Complex variable Fast Multipole Method for modelling hydraulic fracturing in inhomogeneous media

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Abstract

The work aims to develop a numerical method for modelling hydraulic fractures in strongly inhomogeneous rocks. We employ the complex variable (CV) boundary integral equations specially tailored for blocky systems with multiple interfaces. Their numerical implementation is carried out by including the CV boundary element method (BEM) in frames of the fast multipole method (FMM) to solve systems with large (up to millions) number of unknowns. Recurrent analytical quadrature rules are used for increasing the robustness and accuracy of calculations. They are obtained for higher order approximations of the density at straight and circular-arc, ordinary, tip and multi-wedge elements. Conclusions are drawn on the accuracy of the method developed.

1 Introduction

Hydraulic fracturing is one of the major techniques of reservoir stimulation employed by the petroleum and gas industry. In practice this method is used in rock mass, which is strongly inhomogeneous. Meanwhile, analytical studies and available numerical codes do not account for this factor because of extreme mathematical difficulties. The most advanced code by the Schlumberger Company models only a vertical fracture propagating across a few horizontal elastic layers. This work aims to develop a numerical method, which may serve for accounting for strong inhomogeneity. As a basis, we employ boundary integral equations (BIE) in a form specially tailored to model blocky systems with complicated interfacial conditions at contacts of structural elements and containing pores, inclusions and growing cracks [1], [2]. For such structures, solving the BIE by the boundary element method (BEM) is superior over the finite element method. The main difficulty when employing this method for strongly inhomogeneous rock is very large number (up to millions) unknowns. To overcome the difficulty, we follow the line of combining the BEM with the fast multipole method (FMM) (see, e.g. [3]). In this work the combined BEM-FMM is developed for plane problems in complex variables (CV). In contrast with known applications of the BEM-FMM, we employ (i) the mentioned special forms of the BIE, (ii) combinations of circular-arc and straight boundary elements providing continuous tangent when approximating smooth parts of external boundaries and contacts, (iii) special singular elements accounting for singular behaviour of fields near singular points like common apexes of structural elements, (iv) approximations of higher order for circular-arc and straight boundary elements both ordinary and singular, (v) analytical recurrent formulae for integrals defining *influence coefficients* of the BEM and *multipole moments* of the FMM. We focus on the log-type kernels as the most difficult for evaluation. Results of numerical tests highlight efficiency of the CV-BEM-FMM developed.

2 Evaluation of influence coefficients to solve CV-BIE for inhomogeneous media

The CV-BIE, specially derived for solving 2D potential and elasticity problems concerning with blocky systems with multiple interfaces, cracks, inclusions and pores, contain seven standard integrals [1], [2]. After representing the contour by boundary elements, the integrals to be evaluated over a boundary element L_e are:

$$\int\limits_{L_e} \frac{f(\tau)}{\tau - z} d\tau, \int\limits_{L_e} \frac{f(\tau)}{(\tau - z)^2} d\tau, \int\limits_{L_e} f(\tau) \frac{\partial k_1}{\partial z} d\tau, \int\limits_{L_e} f(\tau) \frac{\partial k_2}{\partial z} d\tau,$$

$$\int\limits_{L_e} f(\tau) dk_1(\tau, z) d\tau, \int\limits_{L_e} f(\tau) dk_2(\tau - z) d\tau, \int\limits_{L_e} f(\tau) \ln|\tau - z| ds, \tag{1}$$

where $f(\tau)$ is the density, $z = x + iy$ is the CV coordinate of a filed point, τ is the CV coordinate of an integration point, ds is the length increment of the integration path, $k_1 = Ln(\frac{\tau - z}{\overline{\tau} - \overline{z}})$, $k_2 = \frac{\tau - z}{\overline{\tau} - \overline{z}}$. The recurrent analytical quadrature rules, serving for efficient evaluation of the first six of them over straight and circular-arc, ordinary, tip and multi-wedge boundary elements may be found in [1], [4].

Below we focus on evaluation of the last (log-type) integral in (1), which presents the main difficulty when considering harmonic problems. They arise from the fact that in these problems the density is *real* what complicates using the *complex* variables when deriving recurrence analytical formulae for curvilinear singular boundary elements (see, e.g. [2], [5]). Having this integral evaluated, analytical quadrature rules for singular and hypersingular integrals are obtained by direct differentiation with respect to z, what in its turn serves for efficient evaluation of the remaining integrals.

We consider two major forms of boundary elements, which allow one to represent a smooth part of a contour by a curve with continuous tangent. These are (i) straight and (ii) circular-arc elements. By linear transformation, integration over such an element is reduced to that over a standard element.

(i) A straight element of the length $2l$ is transformed into the standard element along the real axis [-1,1] in the variable τ' by the transformation $\tau = \tau_C + l \exp(i\alpha_C)\tau'$, where τ_C is the center of an element, α_C is its angle with the x-axis. For the points of a transformed straight element we have $\tau' = \overline{\tau'}$, where the overbar denotes complex conjugation.

For a density $f(\tau')$ on a standard straight element, the recurrent quadrature rules allow us using approximation of an arbitrary order, accounting, when appropriate, for power-type asymptotics near end point $\tau' = 1$. As a rule, it is sufficient to use approximation of the second order. Then

$$f(\tau') = \sum_{k=1}^{3} f_k \sum_{j=0}^{2} c_{kj} \tau'^j (1 - \tau')^\beta, \tag{2}$$

where f_k $(k = 1, 2, 3)$ are the nodal values of an approximated function, c_{kj} $(j = 0, 1, 2)$ are the Lagrange coefficients. For an ordinary (non-singular) element, $\beta = 0$. For a singular element, in numerical applications we assume β to be negative in cases when the density enters the log-type integral: $\beta = -\alpha$, with positive rational $\alpha = m/n$ $(m < n)$. When considering singular and hypersingular integrals, the exponent β in the density may be positive rational itself: $\beta = m/n$ $(m < n)$. The value of β for a particular problem is found by using the method suggested in [6] (for details, see [7]).

(ii) A circular-arc element with the angle $2\theta_0$ and radius R is transformed into the standard circular-arc element of unit radius, having the same angle and located symmetrically with respect to the x-axis: $\tau = \tau_C - iR\exp(i\alpha_C)\tau'$. Herein, τ_C is the center of the arc and α_C is the angle of the tangent at the midpoint of the arc with the x-axis. For points of the transformed circular-arc element we have $\tau' = 1/\overline{\tau'}$. The approximation used on the standard circular-arc element is:

$$f(\tau') = \sum_{k=1}^{3} f_k \sum_{j=-1}^{1} \widetilde{c}_{kj}\tau'^{j}\mathrm{Re}((e^{i\theta_0} - \tau')^{\beta}), \tag{3}$$

where $\widetilde{c}_{kj}$ $(j = -1, 0, 1)$ are coefficients of the form-functions at the arc of unit radius (see, e.g. [1]). The exponent β is prescribed similar to that for a straight ordinary or singular element.

Below the approximations (2), (3), suggested for evaluation of influence coefficients, serve us for evaluation of multipoles, as well.

3 Building quad-tree for FMM

A detailed description of the FMM algorithm may be found in [3]. Here we present the specific features of its numerical implementation adjusted to the particular forms of the CV-BIE. They are developed to minimize computer time and memory expense. Firstly, as all our computations are performed in the CV, the input data on the geometry of boundary elements is prescribed in the CV, as well. Thus a CV array of input data contains the CV coordinates of the central points of boundary elements. These data are repeatedly employed in further operations. In particular, an element is assumed to belong to a cell if its central point belongs to the cell.

The input information for building the hierarchical quad-tree consists of (i) these data on the coordinates of central points and (ii) a prescribed maximal number N_{max} of elements, which may be in a leaf.

In our procedure, following the well-known general line (e.g.[3]), we avoid looking through the entire input data matrix. Rather, at each level, only branches are taken into consideration, and for each branch we consider only those points, which belong to the branch. This tends to reduce memory and time expense. To reach this goal, we use special *renumeration* of elements. The renumeration is performed as follows. We consider a parent-branch at some level. For it, we have prescribed the total number M of points belonging to it. The points are numerated in growing order from N_1 to N_2, so that $N_2 = N_1 + M - 1$, and for each point, its number in the starting global numeration is known.

The parent-cell is divided into four child-cells, numerated form 1 to 4. The M points are analized to find the child, to which a point belongs. As a result, we attribute each of M points to a child-cell and find the total number of points in each of the children. Denote the total number of points in the k-th child M_k; obviously, $M = M_1 + M_2 + M_3 + M_4$. If $M_k = 0$, the corresponding child is empty and it is excluded from further analysis. Points of the first non-empty child k_1 with the total number of points M_{k_1} obtain numbers from N_1 to $N_1 + M_{k_1} - 1$; points of the second non-empty child k_2 (if it exists) with the total number of points M_{k_2} obtain numbers from $N_1 + M_{k_1}$ to $N_1 + M_{k_1} + M_{k_2} - 1$, and so on. Note, that in the new numeration, the first element of the first non-empty child-cell has the number N_1, while the last element of the last non-empty child-cell obtains the number N_2. Hence, the renumeration does not influence the numeration of elements in other cells on a

considered level and on all preceeding levels. As a result, for each child, which becomes a parent on the next level, the situation is reproduced: we have prescribed the total number of points belonging to it and numerated in growing order, which does not influence the numeration of points in other cells on the considered and preceeding levels. Finally, the totality of points in all leaves coincides with the points of the input array; now these points are numbered in that order, in which leaves appear in the dividing process.

In the course of dividing, we also save data on the number of a parent of a non-empty cell, total number of leaves and branches at each level, CV coordinates of centroids of non-empty cells, etc. These data are used later on for iterative solving the system of the CV-BEM in subroutines performing standard translations (Moment-to-Moment, Moment-to-Local and Local-to-Local) of the FMM (e.g. [3]). At each of the iterations, the multipole moments of each of leaves should be known. Their evaluation is performed by using analytical recurrence formulae, derived by the authors and presented in the next section.

4 Evaluation of multipole moments

As mentioned, we focus on the integral with log-type kernel. When a collocation point z is far away from a boundary element of integration L_e, the expansion of the potential $G(\tau, z) = -\frac{1}{2\pi}\ln|\tau - z|$ into the Taylor series, yields:

$$-\frac{1}{2\pi}\int\limits_{L_e} f(\tau)\ln|\tau - z|ds \approx \frac{1}{2\pi}\mathrm{Re}\left(\sum_{q=0}^{R_q} O_q(z-\tau_0)\int\limits_{L_e} I_q(\tau-\tau_0)f(\tau)ds\right), \tag{4}$$

where τ_0 is the *global* CV coordinate of the center of that leaf, to which the element L_e belongs, z is the CV coordinate of a collocation point, R_q is the prescribed maximal degree of multipole moments, which are kept in multipole expansions. The functions $O_q(z-\tau_0)$ and $I_q(\tau-\tau_0)$ are defined as (e.g. [3]):

$$O_0(z-\tau_0) = -Ln(z-\tau_0),\ O_q(z-\tau_0) = \frac{(q-1)!}{(z-\tau_0)^q} \text{ for } q \geq 1,$$

$$I_q(\tau-\tau_0) = \frac{(\tau-\tau_0)^q}{q!} \text{ for } q \geq 0.$$

An integral with the integrand $I_q(\tau-\tau_0)f(\tau)$, containing the q-th degree of $\tau-\tau_0$, is called the multipole moment of order q. To evaluate the moments, we use the same transformations of coordinates, which have been employed when evaluating the influence coefficients.

For a straight boundary element, with the density function (2), we obtain:

$$M_{LS}^q(\tau_0') = \frac{1}{q!}\sum_{k=1}^{3} f_k \sum_{j=0}^{2} c_{kj} l (l e^{i\alpha_C})^q \int\limits_{-1}^{1} \tau'^j (1-\tau')^\beta (\tau' - \tau_0')^q d\tau'. \tag{5}$$

For a circular-arc element with the density function (3), we have:

$$M_{LC}^q(\tau_0') = \frac{1}{q!}\sum_{k=1}^{3} f_k \sum_{j=-1}^{1} \widetilde{c}_{kj}(-iR)(-iRe^{i\alpha_C})^q \int\limits_{e^{-i\theta_0}}^{e^{i\theta_0}} \tau'^{j-1}\mathrm{Re}((e^{i\theta_0}-\tau')^\beta)(\tau'-\tau_0')^q d\tau', \tag{6}$$

In both cases, τ_0' is the *local* CV coordinate of the leaf centroid.

Denote M_q^j a typical integral on the right-hand side of equations (5) and (6),

$$M_q^j = \int_a^b w(\tau')\tau'^j(\tau' - \tau_0')^q d\tau', \tag{7}$$

Herein, $w(\tau')$ is a real weight function, accounting for behaviour of the density near the end point b; for an ordinary straight or circular-arc element, $w(\tau') = 1$; for a singular straight element ($b = -a = 1$), $w(\tau') = (1 - \tau')^\beta$; for a singular circular-arc element ($b = a^{-1} = e^{i\theta_0}$), $w(\tau') = \mathrm{Re}((e^{i\theta_0} - \tau')^\beta)$. When using three-node elements, we consider $j = 0, 1, 2$ for a straight element and $j = -2, -1, 0$ for a circular-arc element. As $(\tau' - \tau_0')^q = (\tau' - \tau_0')^{q-1}(\tau' - \tau_0')$, equation (7) yields the recurrence formula:

$$M_q^j = M_{q-1}^{j+1} - \tau_0' M_{q-1}^j. \tag{8}$$

Although equation (8) may be employed for evaluation of all the moments, it does not provide robust procedures for a straight element, because it requires using all the terms M_0^{2+q} when calculating M_q^2 with q running from 1 to R_q. Meanwhile, as shown below, we need to use equation (8) when considering negative degrees $j = -1, -2$, which appear in moments for circular-arc elements.

An alternative, more convenient recurrence equation for non-negative degrees j ($j = 0, 1, 2$) employs binomial representation of τ'^j, written as $\tau'^j = [(\tau' - \tau_0') + \tau_0']^j$. Denote

$$\widetilde{I}_q = \int_a^b w(\tau')(\tau' - \tau_0')^q d\tau'. \tag{9}$$

In many cases these coefficients are promptly evaluated recurrently in an analytical form. Then for $j = 0, 1, 2$ we obtain:

$$M_q^0 = \widetilde{I}_q,\ M_q^1 = \widetilde{I}_{q+1} + \tau_0'\widetilde{I}_q,\ M_q^2 = \widetilde{I}_{q+2} + 2\tau_0'\widetilde{I}_{q+1} + \tau_0'^2\widetilde{I}_q. \tag{10}$$

Equations (9) and (10) provide efficient evaluation of moments for straight elements.

For circular-arc elements, we firstly find

$$M_0^{-1} = \int_a^b w(\tau')\frac{1}{\tau'}d\tau',\ M_0^{-2} = \int_a^b w(\tau')\frac{1}{\tau'^2}d\tau'. \tag{11}$$

Then all higher-order moments are found recurrently by using (8), (9):

$$M_q^0 = \widetilde{I}_q,\ M_q^{-1} = \widetilde{I}_{q-1} - \tau_0' M_{q-1}^{-1},\ M_q^{-2} = M_{q-1}^{-1} - \tau_0' M_{q-1}^{-2}. \tag{12}$$

Equations (9), (11), (12) provide evaluation of moments for circular-arc elements.

5 Numerical experiments

We used exact analytical formulae for integrals over a closed or open circular contour (see, e.g. [1]) to (i) check that the derived formulae and developed procedures were correct, (ii) study the influence of the highest order R_q of moments employed on the accuracy of calculations. We considered both log-type and singular integrals.

For a field point outside a closed contour, the singular integral with constant density is zero. Application of FMM and the moments of the form (5) and (6) gave this result with high accuracy. Specifically, when approximating the contour by four straight or circular-arc boundary elements, under the assumption that the field point is located at a distance 0.05π from the contour, the evaluated value was 10^{-17} for both types of elements. Approximations by a larger number of elements do not affect this accuracy notably.

In another example, we considered an open contour represented by the circular crack with the angle $2\theta_0 = \frac{\pi}{3}$, radius $R = 0.025$, and the center of the circle at the point $\tau_C = 0.3(1+i)$. The length of the crack is $L = 2R\theta_0 = \pi R/3$. The distance from the center of the crack to the field point is $R + nL$ with $n = 1(3, 9)$. The relative distance is $r = \frac{R}{R+nL}$. Table 1 contains the obtained data on the accuracy of the CV-FMM-BEM for various r and R_q.

Table 4: Relative error of CV-BEM-FMM for various relative distance of field point and for various number of moments

	$M_{LC},\ \beta = 0.0$				$M_{SC},\ \beta = 0.0$			
R_q	5	8	12	16	5	8	12	16
r=1/3								
relative error	3E-5	2E-5	7E-7	4E-8	1E-2	6E-4	4E-5	2E-6
r=1/9								
relative error	1E-6	1E-8	1E-8	1E-8	1E-4	3E-7	1E-10	1E-13
r=1/19								
relative error	1E-7	1E-8	1E-8	1E-8	2E-6	1E-9	1E-12	3E-13

The data of Table 1 show that, as could be expected, the accuracy grows with growing distance and the number of moments held in calculations. It can be also seen that to the accuracy commonly provided by the conventional CV-BEM in calculations with double precision (5-6 significant digits, at most), for $r = 1/3$ $(1/9,\ 1/19)$, it is sufficient to hold the moments of the degree $R_q = 5$ $(4,\ 3)$ and $R_q = 12$ $(6,\ 4)$, respectively, for ordinary log-type and singular-type circular-arc element.

6 Acknowledgment

The first and the third of the authors (E. R. and A. L.) gratefully acknowledge the support of the European Research Agency (FP7-PEOPLE-2009-IAPP Marie Curie IAPP transfer of knowledge programme, Project Reference # 251475). The second auther (L. R.-R.) appreciates the support of the Polish Ministry of Science and Higher Education (Scientific Project N N519 440739 for 2010- 2012).

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