Continua traffic flow models

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Abstract

The present research was aimed at mathematical modeling of essentially unsteady-state traffic flows on multilane roads, wherein massive changing of lanes produces an effect on handling capacity of the road segment. The model is based on continua approach. However, it has no analogue with the classical hydrodynamics because momentum equations in the direction of a flow and in orthogonal directions of lane-changing are different. Thus velocity of small disturbances propagation in the traffic flow is different depending on direction: counter flow, co-flow, orthogonal to the flow. Numerical simulations of traffic flows in multilane roads were performed and their results are presented.

Russian Foundation for Basic Research (grant 13-01-12056) is acknowledged for financial support.

1 The continua model of traffic flows.

We introduce the Euler's co-ordinate system with the Ox axis directed along the auto route and the Oy axis directed across the traffic flow. Time is denoted by t. The average flow density $\rho(x, y, z)$ is defined as the relation of the surface of the road occupied by vehicles to the total surface of the road considered.

$$\rho = \frac{S_{tr}}{S} = \frac{hnl}{hL} = \frac{nl}{L}$$

where h is the lane width, L is the sample road length, l is an average vehicle's length plus a minimal distance between jammed vehicles, n is the number of vehicles on the road. With this definition, the density is dimensionless changing from zero to unit.

The flow velocity denoted $V(x, y, z) = (u(x, y, t), v(x, y, t))$ where u can vary from zero to U_{max}, where U_{max} is maximal permitted road velocity. From definitions, it follows that the maximal density $\rho = 1$ relates to the case when vehicles stay bumper to bumper. It is naturally to assume that the traffic jam with $V = 0$ will take place in this case.

Determining the "mass" distributed on a road sample of the area S as:

$$m = \int\limits_S \rho d\sigma$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad (1)$$

Then, we derive the equation for the traffic dynamics. The traffic flow is determined by different factors: drivers reaction on the road situation, drivers activity and vehicles response, technical features of the vehicles. The following assumptions were made in order to develop the model:

- It is the average motion of the traffic described and not the motion of the individual vehicles, that is modelled. Consequently, the model deals with the mean features of the vehicles not accounting for variety in power, inertia, deceleration way length, etc.
- The "natural" reaction of all the drivers is assumed. For example, if a driver sees red lights, or a velocity limitation sign, or a traffic hump ahead, he is expected to decelerate until full stop or until reaching a safe velocity, and not to keep accelerating with further emergency braking.
- It is assumed that the drivers are loyal to the traffic rules. In particular, they accept the velocity limitation regime and try to maintain the safe distance depending on velocity.

The velocity equation for the x-component is then written as follows:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = a; \quad a = max\left\{-a^-; min\left\{a^+; a^{'}\right\}\right\};$$

$$a^{'} = \sigma a_\rho + (1-\sigma)\int_0^Y \omega(y) a_\rho(t,s)ds + \frac{U(\rho)-u}{\tau};$$

$$a_\rho = -\frac{k^2}{\rho}\frac{\partial \rho}{\partial x}.$$

Here, a is the acceleration of the traffic flow; a^+ is the maximal positive acceleration, a^- is the emergency braking deceleration; a^+ and a^- are positive parameters which are determined by technical features of the vehicle. The parameter is the small disturbances propagation velocity ("sound velocity"), as it was shown in [16-18]. The parameter $k > 0$ is the delay time which depends on the finite time of a driver's reaction on the road situation and the vehicle's response. This parameter is responsible for the drivers tendency to keep the vehicles velocity as close as possible to the safe velocity depending on the traffic density $U(\rho)$ [16-18]:

$$U(\rho) = \begin{cases} -kln(\rho), \ u < U_{max} \\ U_{max}, \ u \geq U_{max} \end{cases}$$

The velocity $U(\rho)$ is determined from the dependence of the traffic velocity on density in the "plane wave" when the traffic is starting from the initial conditions ρ_0 and $V = 0$, with account of velocity limitation from above ($u \geq U_{max}$). The value of τ could be different for the cases of acceleration or deceleration to the safe velocity $U(\rho)$:

$$\tau = \begin{cases} \tau^+, \ U(\rho) < u \\ \tau^-, \ U(\rho) \geq u \end{cases}$$

In the formula for $a^{'}$ the first term describes an influence on the driver's reaction in a local situation, the second – a situation ahead the flow and the third – a driver's tendency to drive a car with the velocity which is the safest in each case. If we assume $\sigma = 0$, then the expression for $a^{'}$ will be:

$$a^{'} = \frac{1}{\Delta}\int_x^{x+\delta} a_\rho(t,s)ds + \frac{U(\rho)-u}{\tau}.$$

In this case acceleration is not a local parameter, but depends on its values in the region of length Δ ahead of vehicle, where Δ - is the distance each driver takes intro account on making decisions. This distance depends on road and weather conditions. The last term of the relaxation type takes into account tendency to reach optimal velocity velocity.

If we assume $\sigma = 1$, then the expression for will be:

$$a' = a_\rho + \frac{U(\rho) - u}{\tau}.$$

Now acceleration depends on local situation only. We will consider a case when $\sigma = 1, \tau = \propto$, so the equation of motion for the x-component is:

$$a_x = -\frac{k^2}{\rho}\frac{\partial \rho}{\partial x} \tag{2}$$

The equation of motion for the y-component we can write in such form as for the x one:

$$a_y = -\frac{A^2}{\rho}\frac{\partial \rho}{\partial y} \tag{3}$$

The description for the parameter A will be given below.

In order to understand the physical meaning of parameter A we shall consider the following model. One car is changing it's lane with the density $\rho \neq 0$ to the lane with the density $\rho = 0$. In this case a car has the maximum acceleration a_{max}. So by putting $\frac{\partial \rho}{\partial y}\frac{0-\rho}{h}$ to the velocity equation for the y-component we can obtain $\rho\frac{0-\rho}{h} = -A^2\frac{\partial \rho}{\partial y}$ and then $\frac{A^2}{h} = a_{max}$. The diagram for the $v(y)$ is showed on the Fig.1.

According to it we have: $v_{max} = \sqrt{a_{max}h}$, that is $a_{max} = \frac{v_{max}^2}{h}$ or $A^2 = v_{max}^2$. But in this model $v_{max} = 2v_{av}$ (v_{av} is an average speed of car's changing a lane), so $A^2 = 4v_{av}^2$, where v_{av}^{max} - the average speed for v_{max}.

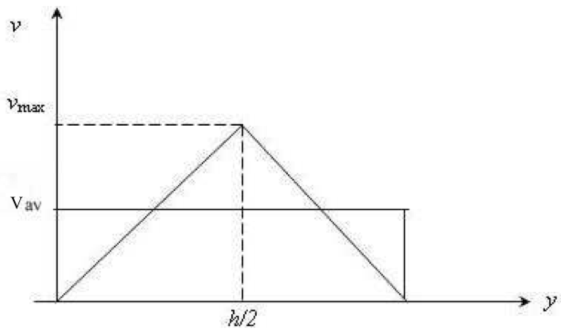

Figure 1.

For description of lane change dynamics we approximately assume that the trajectory of maneuver of lane's changing is assembled of two parts of identical circles (Fig.2). On the Fig.2 the bold line is a trajectory of lane's changing and it begins in the middle of the lane on which the car is going and ends in the middle of next lane.

The centripetal force which acts on the car is: $F = \frac{mV^2}{R_{turn}}$, where R_{turn} is the radius of the turn, V - is the velocity of the car, directed by the tangent to the car's trajectory. Let F be the maximum possible flank force under which car is drivable (not skidding), $F \leq F_*$. Then we may calculate the radius of the turn $R_{turn} = \frac{m}{F_* V^2}$. $v_{av}^{max} = V \sin\theta$. From the similar triangles ΔABC and ΔOAD (Fig.2) derive that $AD = DB$, labeling $AD = b$ will get:

$$\frac{b}{R} = \frac{h/4}{b} \; or \; b^2 = \frac{Rh}{4},$$

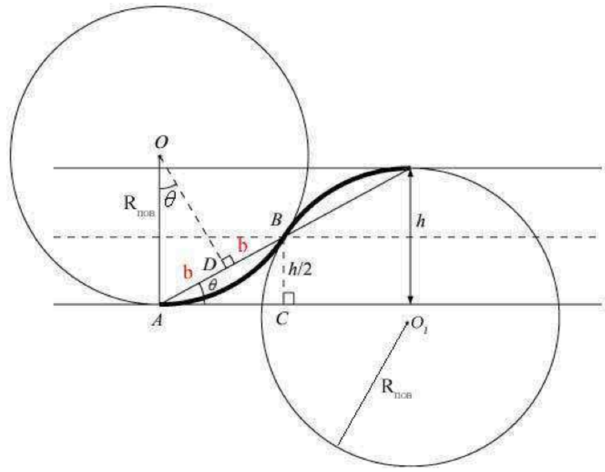

Figure 2.

$$\sin\theta = \frac{h}{4b} = \frac{b}{R} = \frac{\sqrt{Rh}}{2R} = \frac{1}{2}\sqrt{\frac{h}{R}}.$$

Then $v_{av}^{max} = V\sin\theta = \frac{1}{2}V\sqrt{\frac{hF_*}{mV^2}} = \frac{1}{2}\sqrt{\frac{hF_*}{m}}$, that is the average speed for is v_{max}. We obtained above $A^2 = 4v_{av}^2$, so the expression for A is $A^2 = \frac{hF_*}{m}$.

So we get that parameter A^2 depends on the force F_* , assignable to the lane width, mass of the car and the traction of tires. The equations (1),(2),(3) provide a system describing traffic flows on multilane roads:

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{k^2}{\rho}\frac{\partial\rho}{\partial x},$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{A^2}{\rho}\frac{\partial\rho}{\partial y}.$$

2 A problem of a multilane road

The considered problem is the nature of car's behavior on a multilane rectangular road. The traffic flow is divided into two parts on the left boundary by y_{fix} . For $0 \le y \le y_{fix}$ the flow has the bigger value then for $y_{fix} < y \le H$. Boundary conditions for $x = 0$:

$$\left.\begin{array}{l} v = 0, u = u_1, \rho = \rho_1, u_1\rho_1 = q_1 \\ v = 0, u = u_2, \rho = \rho_2, u_2\rho_2 = q_2 \end{array}\right\} u > k$$

or

$$q = \rho u = \begin{cases} q_1, \ y > y_{fix} > 0 \\ q_2, \ y \le y_{fix} \le H \end{cases}$$

Boundary conditions for $x = L$:

$$\frac{\partial\rho u}{\partial x} = 0; \quad \frac{\partial\rho v}{\partial x} = 0.$$

The numerical calculations of the problems were processed using the AUSM method [21]. The mesh had $N_x = 201$ and $N_y = 21$ grid nodes.
$T = 240s$ – calculation time;

$L = 500m$ – the length of the domain;

$H = 12m$ – the width of the domain;

$\rho_0 = 0,01$ – the density of traffic flow on the bounder $x = 0$;

$u_0 = 7.0m/s$ – the x-velocity of traffic flow on the bounder $x = 0$;

$v_0 = 0.0m/s$ – the y-velocity of traffic flow on the bounder $x = 0$;

$U_{max} = 20m/s$ – maximal permitted road velocity;

$k = 9m/s$ – the small disturbances propagation velocity ("sound velocity") on x axis;

$A = 3m/s$ – the analogy of k coefficient on y axis;

$y_{fix} = 3.75m$ – the devisor of the road on 2 parts;

q_0 – a flow on the left boundary for $0 \le y \le y_{fix}$;

q_1 – a flow on the left boundary for $y_{fix} < y \le H$;

The obtain results are pictures for components of velocity and for density for each moment of time.

3 Calculation results

Results for density

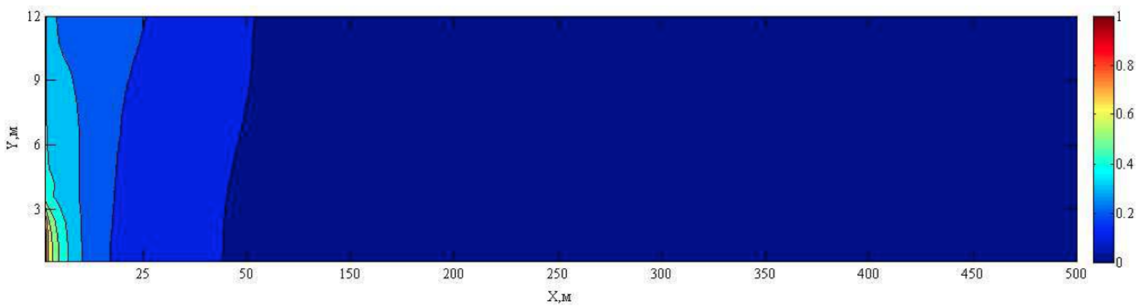

Figure 3: Density, T=5

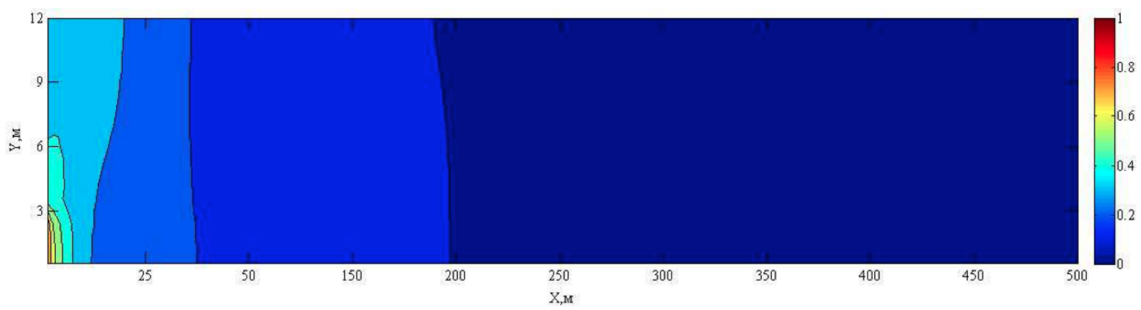

Figure 4: Density, T=10

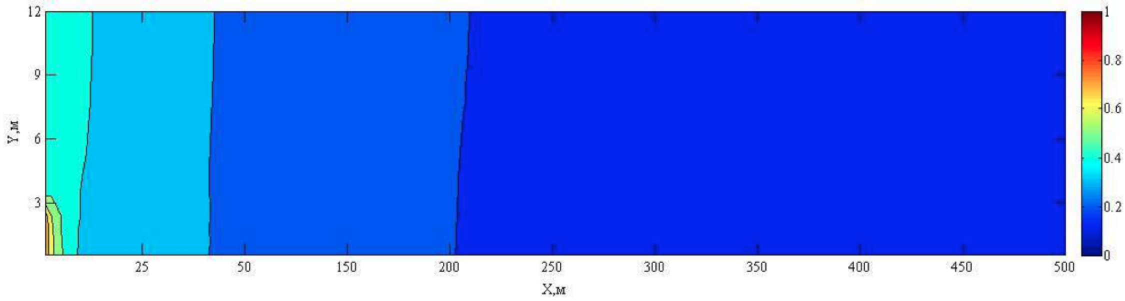

Figure 5: Density, T=30

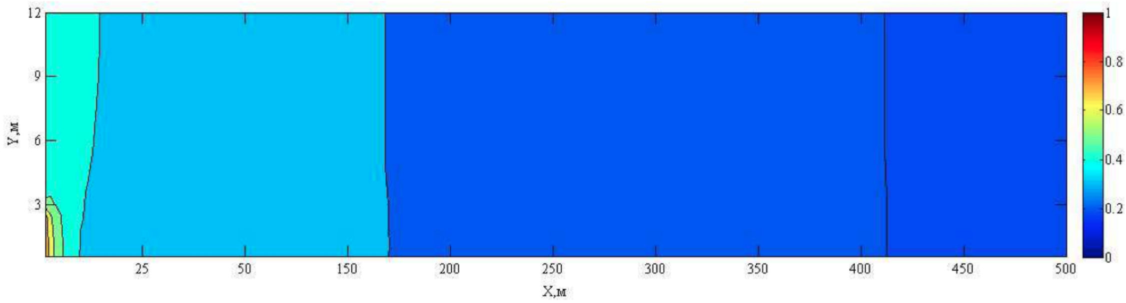

Figure 6: Density, T=60

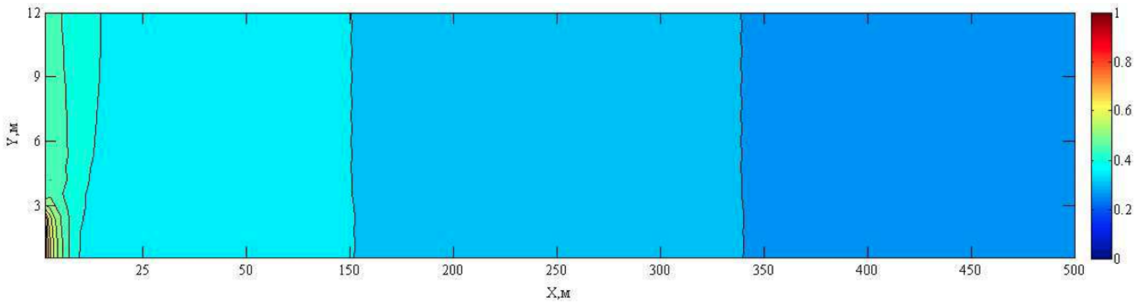

Figure 7: Density, T=120

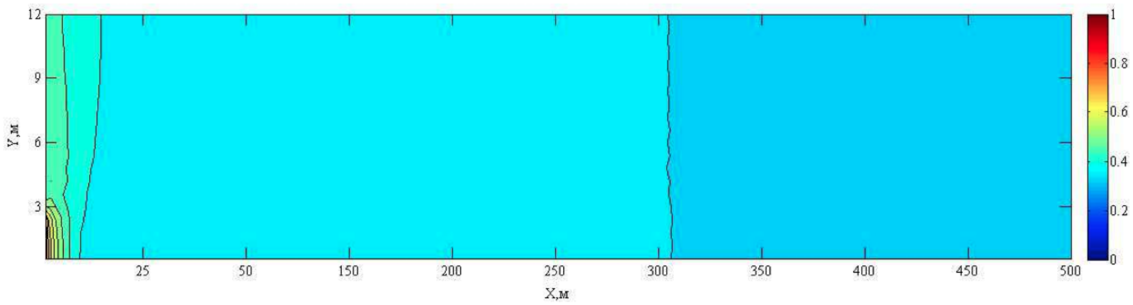

Figure 8: Density, T=240

Results for x-velocity

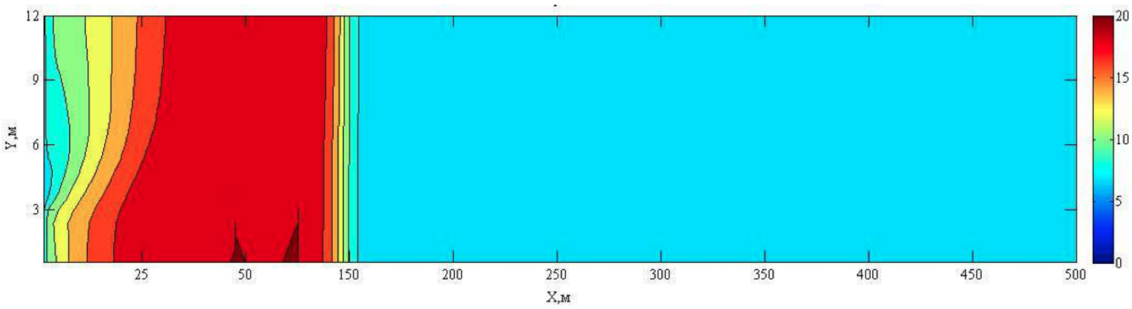

Figure 9: x-velocity, T=5

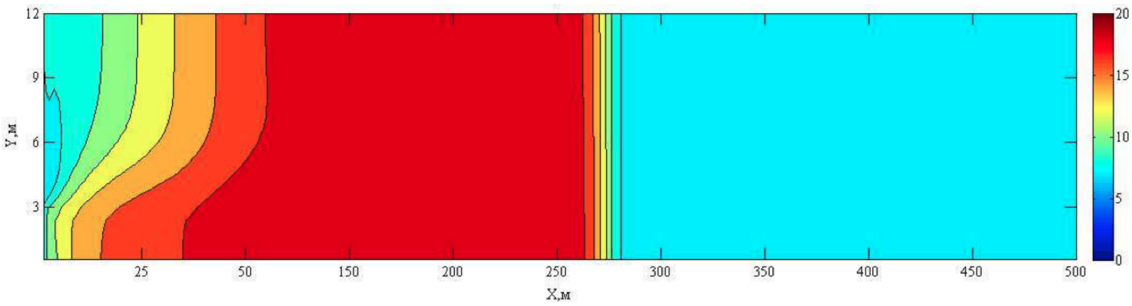

Figure 10: x-velocity, T=10

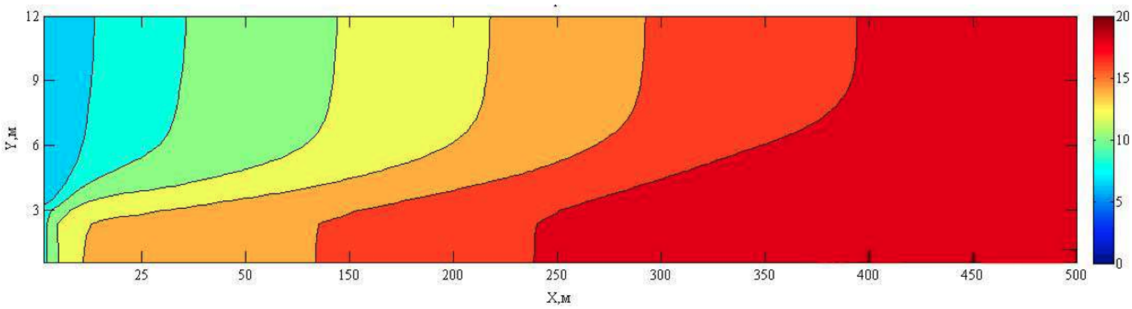

Figure 11: x-velocity, T=30

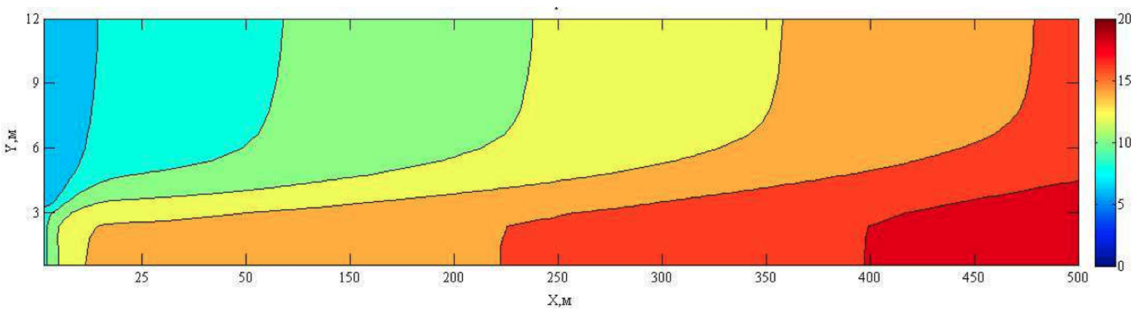

Figure 12: x-velocity, T=60

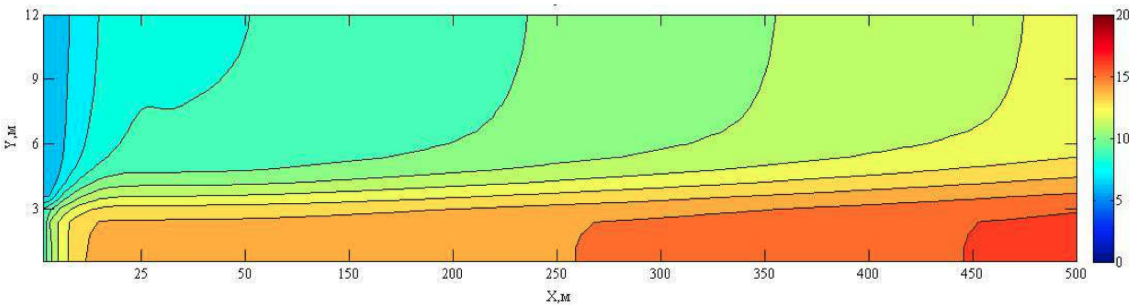

Figure 13: x-velocity, T=120

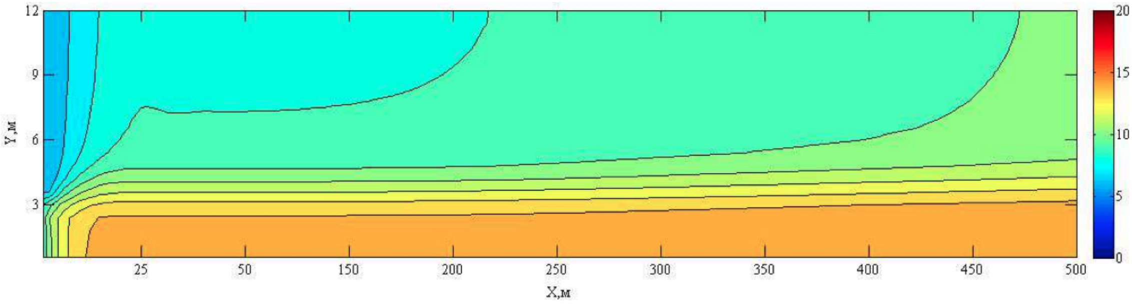

Figure 14: x-velocity, T=240

Results for y-velocity

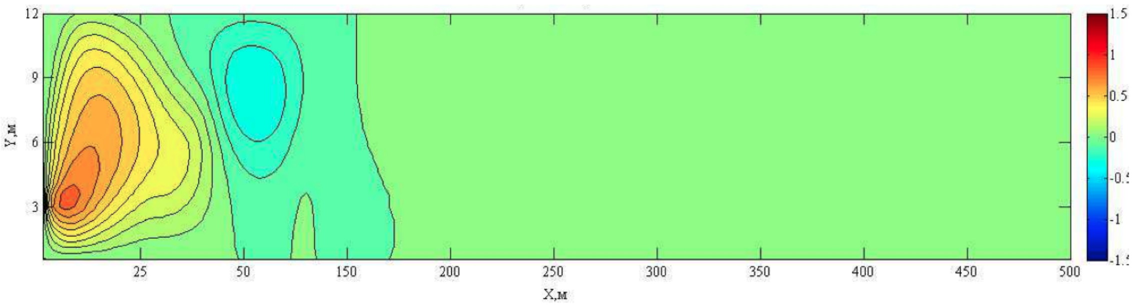

Figure 15: y-velocity, T=5

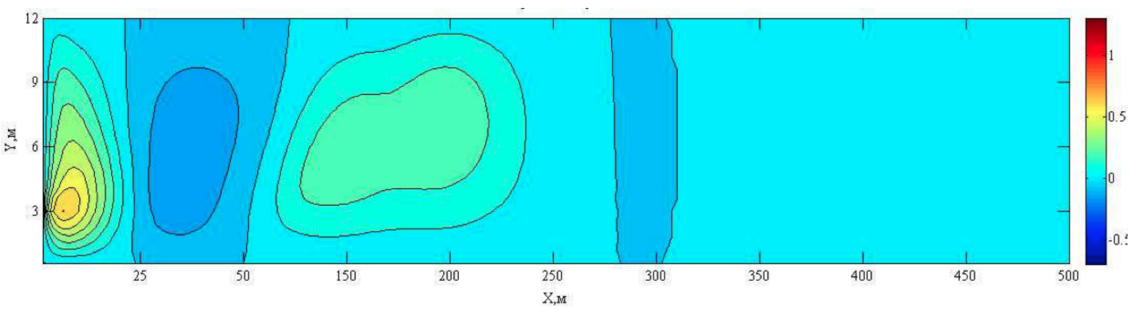

Figure 16: y-velocity, T=10

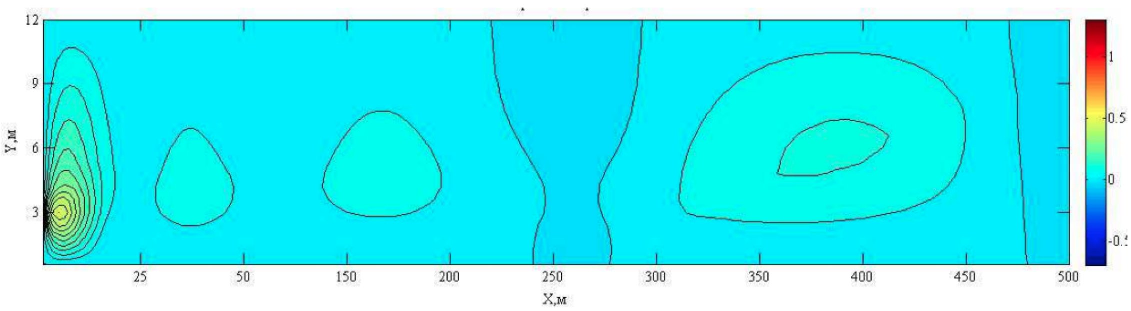

Figure 17: y-velocity, T=30

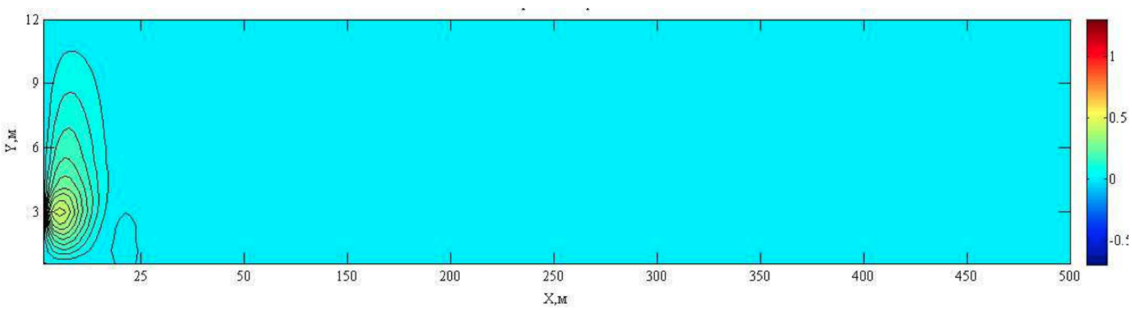

Figure 18: y-velocity, T=60

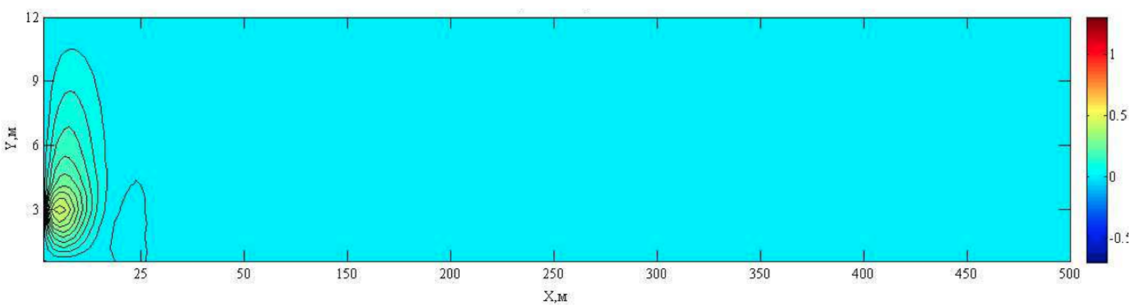

Figure 19: y-velocity, T=120

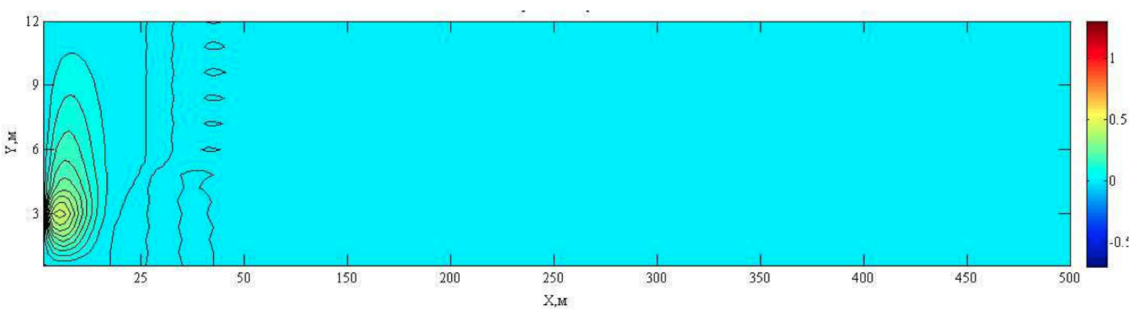

Figure 20: y-velocity, T=240

4 Conclusions

The mathematical model for traffic flows simulations in multi-lane roads is developed. A model problem for traffic evolution in multi-lane road with non-uniform flux in different lanes was regarded. The results show that on entering the road segment high orthogonal fluxes occur in the direction of less dense lanes, which brings to slowing down flow velocity in that lanes and increasing density. The equation of motion at brings to formation of inverse flow from those lanes back. In time flow evolution brings to a more smooth transition zone, however, on coming in contact flow zones of different fluxes always cause formation of an orthogonal flow in the direction of less density lane in the nearest vicinity followed by an inverse flow at some distance.

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