Stability analysis of the 2D electroconvective charged flow between parallel plates using Discontinuous Galerkin Finite Element methods

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Abstract

We explore the capability of Discontinuous Galerkin Finite Element methods to solve numerically the charge transport equation in EHD convective problems. These methods are especially suited to treat purely hyperbolic problems. We compute both the electric and velocity fields in the case of strong injection. The numerical solution compares very well to the analytical solution for the hydrostatic situation, as well as with the theoretical linear stability criterion. The results are very promising for future research of complex electroconvection problems.

1 Introduction

Electrohydrodynamics (EHD) is an interdisciplinary area dealing with the interaction of fluids and electric fields or charges. It lies at the heart of several important industrial important processes[1] . In this paper we analyze the classical problem of the 2D flow between two parallel plates immersed in a dielectric liquid. When a high voltage is applied, the electrodes inject electric charge into the liquid, and the Coulomb force put the liquid into motion. Experiments and theoretical analysis show that the pattern of convection is made of hexagonal cells similar to those of Rayleigh-Bénard convection[2, 3]. The onset of the global motion is controlled by a non dimensional parameter involving the applied electric potential, the mobility of the charge carriers and the properties of the fluid.

The transport of electric charge involves three different mechanisms: drift by the electric field, convection by the velocity and th fluid and diffusion. In EHD, diffusion is only relevant inside a very thin boundary layer near the electrodes, and it is not relevant for phenomena developing in the bulk. So in our case the electric charge is transported only by the electric and velocity fields. The problem becomes purely hyperbolic, and special numerical treatments are needed.

In previous works we have simulated the time evolution of the 2D case using Particle-In-Cell(PIC) methods to deal with the distribution of electric charge[4, 5]. The numerical diffusion introduced by PIC methods is minimal, so they do a good job describing the electric charge distribution. However, they have problems too. They are numerically expensive, as a great number of particles are needed to simulate the problem. This is specially true for 3D problems. Also, special care must be taken to assures the value of the boundary condition for the charge near the injector.

Others methods have been used to simulate EHD problems: FE-FCT[4, 5], finite volumes[6], etc. In this paper, we explore the application of Finite Elements Discontinuous Galerkin methods to solve numerically the charge transport equation. We obtain

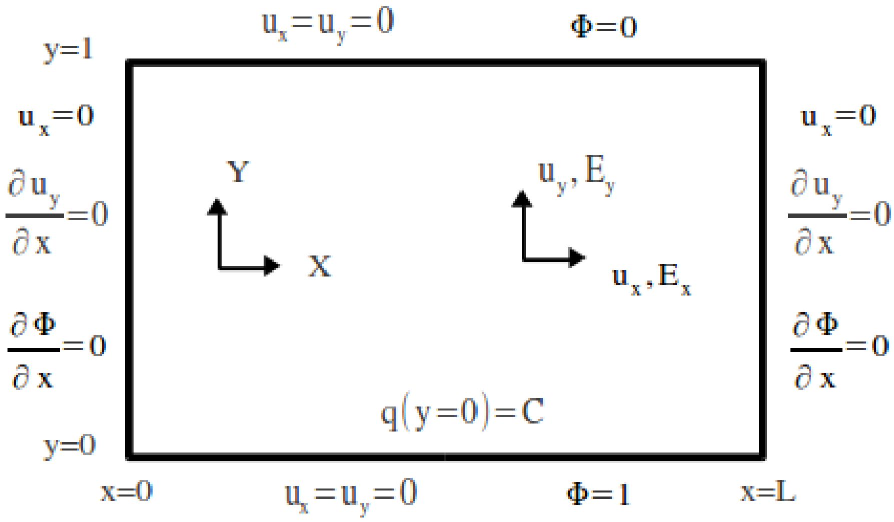

Figure 1: Non-dimensional computational domain and boundary conditions for the problem.

the linear stability analysis criteria for strong injection and compare the computed value with the analytical one[2, 3]. We will see that the Discontinuous Galerkin elements are able to reproduce very accurately the value for the linear stability criterion with much less CPU computing time than PIC methods.

2 Problem formulation

Two plane electrodes a distance d apart immersed in a non-conductive fluid are considered. An electric potential is applied between the plates so that injection of charge occurs. The electric field forces the charges away from the injector and in this way a space charge appears. The Coulomb force pushes the charges and the liquid with them. If the electric potential is high enough all the liquid is put into motion. Here we consider the 2D case, so the system is considered to be infinite along one of the direction parallel to the electrodes.

The fluid is considered to be incompressible, isothermal and insulating with mass density ρ, kinematic viscosity ν and permittivity ε. An electric voltage Φ_0 is applied between the plates. The charge carriers are considered to be of the same type with an ionic mobility K so they migrate along the liquid with a velocity $K\mathbf{E}$, where $\mathbf{E}$ is the electric field. Unipolar autonomous injection is assumed so the density of charge at the injector is constant and equal to q_0, and that the ions discharge instantaneously once they reach the opposite electrode.

There are three mechanisms responsible for the motion of ions: convection by the fluid, drift by the electric field and molecular diffusion. The last one can be neglected [1] so the current density is given by $\mathbf{J} = q(K\mathbf{E} + \mathbf{u})$, $\mathbf{u}$ being the velocity of the fluid and q the electric charge density. The first term represents drift and the second one convection.

The scales for all the involved variables are

$$\begin{array}{lll} x, y \sim d & \Phi \sim \Phi_0 & E \sim \Phi_0/d \\ u \sim K\Phi_0/d & t \sim d^2/K\Phi_0 & p \sim \rho K^2\Phi_0^2/d^2 \\ q \sim \varepsilon\Phi_0/d^2 & & \end{array} \tag{1}$$

p being the pressure.

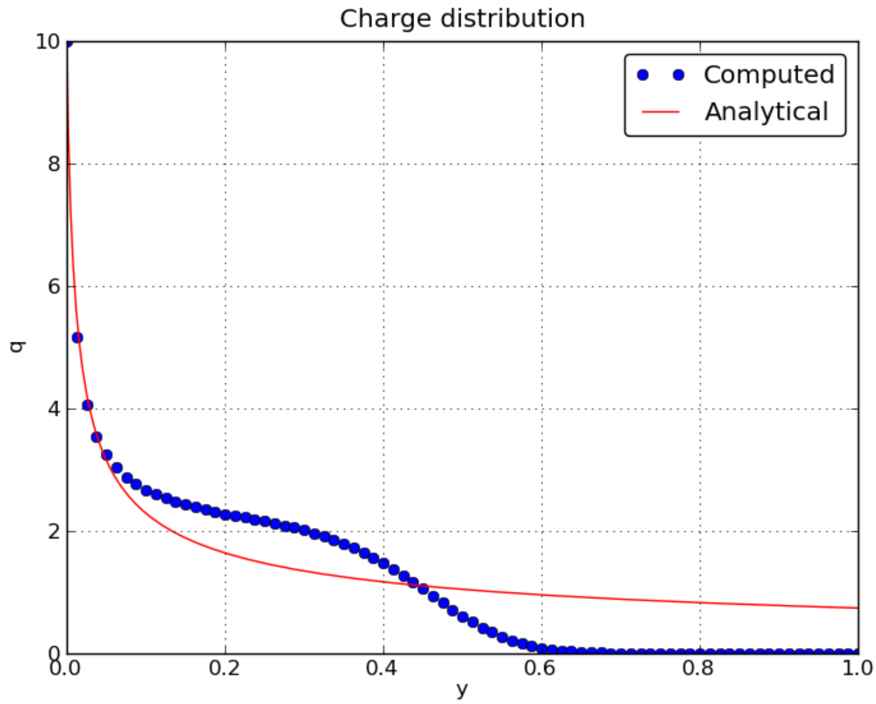

Figure 2: Charge density along the vertical central line of the domain for the hydrostatic solution for $C = 10$ and $t = 0.4$. The analytical solution and the outcome of the numerical simulation are shown.

The non-dimensional equation defining the problem are

$$\nabla^2\Phi = -q, \qquad \mathbf{E} = -\nabla\Phi, \tag{2}$$

$$\nabla \cdot \big[q(\mathbf{u} + \mathbf{E})\big] + \frac{\partial q}{\partial t} = 0, \tag{3}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{4}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{M^2}{T}\nabla^2\mathbf{u} + M^2 q\mathbf{E}, \tag{5}$$

The non-dimensional parameters of the problem are

$$T = \frac{\varepsilon\Phi_0}{\rho\eta K}, \quad C = \frac{q_0 d^2}{\varepsilon\Phi_0}, \quad M = \frac{1}{K}\sqrt{\frac{\varepsilon}{\rho}} \tag{6}$$

T is the ratio of the force term to the viscous term, and will be the stability paramenter. M is the ratio of the hydrodynamic mobilty[7] and C measures the injection strength.

In the linear stability analysis, the threshold value for the onset of the motion depends on the wavelength of the perturbation[2]. The minimum of these values is the absolute linear stability threshold. In the case of strong injection ($C = 10$), the critical wavelength turns out to be $k_{min} = 5.113$. We consider as domain a rectangle of size $L = \pi/k_{min} = 0.614$. This way, we solve the problem in one half of a convective cell. The non-dimensional domain and boundary conditions are shown in figure 1. At the lateral walls the perpendicular components of the electric and velocity field are null. The value of the charge density at the injector (the bottom plate) is C.

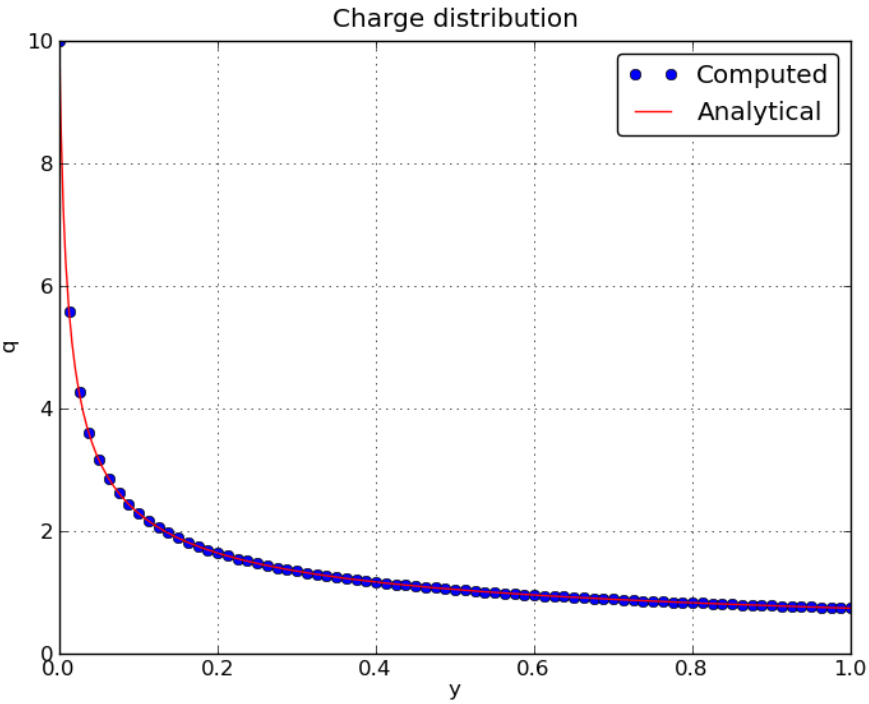

Figure 3: Charge density along the vertical central line of the domain for the hydrostatic solution for $C = 10$ and $t = 5.0$. The analytical solution and the outcome of the numerical simulation are shown.

3 Numerical algorithms

Both the electric field and the velocity field are computed using Continuous Galerkin Finite Elements(CG-FEM). We solve the Navier-Stokes equation using a Incremental Pressure Correction Scheme (IPCS)[8].

As described in the introduction, we use Discontinuous Galerkin Finite Elements (DG-FEM) to solve the charge transport equation. These methods were originally developed to deal with hyperbolic problems, but in recent years have been applied to all kind of problems involving partial differential equations[9]. The key idea is to consider internal degrees of freedom inside every element. The connection between elements is achieved using so called numerical fluxes. In this way, conservation is imposed locally. These methods have proved to be very stable when treating hyperbolic problems, and allow to work with complex geometries, as well as prescribing different orders of approximation inside each element.

We use a structured mesh made of triangles. We consider second order elements for the electric potential(CG-FEM) and the velocity field (CG-FEM). The pressure is approximated using first order CG-FEM, in order to comply with the LBB condition. The IPCS scheme is first order in time. For the charge density we use full upwind second order DG-FEM, with a backward scheme in time in order to enhance stability. The resulting numerical scheme is first order in time.

The algorithms have been implemented using the DOLFIN[10] Python library. This is an interface to FEniCS[11], a framework for automated solution of differential equations by the Finite Element method.

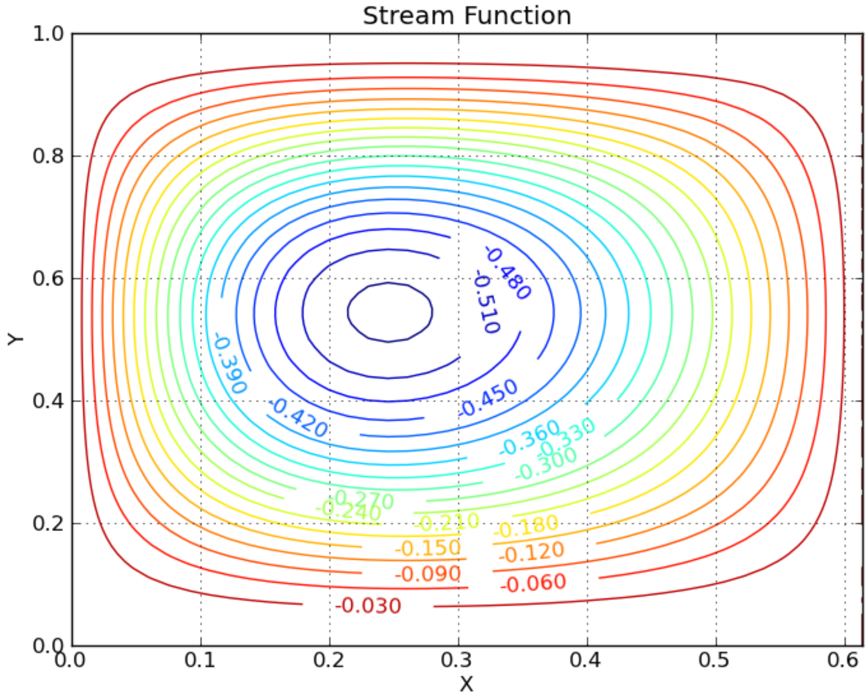

Figure 4: Contour plot of the stream function for for $t = 15$, $C = 10$, $M = 10$, $T = 200$.

4 Results

We present some results from simulations in 2D for the strong injection regime ($C = 10$). The bottom length of the rectangle is $L = 0.614$, which corresponds to the more unstable wavelength according to the linear stability analysis[2]. The mesh has 40 regular intervals along the X direction and 50 intervals along the Y direction, smaller near the injector at the bottom and coarser near the collector at the top. The mesh has 2091 nodes and 4000 triangular elements. The time step is $dt = 0.01$ for all simulations.

4.1 Hydrostatic regime

In order to verify the ability of the DG-FEM method to simulate the charge distribution, we have run simulations without computing the velocity. The results are compared with the analytical solution for $C = 10$

Figure 2 shows the charge density along a vertical line at non-dimensional time $t = 0.4$. The front of charge advancing towards the top electrode can be seen. The steady analytical solution is also plotted. The DG-FEM is able to describe this front of charge with no spurious oscillation near the region of the steepest gradient.

Figure 3 shows the computed and analytical charge densities along a vertical line when the steady state has been reached. The maximum difference between the computed value of the charge density and the analytical solution is less 0.5%. Also the values of the electric current computed at the injector and the collector differ in less than 0.1%.

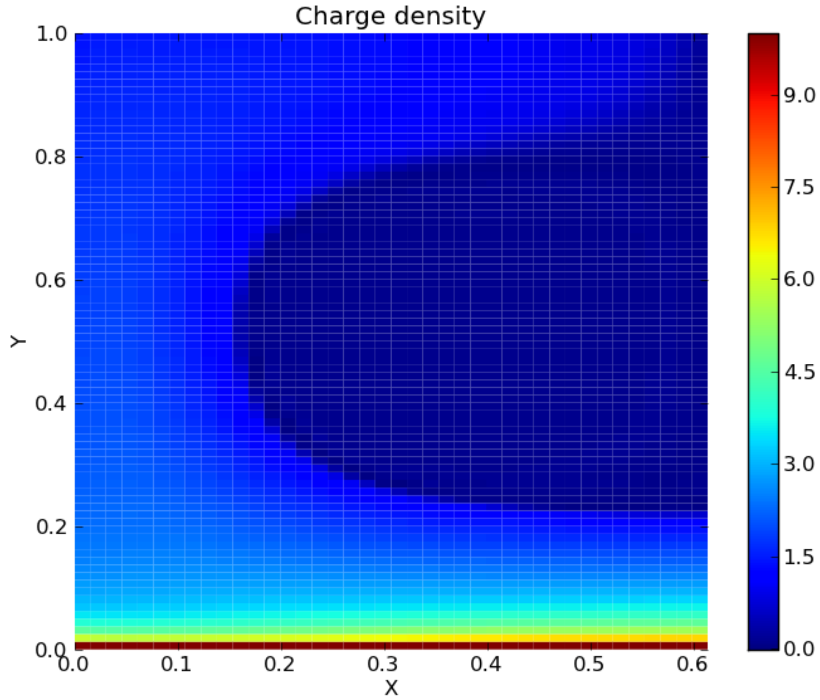

Figure 5: Charge density for $t = 15$, $C = 10$, $M = 10$, $T = 200$. The central region of the convective is void of electric charge.

4.2 Linear stability in the strong injection regime

In order to get the threshold value of the linear instability, we have run a set of simulations changing the value of the stability parameter T for different fixed values of the mobility parameter M. As an initial condition for the charge density, we set the analytical profile for the hydrostatic regime. Then we compute the electric field and solve the Navier-Stokes equation. The electric and velocity fields obtained are used to advance the charge density. The process is repeated iteratively in time. All the simulations were done for the strong injection regime, $C = 10$.

If the value of T is greater than the critical value T_c a velocity roll appears, with a maximum velocity greater than the electric field ($v_{max} = 4$ for $T = 200$ and $M = 10$). The velocity roll pushes the charge away from the central region, where a region void of electric charge appears. Figure 4 show the contour plot of the stream function for $M = 10$ and $t = 15$. The velocity roll is fully developed here. Figure 5 shows the distribution of electric charge density for $M = 10$ and $t = 15$. The central region empty of charge is clearly seen.

Figure 6 plots the evolution in time for $M = 10$ and several values of T of the global angular momentum of the convective cell, computed as $Lam = \int |(\mathbf{r} \ - \mathbf{r}_0) \times \mathbf{u}| \, d\mathrm{S}$, where $\mathbf{r}_0$ points to the center of the domain. This magnitude gives an idea of the strength of the velocity roll. For all values of T the growth becomes exponential in a certain interval of time (this corresponds to the linear sections of the curve in the figure, as the scale of the Y axis is logarithmic). In the linear stability analysis, in this region the angular momentum is considered to depend on time as $L_{am} = A\, e^{\sigma t}$, where σ is the growth factor. The value of σ depends on T. The critical value T_C for the onset of the instability corresponds to $\sigma = 0$. Using a quadratic fit for the function $\sigma(T)$ the value of T_c is obtained for the

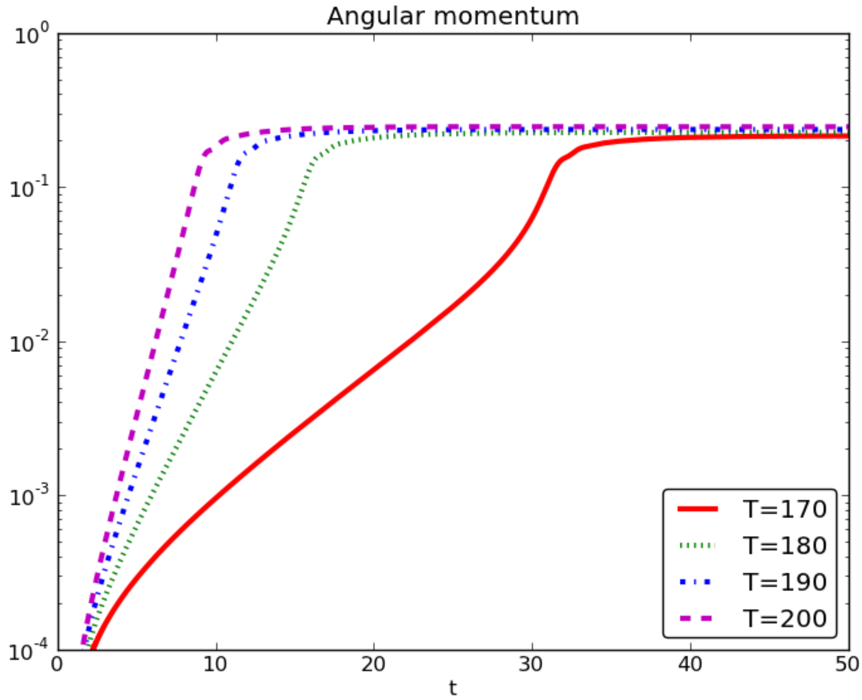

Figure 6: Evolution in time of the total angular momentum for $C = 10$ and $M = 10$. The critical value of T can be estimated from the regions where linear growth is observed (the scale is logarithmic for the Y axis)

different values of M. Table 6 shows the values of T_c obtained from our simulations for three different values of M.

These number are to be compared with $T_c^a = 164.1$, the critical value obtained from the linear stability analysis, independent of the value of M[2]. The computed values are very close to this theoretical number. The relative difference is only of 1.4%, which is consistent with the analytical result.

5 Conclusions

We have explored the possibility of using Discontinuous Galerkin Finite Element methods (DG-FEM) to solve numerically the charge transport equation in the 2D EHD convection between parallel plates in the strong injection regime. These methods are specially suited to deal with hyperbolic problems, as it is this case due to the negligible charge diffusion. We have used Continuous Galerkin Finite Element methods to solve the electric and hydrodynamic problems.

In the hydrostatic regime, the DG-FEM method is able to describe the advancing front of charge without spurious oscillations, and reproduces with a very good precision the analytical solution.

When the whole problem is considered, computing both the electric and velocity fields, the critical value for the onset of the linear instability obtained from the computation agrees extremely well with the analytical solution obtained from the linear stability analysis. It turns out to be essentially independent of the value of the mobility parameter, as it is

M	T_c
5	164.0
10	161.7
20	161.7

Table 6: Critical values for the onset of instability from the simulations for several values of M. The value obtained from the linear stability analysis is $T_c^a = 164.1$

predicted by the theory.

Further work is needed to validate the capacity of the DG-FEM methods to deal with EHD problems. In particular, the long term evolution of the charge density distribution has to be examined, in order to analyze the possible influence of the numerical diffusion that the method introduces, even if it is small. Also extensions to the 2D weak injection case, in the first place, and to the 3D dimensional problem is envisioned in future works.

6 Acknowledgements

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