations numerically for various values of parameters a, b and Δ. The similar equations can be derived for the trajectory of the point B. Fig 3-4. shows the trajectories of points A and B on the fixed plane. The bottom curve is the trajectory of point A and the upper curve is a trajectory of point B.

Note that the system (4)-(6) can be solved for the particular values of parameters. For the Oloid ($a = b$, $\Delta = a/2$) this system is solved in terms of elementary functions [4, 5]. For the Two-Circle-Roller ($a = b$, $\Delta = a/\sqrt{2}$) it is solved with the help of elliptic integrals of the third kind [2].

4 Equilibria of the body and their stability

Having expressions for the vector $\overrightarrow{GA} = \boldsymbol{r}_1$ and the normal vector to the fixed plane $\boldsymbol{n}$, we can easily find expression for the potential energy of the rigid body consisting of two elliptical disks:

$$\begin{aligned} V &= Mgz_G = -Mg\left(\overrightarrow{GA}\cdot\boldsymbol{n}\right) = \\ &= \frac{Mgb\left(\Delta b\cos\theta + a\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\right)}{\sqrt{2a^4\sin^2\theta + b^4\cos^2\theta + 4ab\Delta\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + 4b^2\Delta^2\cos^2\theta}}.\end{aligned}$$

Critical points of the potential energy correspond to the equilibria of the system. The derivative of the potential energy has the form:

$$\begin{aligned} V'_\theta &= \frac{Mga^3b^3\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\times \\ &\times\frac{\left(b^2 - 2a^2 + 2\Delta^2\right)}{\left(2a^4\sin^2\theta + b^4\cos^2\theta + 4ab\Delta\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + 4b^2\Delta^2\cos^2\theta\right)^{\frac{3}{2}}}.\end{aligned}$$

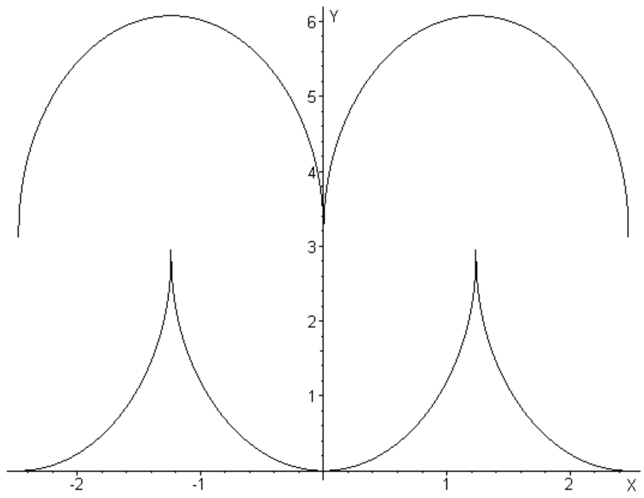

Figure 3: Trajectories of the ground contact points A (bottom curve) and B (upper curve). The values of parameters are: $a = 2$, $\Delta = 1$, $b = 1$.

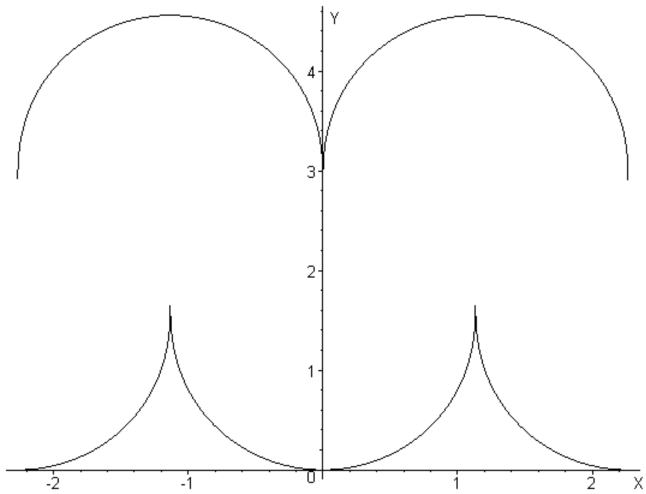

Figure 4: Trajectories of the ground contact points A (bottom curve) and B (upper curve). The values of parameters are: $a = \frac{\sqrt{3}}{\sqrt{2}}$, $\Delta = 1$, $b = 1$.

Thus the rigid body consisting of two elliptical disks has two equilibria: $\theta = 0$ and $\theta = \pi/2$. The sign of the second derivative of V, calculating at the corresponding equilibrium gives the conditions of stability of this equilibrium. Therefore for the equilibrium $\theta = 0$ we have

$$V''\Big|_{\theta=0} = \frac{Mga^3\left(b^2 - 2a^2 + 2\Delta^2\right)}{b\left(b^2 + 4a\Delta + 4\Delta^2\right)^{\frac{3}{2}}}.$$

This means that the equilibrium $\theta = 0$ is stable when $b^2 - 2a^2 + 2\Delta^2 > 0$ and unstable when $b^2 - 2a^2 + 2\Delta^2 < 0$. Similarly for the equilibrium $\theta = \pi/2$ we have

$$V''\Big|_{\theta=\pi/2} = -\frac{\sqrt{2}Mgb^3\left(b^2 - 2a^2 + 2\Delta^2\right)}{4a^4} > 0.$$

and the equilibrium $\theta = \pi/2$ is stable when $b^2 - 2a^2 + 2\Delta^2 < 0$ and unstable when $b^2 - 2a^2 + 2\Delta^2 > 0$.

The case when $b^2 - 2a^2 + 2\Delta^2 = 0$ or

$$\Delta = \sqrt{a^2 - \frac{b^2}{2}}$$

corresponds to neutral equilibrium. In this case the potential energy of the body is constant: the center of mass of the body moves at a constant height.

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