Scattering of SH wave in half-space with cavity of arbitrary shape and crack

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Abstract

We present a method to study the scattering of SH wave and the ground motion in the half-space containing a shallow buried cavity of arbitrary shape and a linear crack in any orientation. The methods of complex function and multi-polar coordinate system are utilized to construct a suitable Green's function, which is used for computing stress caused by the line source. Then a crack can be built with the crack-division technique, by loading force that are equal but opposite to the stress at the crack. The dynamic stress concentration factor around the cavity and the dynamic stress intensity factor at crack tips under varied conditions are calculated, and the results indicates that: the dynamic stress concentration factor can be changed greater with the difference of the shape of the hole, the incident angle, and/or the wave number, compared to the variety of the parameters of the crack. In general, the dynamic stress intensity factor at the crack tip is as the change of the wave number to a periodic fluctuation.

Key words: cavity of arbitrary shape, crack, scattering of SH wave, method of complex function, Green's function

1 Introduction

The problem of scattering of elastic waves by all kinds of defects is of considerable importance, in geophysics, fracture mechanics, explosion mechanics, ultrasonic testing and other areas, and it is therefore naturally to receive much attention[1]. Defects, such as cavities, cracks, inclusions and linings, can be existed individually [2-8] or collectively [9-10]. In principle, the defects with arbitrary shapes can be conformal mapped into circular ones, then the dynamic response of elastic wave can be obtained. However, when the defects are spindly, just like cracks, it cannot be calculated correctly using the wave function expansion method due to the extremely divergence of Bessel function. So we must take other measures, for example the ray method and the Integral Equation Method [9, 11-13], though they are relatively cumbersome. The interaction of cavities and cracks by SH wave has been studied by some researchers. For example, Liu et al. [14] has treated the scattering of SH wave by a crack originating at a circular hole edge. An approximate solution for the scattering of SH wave by a crack inside a circular inclusion was given by Lu et al. [15]. Until now, few efforts have been made to study the interaction of cavities and cracks when the crack is placed in any position and direction, especially when the cavity is of arbitrary shape.

In this paper, we study the scattering of SH wave and the ground motion in the half-space containing a shallow buried cavity of arbitrary shape and a linear crack in any orientation, using the Green's function method and multi-polar coordinate system. The

crack is built with the crack-division technique, by loading force that are equal but opposite to the stress at the crack. The dynamic stress concentration factor around the cavity and the dynamic stress intensity factor at crack tips under varied conditions are calculated.

2 Fundamental Equations

By introducing the complex variable $z = x + iy$, $\bar{z} = x - iy$, the governing equation of the steady-state SH wave propagation can be expressed in the complex plane $(z, \bar{z})$as

$$\frac{\partial^2 W}{\partial z \partial \bar{z}} + \frac{k^2}{4} W = 0 \tag{1}$$

where k is the wave number, W is the amplitude of the displacement. The corresponding expressions of stress are given by

$$\tau_{r3} = \mu(\frac{\partial W}{\partial z} e^{i\theta} + \frac{\partial W}{\partial \bar{z}} e^{-i\theta}), \tau_{\theta 3} = i\mu(\frac{\partial W}{\partial z} e^{i\theta} - \frac{\partial W}{\partial \bar{z}} e^{-i\theta}) \tag{2}$$

where '3' represents the out-of-plane direction.

The method of conformal mapping $Z = w(\eta)$, $\eta = e^{i\theta}$ is used, the external region of the z plane is mapped into the external region with a unit circle in η plane, and the conformal mapping function can be generally expressed as

$$Z = w(\eta) = \frac{l\eta + m\eta^{-1}}{1 - n\eta^{-1}}, (|n| < 1) \tag{3}$$

where $l = l_1 + il_2$, $m = m_1 + im_2$, $n = n_1 + in_2$. In particular, for elliptical holes, $l = \frac{a+b}{2}$, $m = \frac{a-b}{2}$,$n = 0$, where a and b are the length of the major and minor axis, respectively; for square holes, the mapping function is $Z = w(\eta) = 1.11a(\eta - 1/9\eta^3)$, where a is half of the length of the side; and for triangular holes, $Z = w(\eta) = c(\eta + m\eta^{-n})$, $(c > 0, 0 \leq m < \frac{1}{n})$.

Substituting the mapping function (3) into Eq.(1) and Eq.(2), we obtain

$$\frac{1}{w'(\eta)\bar{w}'(\eta)} \frac{\partial^2 W}{\partial \eta \partial \bar{\eta}} + \frac{k^2}{4} W = 0 \tag{4}$$

$$\tau_{r3} = \frac{\mu}{|w'(\eta)|}(\eta \frac{\partial W}{\partial \eta} + \bar{\eta} \frac{\partial W}{\partial \bar{\eta}}), \tau_{\theta 3} = \frac{i\mu}{|w'(\eta)|}(\eta \frac{\partial W}{\partial \eta} - \bar{\eta} \frac{\partial W}{\partial \bar{\eta}}) \tag{5}$$

Boundary conditions can be expressed as follows

$$\tau_{r3} = 0, \quad when \quad \eta = e^{i\theta} \tag{6}$$

$$\tau = 0, \quad at \quad the \quad surface \quad of \quad half \quad space \tag{7}$$

3 Green's Function

The Green's function here should meet Eq.(4), Eq.(6)and Eq.(7). In a complete half-space, the disturbance impacted by the line source loading $\delta(\underline{r} - \underline{r_0})$ can be described in this form

$$G^{(i)} = \frac{i}{4\mu} H_0^{(1)}(k|w(\eta)| - w(\eta_0)|) \tag{8}$$

where $H_0^{(1)}(\cdot)$ is the zero-order Hankel function of the first kind. The reflected wave is generated due to the surface of half-space, and it can be derived directly according to the characteristics of SH-wave,

$$G^{(r)} = \frac{i}{4\mu} H_0^{(1)}(k|w(\eta)| - \bar{w}(\eta_0)| - 2ih) \tag{9}$$

The expression of the scattered wave excited by the elliptic cavity should meet the governing equation (4), as well as the stress free condition at the surface of the half-place. It can be constructed in plane

$$G^{(s)} = \sum_{n=-\infty}^{+\infty} A_n\{H_n^{(1)}[k|w(\eta)|][\frac{w(\eta)}{\bar{w}(\eta)}]^n + H_n^{(1)}[k|w(\eta)-2ih|][\frac{w(\eta)-2ih}{\bar{w}(\eta)-2ih}]^n\} \tag{10}$$

The symmetry of the SH-wave, the method of complex function and multi-polar coordinate system are used in Eq.(14), in which A_n are unknown coefficients to be determined by the boundary condition around the edge of the cavity, $H_0^{(1)}(\cdot)$ is the order-n Hankel function of the first kind. Then we can get the Green's function by compositing those three components

$$G = G^{(i)} + G^{(r)} + G^{(s)} \tag{11}$$

Substituting Eq.(5) and Eq.(11) into Eq.(9), we have

$$\sum_{n=-\infty}^{\infty} A_n\varepsilon_n = \varepsilon \tag{12}$$

where

$$\begin{aligned}
\varepsilon_n &= \frac{\mu k}{2}\{[H_{n-1}^{(1)}(k|w(\eta)|)[\frac{w(\eta)}{|w(\eta)|}]^{n-1} - H_{n+1}^{(1)}(k|w(\eta)-2ih|)[\frac{w(\eta)-2ih}{|w(\eta)-2ih|}]^{-(n+1)}] \\
&\cdot\frac{\eta w'(\eta)}{R|w'(\eta)|} - [H_{n+1}^{(1)}(k|w(\eta)|)[\frac{w(\eta)}{|w(\eta)|}]^{n+1} - H_{n-1}^{(1)}(k|w(\eta)-2ih|) \\
&\cdot[\frac{w(\eta)-2ih}{|w(\eta)-2ih|}]^{-(n-1)}]\frac{\bar{\eta}\bar{w}'(\eta)}{R|w'(\eta)}\} \\
\varepsilon &= \frac{ik}{8}H_1^{(1)}(k|w(\eta)-w(\eta_0)|)\cdot[\frac{\bar{w}(\eta)-\bar{w}(\eta_0)}{|w(\eta)-w(\eta_0)|}\frac{\eta w'(\eta)}{R|w'(\eta)|} + \frac{w(\eta)-w(\eta_0)}{|w(\eta)-w(\eta_0)|} \\
&\cdot\frac{\bar{\eta}\bar{w}'(\eta)}{R|w'(\eta)|}] + \frac{ik}{8}\cdot H_1^{(1)}(k|w(\eta)-\bar{w}(\eta_0)-2ih|)\cdot[\frac{\bar{w}(\eta)-\bar{w}(\eta_0)+2ih}{|w(\eta)-\bar{w}(\eta_0)-2ih|}\frac{\eta w'(\eta)}{R|w'(\eta)|} - \\
&\frac{w(\eta)-\bar{w}(\eta_0)-2ih}{|w(\eta)-\bar{w}(\eta_0)-2ih|}\frac{\bar{\eta}\bar{w}'(\eta)}{R|w'(\eta)|}]
\end{aligned}$$

Eq.(12) is a function containing infinite unknown coefficients A_n, we can transform it into infinite algebraic equations by multiplying $e^{-m\theta}$ on both sides of Eq.(12) and integrating over the interval $(-\pi, \pi)$, then we can get

$$\sum_{n=-\infty}^{\infty} A_n\varepsilon_{mn} = \varepsilon_m \tag{13}$$

where $\varepsilon_{mn} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\varepsilon_n e^{-m\theta}d\theta$, $\varepsilon_m = \frac{1}{2\pi}\int_{-\pi}^{\pi}\varepsilon e^{-m\theta}d\theta$. By intercepting finite terms, Eq.(13) converts to algebraic equations containing unknown coefficients A_n, which can be solved by the Gaussian method, then we can get the Green's function G.

4 The scattering of SH wave by the cavity and the crack

4.1 The first kind problem (SH wave incidences from below)

Just shown as Fig.1, there is an irregular cavity and a linear crack of arbitrary position and direction in half space, and SH wave incidences from below. It can be regarded as

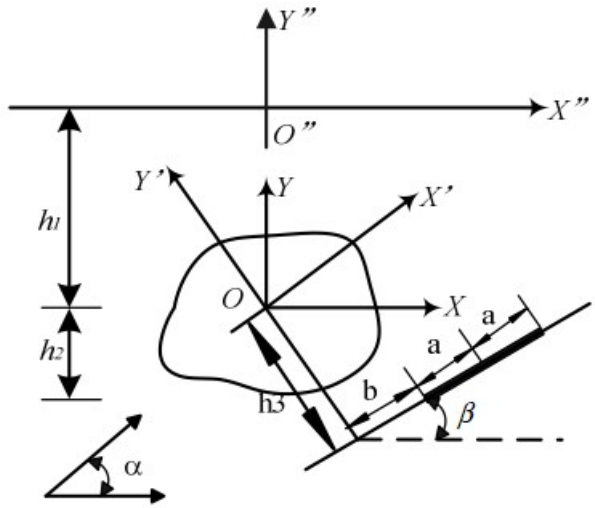

Figure 1: Model of half space with an arbitrary-shaped cavity and an crack when SH-wave incidences below

the Seismic problems of subsurface structures. To solve the problem, we should set up 3 coordinates XOY, $X'O'Y'$ and $X''O''Y''$, the relationship of them are

$$x' = x\cos\beta + y\sin\beta, y' = y\cos\beta - x\sin\beta,$$
$$x'' = x, y'' = y - h_1, h_3 = \frac{h_2 + b\sin\beta}{\cos\beta}$$

The incident wave and the reflected wave can be expressed as

$$W_1^{(i)} = W_0 \exp\{\frac{ik}{2}[(w(\eta) - ih)e^{-i\alpha_0} + (\bar{w}(\eta) + ih)e^{i\alpha_0}]\} \tag{14}$$

$$W_1^{(r)} = W_0 \exp\{\frac{ik}{2}[(w(\eta) - ih)e^{i\alpha_0} + (\bar{w}(\eta) + ih)e^{-i\alpha_0}]\} \tag{15}$$

In the same turn, the corresponding expressions of stress can be obtained by Eq.(5). When the displacement and the stress are obtained, then a crack can be built with the crack-division technique, by loading force that are equal but opposite to the stress at the crack. The additional wave field caused by the crack is

$$W_1^{(c)} = -\int_{(b,-h_3)}^{(2a+b,h_3)} \tau_{\theta z,1} G_1 dz'$$

Finally, in half space containing an irregular cavity and a linear crack of arbitrary position and direction, the wave field of the first kind problem is

$$W_1 = W_1^{(i)} + W_1^{(r)} + W_1^{(s)} + W_1^{(c)} \tag{16}$$

4.2 The second kind problem (SH wave incidences vertically from upside)

Just shown as Fig.2, there is an irregular cavity and a linear crack of arbitrary position and direction in half space, and SH wave incidences vertically from upside. It can be regarded as the anti-explosion problem of subsurface structures. For this problem, there

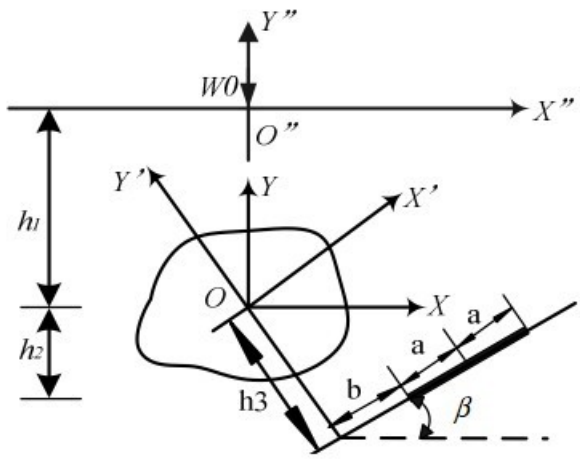

Figure 2: Model of half space with an arbitrary-shaped cavity and an crack when SH-wave incidences from upside

is no reflected wave, and the incident wave can be written as

$$W_2^{(i)} = W_0 \exp\{\frac{-k}{2}[(w(\eta) - ih) - (\bar{w}(\eta) + ih)]\} \tag{17}$$

Using similar methods, we can calculate the wave field of the second kind problem.

5 Dynamic Stress Concentration Factor (DSCF) and Dynamic Stress Intensity Factor (DSIF)

The dynamic stress around a shallow-buried cavity can be described by the dynamic stress concentration factor (DSCF) in the presence of the steady incident SH-wave

$$\tau_{\theta 3}^* = |\tau_{\theta 3}/\tau_0| \tag{18}$$

where $\tau_{\theta 3}$ is the hoop stress around the cavity, and $\tau_0 = \mu k W_0$ is the amplitude peak of the incident stress.

By picking the stress close enough to the crack tip as the nominal stress, we can get the dynamic stress intensity factor (DSIF) at the crack tip

$$K_3 = \frac{\tau_{r3}|_{\underline{r}\to\underline{r_1}}}{\tau_0 Q} \tag{19}$$

where $\tau_{r3}|_{\underline{r}\to\underline{r_1}}$ is the nominal stress of tiny distance from the crack tip, $Q = \sqrt{b/2}$ is the characteristic parameter,b is the length of the crack.

6 Numerical Results and Discussion

In this section, numerical examples are provided to show the distribution of the DSCF around the cavity and the DSIF at the crack tip with the variation of various parameters, such as the wave number k, the incident angle α, the shape of the cavity, the distance of the center of the circular cavity to the horizontal interface, the length of the crack etc.. Then we specialize the problem by setting the cavity as a circular one or by removing the crack, and the results are in conformity with published work available in the literature [6,7,10].

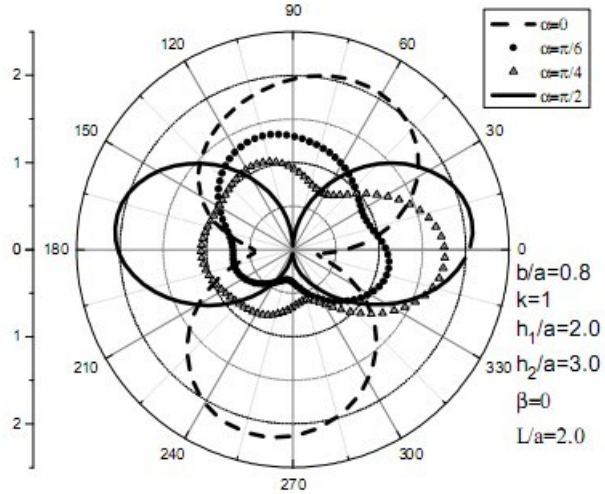

Figure 3: the DSCF around an ellipse cavity when SH-wave incidences from different directions

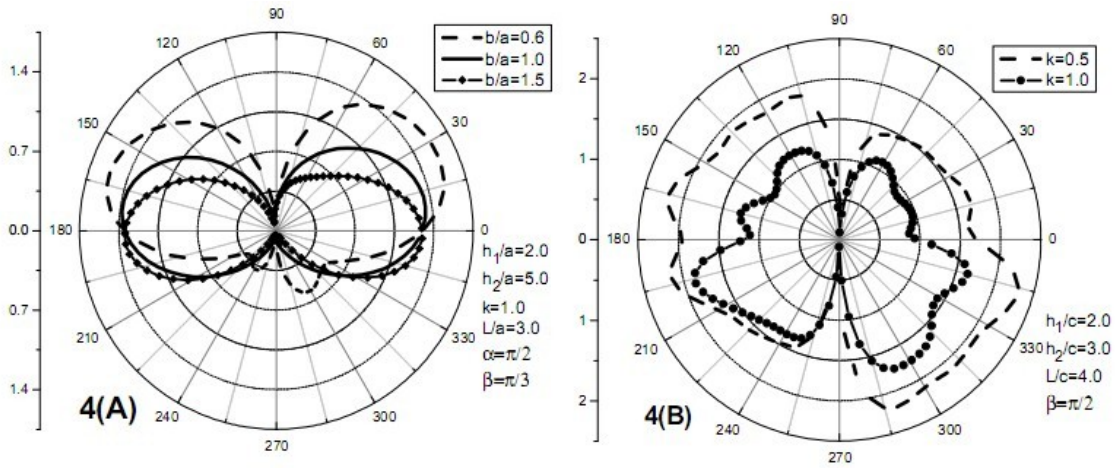

Figure 4: the DSCF around different shaped cavities for different wave numbers.

6.1 The DSCF around the cavity

(1) The results presented in Fig.3 show the variation of DSCF around the elliptic cavity with respect to the incident angle. We can see from it that the DSCF around the cavity varies widely, both for its amplitude and its shape, under deferent incident angle. (2) The distribution of DSCF around the cavity with respect to the change of the shape of the hole and the wave number is shown in Fig.4, where Fig.4(A) is the DSCF around a ellipse hole when the incident angle is 90°, Fig.4(B) shows the DSCF around a triangular cavity when the incident angle is 0°. The results indicate that the shape of the DSCF around the cavity varies widely for different cavities, and the amplitude of the DSCF can change a lot under different wave numbers. (3) Fig.5 shows the variation of DSCF around the elliptic cavity with respect to the distance of the center of cavity to the horizontal interface. Fig.5(A) shows the DSCF under the condition of h_2/a is 5, 13, and with no crack, respectively. Fig.5(B) gives the calculation results of the variation of DSCF at $\theta = 90°$ vs. the depth of the crack. The results indicate that the value of DSCF decreases periodically with

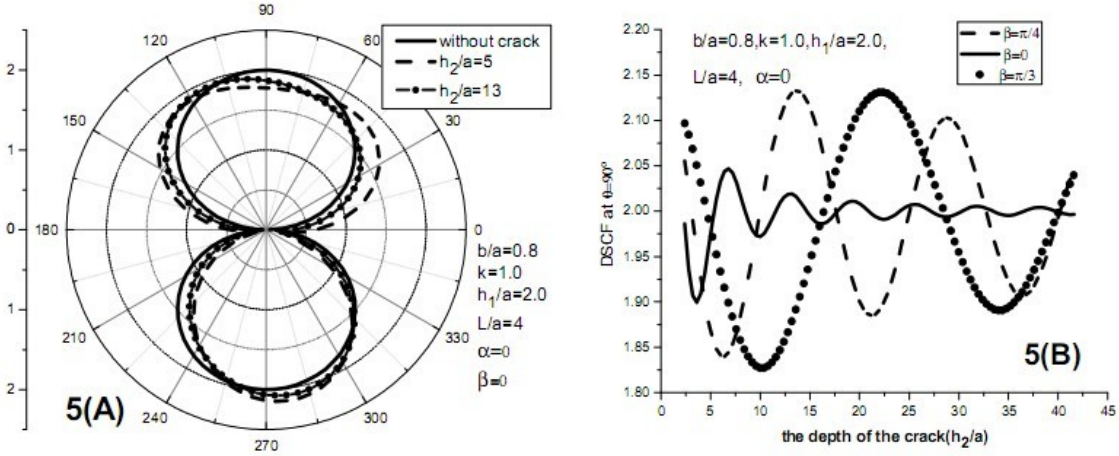

Figure 5: the DSCF with the variation of the depth of the crack.

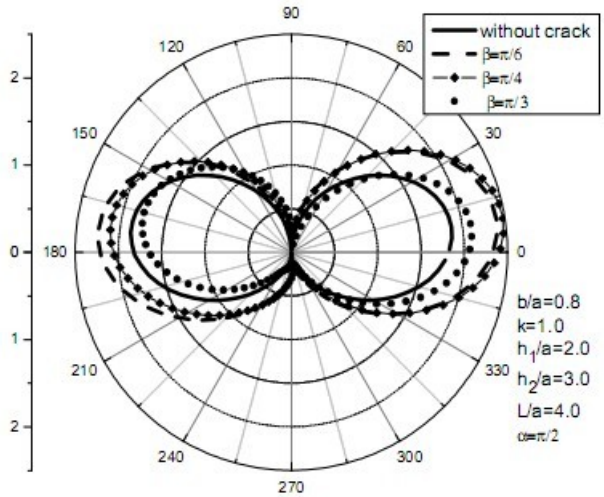

Figure 6: the DSCF around the cavity with different inclining angles of the crack

increasing distance, but the degradation is slight and related to the direction of the crack. (4) The magnitude of DSCF vs. inclining angles of the crack is plotted in Fig.6, and we can see that the effect of inclining angles to DSCF is less than those parameters such as the wave number, the incident angle and the shape of cavity.

6.2 the DSIF at the crack tip

(1) Fig.7 and Fig.8 demonstrate the effect of inclining angles of the crack and the incident angle on the DSIF, respectively. Fig.9 presents a qualitative analysis on the variation of DSIF vs. wave number with different length of the crack. This analysis indicates that the dynamic stress intensity factor at the crack tip is as the change of the wave number to a periodic fluctuation, and the change is more violent when the crack is longer.

(2) The variation of DSIF vs. the depth of the crack with different length of the crack is plotted in Fig.10, and the wave incidences vertically from upside. Generally speaking, the DSIF at the crack tip decrease gradually with the increase of crack depth, and the DSIF is proportional to the length of the crack.

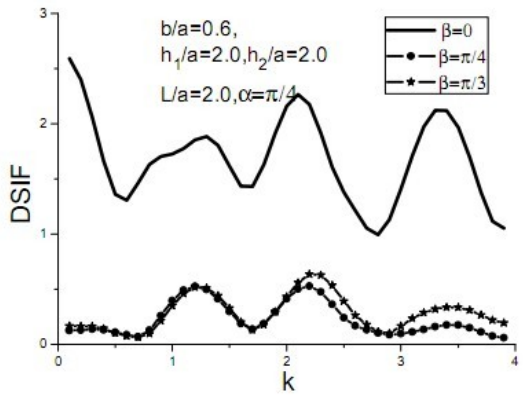

Figure 7: Variation of DSIF vs. wave number with different angles of the crack

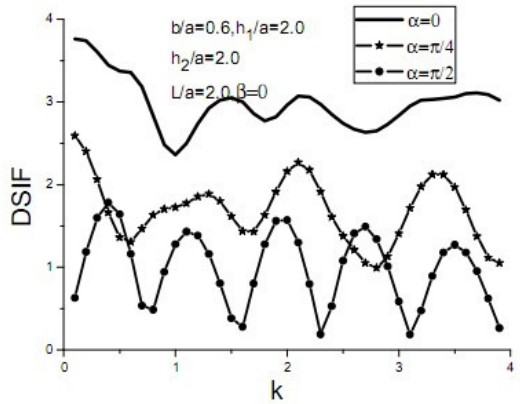

Figure 8: Variation of DSIF vs. wave number with different directions of the incident SH-wave

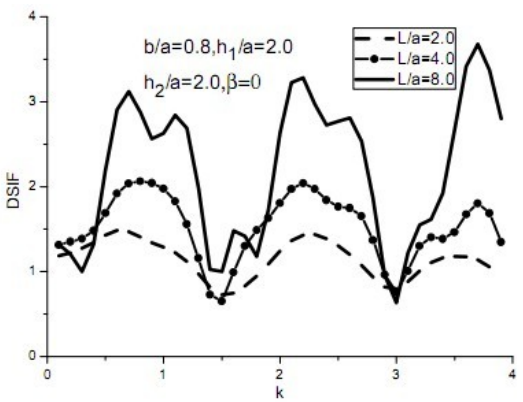

Figure 9: Variation of DSIF vs. wave number with different length of the crack

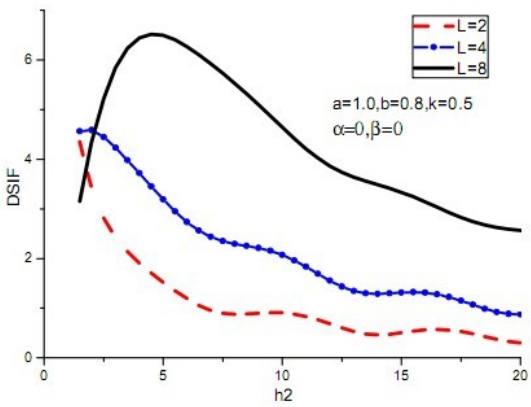

Figure 10: Variation of DSIF vs. depth of the crack with different length of the crack

7 Concluding remarks

This paper presented an analytical method to solve the scattering of SH wave by a cavity of arbitrary shape and a crack at arbitrary position and direction. The methods of complex function and multi-polar coordinate system have been used, and the crack is built with the crack-division technique. The numerical results were presented for both DSCF and DSIF under varied conditions, the results indicates that: the dynamic stress concentration factor can be changed greater with the difference of the shape of the hole, the incident angle, and/or the wave number, compared to the variety of the parameters of the crack. In general, the dynamic stress intensity factor at the crack tip is as the change of the wave number to a periodic fluctuation. An accurate investigation of the response is certainly an aid in the successful estimation of the detection of location, length, and depth of the crack. It is also a prerequisite for the inverse problem, and the method in the paper could be used to study some other correlative problem.

Acknowledgements

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