Analysis of one-dimensional systems with spatially modulated parameters as dissemination of the concept of vibrational mechanics

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Abstract

The paper is devoted to the study of averaged processes, which take place in fast varying periodic structures. Particularly, oscillations of a string and bending oscillations of a beam with variable cross-sections are considered. The method of direct separation of motions and the concept of vibrational mechanics [1,2] are used for the analysis of these systems. Thus, the applicability range of this method is broadened. The influence of "fast" spatial modulations on the effective values of systems parameters is revealed and described. Particularly, it is obtained that the variability of beam's cross-section, along with other effects, leads to the emergence of the additional "slow" longitudinal force. It is noted that the order of equations, which describe the averaged processes in the examined periodic structures, under certain conditions may be different from the order of corresponding equations, which characterize systems in the absence of modulations.

1 Introduction

Continua composed of periodically repeated elements (cells) are used in many fields of science and technology. Examples of such continua are composite materials, consisting of alternating volumes of substances with different properties. Use of these materials enables, in particular, to achieve the effect of heat or sound isolation. The simplest example is material such as felt, which has long been used for these purposes. Various frame structures, e.g. building frames and trusses of bridges, cranes and industrial constructions, railway tracks and compound pipes are also periodic systems. Study of such structures is of particular interest in connection with the problem of optimization of their properties.

Widely used approaches for wave examination in periodic systems are methods, based on the utilization of the Floquet theory [3,4,5]. The so-called frequency stop bands, i.e. frequencies ranges, in which a wave does not propagate through the considered structure, can be determined by its means. However, in the framework of this theory it is problematic to incorporate the (external) boundary conditions. Numerical approaches to study and optimization of periodic structures, particularly, WFE method [6,7] and topology optimization method [8,9,10], are based on finite-element models of the considered structures, and only approximate solution of a problem can be obtained by their means.

The present paper is devoted to the analysis of the averaged processes in spatially periodic structures, particularly, to the study of long wave propagation in a string and a beam with variable cross-sections. The aim of the research is to identify the effective (averaged) properties of the considered systems in relation to these "slow" processes. The averaging procedure for stationary and non-stationary processes in periodic structures (composite materials) based on the multiple scales method [11] combined with the averaging method [12] was proposed in the monograph [13]. In the present paper the method of direct separation of motions (MDSM) is used for the analysis of the considered mechanical systems. This method facilitates solution of various challenging problems of action of high frequency vibrations on nonlinear mechanical systems [1,2]. The advantages of the MDSM over methods [11,12] are the simplicity in application and the transparency of the physical interpretation. Thus, the applicability range of the MDSM is broadened in the paper.

2 Oscillations of a string with variable cross-section

The equation of oscillations of a string with variable cross-section $S = S_0(1 + \alpha \sin kx)$, $0 \le \alpha < 1$,$k >> {}^{\pi}/_{l}$, where l is the length of the string, is considered

$$\rho S \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(T \frac{\partial u}{\partial x}\right) = 0 \tag{1}$$

here ρ is the density of the string material, T is the tension force, $u(x,t)$ is a lateral deflection of the string. The boundary conditions are homogeneous: $u|_{x=0} = u|_{x=l} = 0$. Partial solution of equation (1) is sought in the form $u(x,t) = A(x)B(t)$. The following equations for the new variables $A(x)$ and $B(t)$ are obtained

$$\ddot{B} = -C_1 B; \frac{d^2 A}{dx^2} + \frac{C_1 \rho S_0}{T}(1 + \alpha \sin kx)A = 0 \tag{2}$$

Here C_1 is a constant. The concept of vibrational mechanics and the MDSM [1,2] are used for solving the second of these equations in the following manner: its solutions are searched in the form

$$A = A_1(x) + \psi(x, kx) \tag{3}$$

where A_1 is "slowly varying", and ψ is "fast varying", 2π-periodic in dimensionless (“fast”) spatial coordinate $x_1 = kx$ variable, with period x_1 average being equal to zero: $\langle \psi(x, x_1) \rangle = 0$. Angle brackets denote averaging by x_1. The following equation is obtained for variable A_1 by averaging the second of equations (2) by x_1

$$\frac{d^2 A_1}{dx^2} + \frac{C_1 \rho S_0}{T} A_1 = -\frac{C_1 \rho S_0}{T} \alpha \langle \psi \sin x_1 \rangle \tag{4}$$

The boundary conditions are $A_1|_{x=0} = A_1|_{x=l} = 0$. The equation for variable ψ is available from equation (2) by subtracting equation (4). Its solution has the form

$$\psi = A_1 \alpha \left(f_1(C_1\mu) \sin x_1 + \alpha f_2(C_1\mu) \cos 2x_1 + \alpha^2 f_3(C_1\mu) \sin 3x_1 + O(\alpha^3) + ... \right) \tag{5}$$

where $\mu = \rho S_0 \big/ (Tk^2) << 1$, $f_1, f_2, f_3...$ are functions of $C_1\mu$, expressions for which depend on the number of retained terms in series (5). Employing the obtained relation for variable ψ, the following equation is composed for variable A_1

$$\frac{d^2 A_1}{dx^2} + \frac{C_1 \rho S_0}{T} \left(1 + \alpha^2 \frac{f_1(C_1\mu)}{2}\right) A_1 = 0 \tag{6}$$

Based on this equation, it may be concluded that modulations of string cross-sectional area lead to a change of the effective (averaged) value of this parameter. To satisfy the boundary conditions for variable A_1 the following equality should hold true

$$\sqrt{\frac{C_1\rho S_0}{T}\left(1+\alpha^2\frac{f_1(C_1\mu)}{2}\right)}=\frac{n\pi}{l} \tag{7}$$

Retaining only the first term in series (5), that is justified for small α ($\alpha < 0.5$), we obtain expression $f_1(C_1\mu)=\frac{C_1\mu}{1-C_1\mu}$ for function f_1, while equation (7) will have two roots, one of which is much greater than the other. So, formally the two values of the string oscillations frequency $\sqrt{C_{1(1)}}$, $\sqrt{C_{1(2)}}$ correspond to the one value of the wave number $n\pi/l$. Such result indicates a change of the order of the considered equation (1) owing to modulations of the string cross-section. The accounting of the other terms in solution (5) leads to the emergence of the additional values of the frequency $\sqrt{C_1}$, which correspond to the one wave number $n\pi/l$, every next of which exceeds the previous.

Series of numerical experiments was conducted to verify this conclusion. As an illustrative example, the dependence of the acceleration of the middle of the string $\ddot{u}(l/2,t)$ on time at parameters $\rho S_0/T = 1\,(\mathrm{s/sm})^2$, $l = 5$ sm, $\alpha = 0.5$, $k = 2.6$ 1/sm and simple initial conditions $u|_{t=0} = \sin \pi x/l$, $\dot{u}|_{t=0} = 0$ is presented in Figure 1.

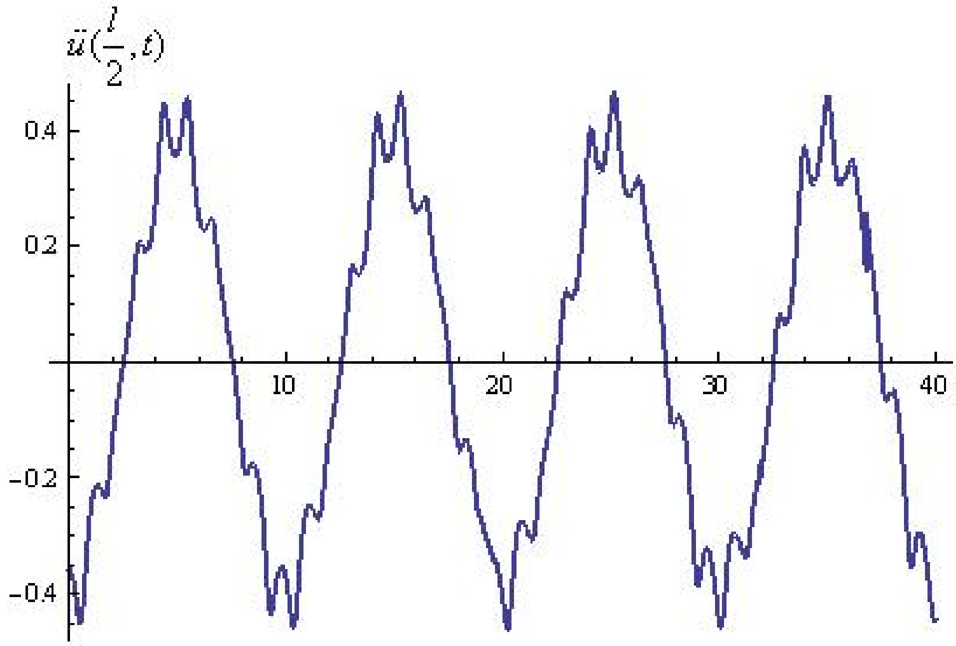

Figure 1: The dependence of the acceleration of the middle of the string on time.

As is seen from Figure 1, modulations of string cross-section indeed lead to the emergence of a spectrum of additional eigenfrequencies, which corresponds to the same wave number $n\pi/l$. Thereby, analitical results were confirmed by numerical experiments.

3 Bending oscillations of a beam with variable cross-section

We consider bending oscillations of a beam with variable cross-section, which are described by the equation

$$\rho S \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EJ\frac{\partial^2 w}{\partial x^2}\right) = 0 \tag{8}$$

Here ρ is the density and E is the Young's modulus of the beam's material, $w(x,t)$ is beam's deflection, $S = S(x) = S_0(1 + \alpha \sin kx)$ is the cross-sectional area, $J = J(x) = J_0(1 + \alpha_1 \sin kx)$ is the moment of inertia of the cross-section. Employing the approach, used above, for solving equation (8), we compose the equation for the slow variable A_1. Based on this equation it may be concluded that modulations of beam's cross-section lead to: the decrease of the effective moment of inertia of its cross-section, the emergence of the additional "slow" longitudinal force, and the increase of the effective cross-sectional area. As in the case of a string, satisfying the boundary conditions, we obtain an equation for the constant C_1, which has a number of roots.

Based on these results the conclusion is drawn that modulations of the beam cross-section lead, firstly, to the change of the value of the fundamental eigenfrequency $\sqrt{C_{1(1)}}$ in comparison with its non-modulated value, and, secondly, to the emergence of a spectrum of additional eigenfrequencies, which corresponds to the same wave number $n\pi/l$, i.e. to a change of the order of the initial differential equation.

4 Conclusions

The equations, which describe the averaged, long-wave processes in a beam and a string with variable cross-sections, are derived by means of the MDSM. Thus, the applicability range of this method is broadened to the cases, when separation of variables can be performed not in time, but, rather, in spatial coordinate. Based on the derived equations the influence of "fast" spatial modulations on the systems' effective (averaged) parameters is detected. Particularly, it is shown that modulations of string cross-sectional area lead to the increase of the effective value of this parameter, and the variability of beam's cross-section, along with other effects, leads to the emergence of the additional "slow" longitudinal force. It is noted that the order of differential equations, which describe the averaged (slow) processes in the considered periodic structures, under certain conditions may differ from the order of the corresponding equations in the absence of modulations.

Acknowledgements

The author is grateful to Professor I.I. Blekhman for specifying the research subject and comments to the paper.

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