Kinematic fluid dynamos examined by toroidal-poloidal decompositions—An example of combining continuum mechanics and electrodynamic field theory

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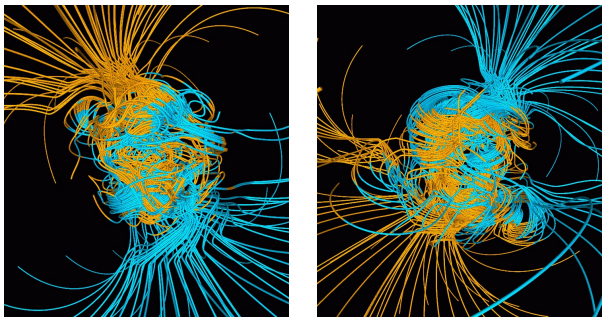
Abstract

This paper presents a rational approach to the problem of kinematic dynamos in spherical cavities and the related induction equation. An operator notation using poloidal-toroidal decompositions is developed in order to analyze the governing differential equation. A solution of the induction equation is sought by using series expansions. Applying projection methods leads to a fully analytic system of differential equations in one coordinate, *i.e.*, the radius, for the series coefficients.

1 Introduction

Larmor proposed in 1919 that the magnetic field of large astronomical objects, such as the Earth or the Sun, is generated by fluid flow in the interior. This is due to self-excitation processes caused by coupling of fluid- and electromagnetic fields, *cf.*, (Merrill, 1998). The coupling is described by additional terms in Maxwell's equations and the equation of linear momentum. The transfer of kinetic energy to electromagnetic energy can lead to an amplification of the magnetic field. This process is a.k.a. dynamo action.



(a) Before dipole reversal. (b) After dipole reversal.

Figure 1: Simulation of a geodynamo in reversal by Glatzmaier, from (Glatzmaier, 2015). Magnetic field lines are shown. Blue/yellow colors indicate the field is directed inward/outward.

By paleomagnetic investigations, it is known that the Earth's magnetic field is reversing, *i.e.*, changing its polarity, *cf.*, (Merrill, 1998). In order to understand and predict the magnetic field's reversal, the so-called geodynamo is used as a modell. A geodynamo simulation by Glatzmaier illustrating the field reversal is shown in Fig. 1. The origin of the Earth's magnetic field and its reversal is a topic of past and current research. Some aspects of it will be presented in this paper.

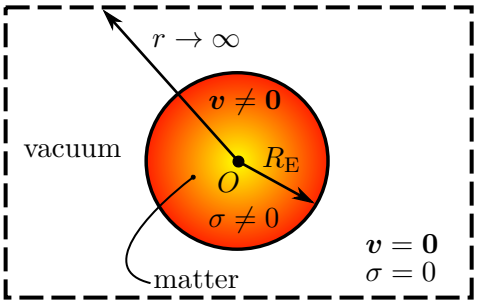


Figure 2: Sketch of the considered problem.

2 Induction equation—derivation and discussion

The considered model problem is sketched in Fig. 2. It consists of a spherical cavity filled with a conducting fluid surrounded by vacuum. The governing field equations of electrodynamics of the problem are given by MAXWELL's equations. Suppose the fluid is neither polarizable, *i.e.*, $\boldsymbol{P} = \boldsymbol{0}$, nor magnetizable, *i.e.*, $\boldsymbol{M} = \boldsymbol{0}$. In an inertial frame of reference the field equations and related jump conditions read:

$$\nabla \cdot \boldsymbol{B} = 0\,, \qquad [\![\boldsymbol{B}]\!] \cdot \boldsymbol{e} = 0\,, \tag{1a}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = \boldsymbol{0}\,, \qquad [\![\boldsymbol{E}]\!] \times \boldsymbol{e} + [\![\boldsymbol{B}]\!]\, w_{\perp} = \boldsymbol{0} \tag{1b}$$

$$\varepsilon_0 \nabla \cdot \boldsymbol{E} = 0\,, \qquad \varepsilon_0\, [\![\boldsymbol{E}]\!] \cdot \boldsymbol{e} = \underset{A}{q^{\mathrm{f}}}\,, \tag{1c}$$

$$\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} - \frac{1}{\mu_0} \nabla \times \boldsymbol{B} = -\boldsymbol{J}^{\mathrm{f}}\,, \qquad -\frac{1}{\mu_0}\, [\![\boldsymbol{B}]\!] \times \boldsymbol{e} + \varepsilon_0\, [\![\boldsymbol{E}]\!]\, w_{\perp} = \underset{L}{\boldsymbol{J}^{\mathrm{f}}}\,. \tag{1d}$$

Note that the MAXWELL-LORENTZ aether relations have already been inserted in the system above. Moreover, OHM's law for a *moving* conductor has been used as a constitutive law. Hence, for the diffusive current $\boldsymbol{j}^{\mathrm{f}}$ we have, *cf.*, (Jackson, 1998):

$$\boldsymbol{j}^{\mathrm{f}} = \sigma\,(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})\,. \tag{2}$$

After performing a scaling analysis and by assuming $\|\boldsymbol{v}\| \ll c$ it may be shown that, *cf.*, (Kovetz, 2000),

$$\mathcal{O}\left(\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right) \ll \mathcal{O}\left(\frac{1}{\mu_0} \nabla \times \boldsymbol{B}\right)\,, \quad \mathcal{O}\left(q^{\mathrm{f}} \boldsymbol{v}\right) \ll \mathcal{O}\left(\frac{1}{\mu_0} \nabla \times \boldsymbol{B}\right)\,, \tag{3}$$

where the LANDAU symbol $\mathcal{O}$ indicates the order of magnitude. This shows that the convective current is negligibly small w.r.t. the left-hand right of (1d). Hence, the total current $\boldsymbol{J}^{\mathrm{f}}$ is given by the diffusive OHMic current $\boldsymbol{j}^{\mathrm{f}}$ only. Application of Eqns. (2) and (3) to Eqn. (1d) leads to:

$$\frac{1}{\mu_0} \nabla \times \boldsymbol{B} = \sigma\,(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \Leftrightarrow \boldsymbol{E} = \frac{1}{\sigma \mu_0} \nabla \times \boldsymbol{B} - \boldsymbol{v} \times \boldsymbol{B}\,. \tag{4}$$

This analysis is part of the so-called magnetohydrodynamic approximation. If, in addition, we assume that the conductivity σ is constant, which corresponds to a homogeneous conductor, the induction equation is obtained by expressing the curl of the electric field $\boldsymbol{E}$ in Eqn. (1b) through Eqn. (4). Hence, the induction reads, *cf.*, (Kovetz, 2000):

$$\frac{\partial \boldsymbol{B}}{\partial t} + \frac{1}{\sigma \mu_0} \nabla \times \nabla \times \boldsymbol{B} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B})\,. \tag{5}$$

This equation can be interpreted as an evolution equation for the magnetic field $\boldsymbol{B}$. The velocity $\boldsymbol{v}$ is present in the equation above. Consequently, the velocity influences the evolution of the magnetic flux density. From mechanics it is known that the evolution of the velocity is described by the balance of linear momentum. This shows that the equation above can, in general, only be solved in combination with the balance of linear momentum. In this paper we consider so-called kinematic dynamos, which indicates that the velocity field is *prescribed*. Therefore, the balance equations of mechanics are not considered any further.

In what follows we consider the non-dimensional form of the induction equation. It reads:[1]

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \nabla \times \boldsymbol{B} = Re_{\text{mag.}} \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \ , \tag{6}$$

in which $Re_{\text{mag.}}$ is the so-called magnetic Reynolds number. The magnetic Reynolds number relates electromagnetic diffusion to fluid-dynamic transport. For a static velocity field the induction equation can be formulated as a generalized eigenvalue problem. A solution ansatz in exponential form in time leads to:

$$\lambda \boldsymbol{B} + \nabla \times \nabla \times \boldsymbol{B} = Re_{\text{mag.}} \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \ , \tag{7}$$

in which $\boldsymbol{B}$ is a function of space variables only and λ is the eigenvalue. Eigenvalues with positive/negative real part indicate exponential growth/decay of the related eigenmode $\boldsymbol{B}$. Static solutions both for the magnetic and for the velocity field correspond to $\lambda = 0$. An exponential growth indicates an amplification of a so-called seed field by a given fluid flow. This would demonstrate that dynamo action may be possible.

In order to solve the eigenvalue problem, boundary conditions for the magnetic field at the boundary between the material interior and vacuous exterior, *i.e.*, at $r = 1$, are necessary. It can be shown that for the case of a fixed boundary, $w_\perp = 0$, a continuous transition of the magnetic field results. Hence,

$$[\![\boldsymbol{B}]\!] = \boldsymbol{0} \, . \tag{8}$$

In the presented model the cavity is the only source. Therefore, in the exterior, which is under vacuum, electromagnetic waves travel from the boundary on to infinity. Because of the fact that for the source, *i.e.*, the material interior, the magnetohydrodynamic approximation applies, it also holds for the exterior. In doing so the wave character of the fields is broken. Applying the magnetohydrodynamic approximation to Maxwell's system in vacuum gives:

$$\nabla \cdot \boldsymbol{B} = 0 \, , \quad \nabla \times \boldsymbol{B} = \boldsymbol{0} \, . \tag{9}$$

The solution of these equations is given by a scalar potential ψ describing the spatial behavior of the magnetic field. The time dependence is given by the source, *i.e.*, the material interior. The potential is obtained through Laplace's equation and in spherical coordinates its gradient, which is nothing else but the magnetic field, is given by the series:

$$\boldsymbol{B}(t, r, \theta, \varphi) = \sum_{n=0}^{\infty} r^{-(n+2)} \sum_{m=-n}^{n} c_n^m(t) \left(-(n+1) Y_n^m(\theta, \varphi) \boldsymbol{e}_r + \nabla_{\theta,\varphi} Y_n^m \right) \, . \tag{10}$$

In conclusion, the exterior solution is determined up to time-dependent series coefficients c_n^m, which can be determined by the boundary condition discussed above.

[1] The choice of reference quantities, *i.e.*, reference length, velocity, and magnetic flux density, inherently determines the form of the non-dimensional induction equation. Other reference quantities may lead to $Re_{\text{mag.}}^{-1}$ in front of the double-curl term.

3 Spherical harmonics and poloidal-toroidal decompositions

3.1 Properties of spherical harmonics

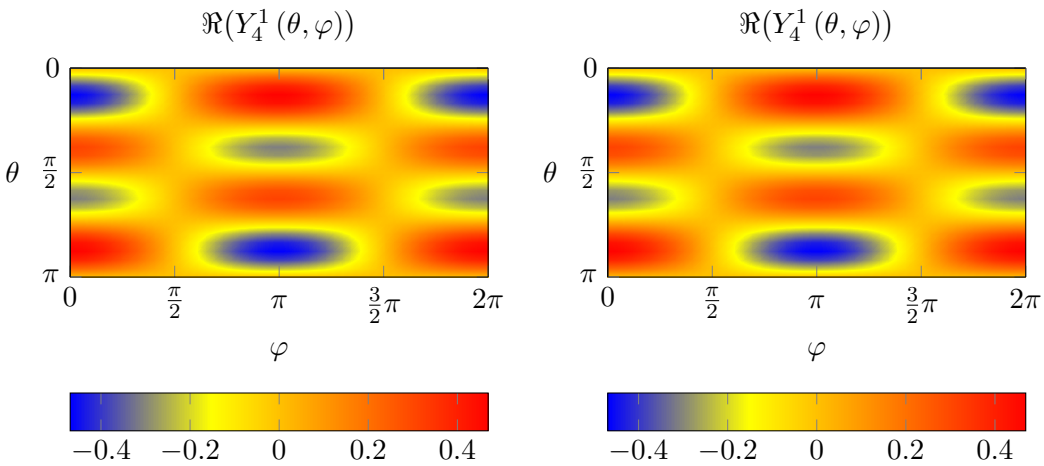
We introduce the scalar or inner product of two arbitrary functions f and g on a spherical surface of radius r as:

$$\langle f, g\rangle_{\partial B_r} := \oiint\limits_{\partial B_r} f(\boldsymbol{x})\,\overline{g}(\boldsymbol{x})\,\frac{1}{\|\boldsymbol{x}\|^2}\,\mathrm{d}A = \int\limits_0^{2\pi}\int\limits_0^{\pi} f(r,\theta,\varphi)\,\overline{g}(r,\theta,\varphi)\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi\,. \tag{11}$$

In the context of the HELMHOLTZ equation spherical harmonics Y_n^m occur as the angular part of the solution. In this paper complex spherical harmonics are used and defined by:

$$Y_n^m(\theta,\varphi) := N_n^m \exp(\mathrm{i}m\varphi) P_n^{|m|}(\cos(\theta))\ . \tag{12}$$

The symbol $P_n^{|m|}$ represents the associated LEGENDRE polynomials. As an example two spherical harmonics are plotted in Fig. 3. If the normalization factor N_n^m is chosen



(a) Real part for $m = 2$, $n = 2$.

(b) Real part for $m = 1$, $n = 4$.

Figure 3: Plot of spherical harmonics Y_n^m.

appropriately, the spherical harmonics constitute an orthonormal system with respect to the scalar product defined above. Accordingly, the following relation holds:

$$\left\langle Y_n^m, Y_p^o\right\rangle_{\partial B_r} = \delta^{mo}\delta_{np}\,. \tag{13}$$

This orthogonality relation allows for a series expansion of scalar functions in spherical harmonics. In general, the series coefficients may be radially dependent. By further inspection of the HELMHOLTZ equation it can be shown that the spherical harmonics represent the eigenfunctions of the LAPLACE operator on a spherical surface, *viz.*:

$$\Delta_{\theta,\varphi} Y_n^m = -n(n+1)\,Y_n^m(\theta,\varphi)\ . \tag{14}$$

3.2 Poloidal-toroidal decomposition

It can be shown that every solenoidal, *i.e.*, divergence-free, vector field $\boldsymbol{F}$ can be further decomposed in a toroidal vector field $\boldsymbol{M}$[2] and a poloidal vector field $\boldsymbol{N}$, *cf.*, (Stratton,

[2]For reasons of notation we use $\boldsymbol{M}$ for toroidal field. This symbol does not represent the magnetization.

2007). Thus, we may write for arbitrary $\boldsymbol{F}$ obeying $\nabla \cdot \boldsymbol{F} = 0$:

$$\boldsymbol{F} = \boldsymbol{M} + \boldsymbol{N} \,. \tag{15}$$

In spherical coordinates, the summands are given by a series of spherical harmonics, *i.e.*:

$$\boldsymbol{M} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \boldsymbol{m}_n^m \,, \quad \boldsymbol{N} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \boldsymbol{n}_n^m \,. \tag{16}$$

The so-called toroidal and poloidal modes are given by:

$$\boldsymbol{m}_n^m := \nabla \times [r m_n^m(r)\, Y_n^m(\theta,\varphi)\, \boldsymbol{e}_r] = m_n^m(r)\, \boldsymbol{\mathcal{D}}_{\theta,\varphi}[Y_n^m] \,, \tag{17a}$$

$$\begin{aligned} \boldsymbol{n}_n^m :=\ & \nabla \times \nabla \times [r n_n^m(r)\, Y_n^m(\theta,\varphi)\, \boldsymbol{e}_r] \\ =\ & \frac{n(n+1)}{r} n_n^m(r)\, Y_n^m(\theta,\varphi)\, \boldsymbol{e}_r + D_r^{(1)}[n_n^m]\, \nabla_{\theta,\varphi} Y_n^m \,. \end{aligned} \tag{17b}$$

The symbol $\nabla_{\theta,\varphi}$ represents the gradient on a spherical surface.[3] Moreover, the operators introduced in the equations above are defined as follows:

$$\boldsymbol{\mathcal{D}}_{\theta,\varphi}[f] := \frac{1}{\sin(\theta)} \frac{\partial f}{\partial \varphi} \boldsymbol{e}_\theta - \frac{\partial f}{\partial \theta} \boldsymbol{e}_\varphi \,, \quad D_r^{(1)}[f] := \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} [r f(r)] \,. \tag{18}$$

The special choices for the toroidal and poloidal fields are motivated by the problem of the vectorial Helmholtz equation, *cf.*, (Glane, 2015). As an example, a toroidal vector field is shown in Fig. 4b.

3.3 Orthogonality of toroidal and poloidal vector fields

The orthogonality of toroidal and poloidal vector fields is analyzed below with respect to the inner product of two arbitrary vector fields $\boldsymbol{f}$ and $\boldsymbol{g}$:

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle_{\partial B_r} := \oiint_{\partial B_r} \boldsymbol{f}(\boldsymbol{x}) \cdot \overline{\boldsymbol{g}}(\boldsymbol{x}) \frac{1}{\|\boldsymbol{x}\|^2} \,\mathrm{d}A = \frac{1}{r^2} \oiint_{\partial B_r} \boldsymbol{f}(\boldsymbol{x}) \cdot \overline{\boldsymbol{g}}(\boldsymbol{x}) \,\mathrm{d}A \,. \tag{19}$$

We consider the operators $\nabla_{\theta,\varphi}$ and $\boldsymbol{\mathcal{D}}_{\theta,\varphi}$. By using the product rule for the divergence, we may write:

$$\nabla_{\theta,\varphi} f \cdot \boldsymbol{\mathcal{D}}_{\theta,\varphi}[g] = \nabla_{\theta,\varphi} \cdot (f \boldsymbol{\mathcal{D}}_{\theta,\varphi}[g]) - f \nabla_{\theta,\varphi} \cdot \boldsymbol{\mathcal{D}}_{\theta,\varphi}[g] \,. \tag{20}$$

After a simple calculation we conclude that:

$$\nabla_{\theta,\varphi} \cdot \boldsymbol{\mathcal{D}}_{\theta,\varphi}[g] = \frac{1}{\sin(\theta)} \left(\frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{1}{\sin(\theta)} \frac{\partial g}{\partial \varphi} \right) - \frac{\partial^2 g}{\partial \varphi \partial \theta} \right) = 0 \,. \tag{21}$$

Hence, the scalar product of the considered operators is:

$$\langle \nabla_{\theta,\varphi} f, \boldsymbol{\mathcal{D}}_{\theta,\varphi}[g] \rangle_{\partial B_r} = \frac{1}{r^2} \oiint_{\partial B_r} \nabla_{\theta,\varphi} \cdot (f \boldsymbol{\mathcal{D}}_{\theta,\varphi}[\overline{g}]) \,\mathrm{d}A = 0 \,. \tag{22}$$

[3] The gradient of a function f on a spherical surface is given by:

$$\nabla_{\theta,\varphi} f = \frac{\partial f}{\partial \theta} \boldsymbol{e}_\theta + \frac{1}{\sin(\theta)} \frac{\partial f}{\partial \varphi} \boldsymbol{e}_\varphi \,.$$

In the last step GAUSS' integral theorem was applied to a closed surface, see (Elsasser, 1946). Then the boundary integral vanishes because the field of integration is an empty set. By using the product rule and GAUSS' integral theorem, we may write:

$$\left\langle \nabla_{\theta,\varphi} Y_n^m, \nabla_{\theta,\varphi} Y_p^o \right\rangle_{\partial B_r} = \frac{1}{r^2} \oiint_{\partial B_r} \left(\nabla_{\theta,\varphi} \cdot \left(Y_n^m \nabla_{\theta,\varphi} \overline{Y_p^o} \right) - Y_n^m \Delta_{\theta,\varphi} \overline{Y_p^o} \right) \mathrm{d}A$$
$$= p\,(p+1) \left\langle Y_n^m, Y_p^o \right\rangle_{\partial B_r} . \quad (23)$$

This shows that the pairwise orthogonality of the operator $\nabla_{\theta,\varphi}$ is reduced to the orthogonality of the spherical harmonics. The same holds for the operator $\boldsymbol{\mathcal{D}}_{\theta,\varphi}$. Because of the orthogonality of the constitutive operators and spherical harmonics the toroidal and poloidal modes constitute a fully orthogonal set. Hence, in spherical coordinates every solenoidal vector field may be expanded in these orthogonal components, *cf.*, (Stratton, 2007).

4 Projection method

In this section we apply the so-called projection method to the induction equation, *i.e.*, the eigenvalue problem (7). This procedure is also presented in (Bullard, 1954). We suppose that the material is compressible. Hence, $\nabla \cdot \boldsymbol{v} = 0$ and the velocity may be expanded in toroidal and poloidal vector fields as well, *cf.*, (Bullard, 1954; Elsasser, 1946). We write the magnetic and the velocity field such that:

$$\boldsymbol{B} = \sum_{j=0}^{\infty} \sum_{i=-j}^{j} \left(\boldsymbol{m}_j^i + \boldsymbol{n}_j^i \right) , \quad \boldsymbol{v} = \sum_{l=0}^{\infty} \sum_{k=-l}^{l} \left(\boldsymbol{o}_l^k + \boldsymbol{p}_l^k \right) , \quad (24)$$

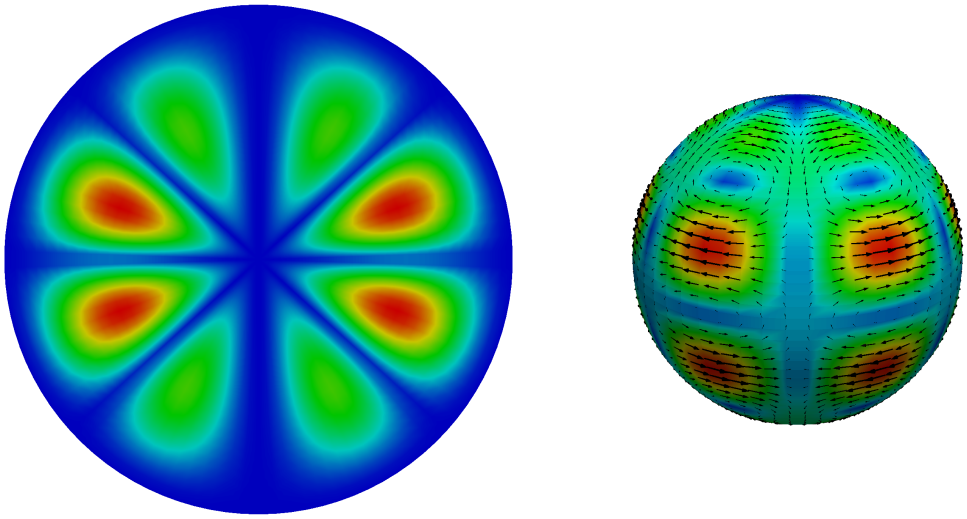
where $\boldsymbol{o}_l^k / \boldsymbol{p}_l^k$ represent the toroidal/poloidal modes similar to Eqn. (17). As an example, Fig. 4 shows a prescribed toroidal velocity field. We define the so-called toroidal/poloidal filters $\tilde{\boldsymbol{m}}_n^m$, $\tilde{\boldsymbol{n}}_n^m$ as:

$$\tilde{\boldsymbol{m}}_n^m := \boldsymbol{\mathcal{D}}_{\theta,\varphi}[Y_n^m] , \quad \tilde{\boldsymbol{n}}_n^m := \frac{n\,(n+1)}{r} Y_n^m\,(\theta,\varphi)\, \boldsymbol{e}_r . \quad (25)$$

Due to the orthogonality of toroidal and poloidal fields or rather their constitutive operators, the scalar product of the magnetic field and the toroidal/poloidal filters gives:

$$\langle \boldsymbol{B}, \tilde{\boldsymbol{m}}_n^m \rangle_{\partial B_r} = j\,(j+1)\, m_j^i\,(r)\, \delta^{im} \delta_{jn} , \quad (26a)$$
$$\langle \boldsymbol{B}, \tilde{\boldsymbol{n}}_n^m \rangle_{\partial B_r} = \frac{j^2\,(j+1)^2}{r^2} n_j^i\,(r)\, \delta^{im} \delta_{jn} . \quad (26b)$$

This operation is called the projection onto a toroidal or rather poloidal mode. The KRONECKER-δ-property may be interpreted as a fully decoupled term. Note that by virtue of construction the curl of a toroidal field is a poloidal field and vice versa. It follows that the double curl of either a toroidal or poloidal field is again a toroidal or poloidal field.



(a) Magnitude in the x-z-plane.

(b) Toroidal vector field on a spherical surface of radius $1/\sqrt{2}$. Colors indicate magnitudes.

Figure 4: Toroidal velocity field with radial function $r^2\left(1-r^2\right)^2$. Vortices indicate a complex 3D fluid flow inside the spherical cavity.

The projection of the double curl term in equation (7) gives:[4]

$$\langle \nabla \times \nabla \times \boldsymbol{B}, \tilde{\boldsymbol{m}}_n^m \rangle_{\partial B_r} = j\left(j+1\right)\left(j\left(j+1\right)\frac{m_j^i\left(r\right)}{r^2} - D_r^{(2)}\left[m_j^i\right]\right)\delta^{im}\delta_{jn}\,, \tag{27a}$$

$$\langle \nabla \times \nabla \times \boldsymbol{B}, \tilde{\boldsymbol{n}}_n^m \rangle_{\partial B_r} = \frac{j^2\left(j+1\right)^2}{r^2}\left(D_r^{(2)}\left[n_j^i\right] - j\left(j+1\right)\frac{n_j^i\left(r\right)}{r^2}\right)\delta^{im}\delta_{jn}\,. \tag{27b}$$

Since the Kronecker-δ-property applies again, the double terms do not cause a coupling of the modes.

4.1 Velocity dependent terms

The remaining term in Eqn. (7) is the velocity dependent term. The more complicated nature of this term necessitates a stepwise procedure using operator notation:

1. Exploit linearity of the cross product, the curl, as well as the scalar product;
2. Calculate the cross products (4 terms);
3. Calculate the curl of the cross products (4 terms);
4. Apply the projection to these 4 terms (8 terms).

The resulting terms contain non-linear expressions in terms of spherical harmonics and the operators introduced above. As an example we have:

$$\nabla \times \left(\boldsymbol{o}_l^k \times \boldsymbol{m}_j^i\right) = \frac{o_l^k\left(r\right)m_j^i\left(r\right)}{r}\boldsymbol{\mathcal{D}}_{\theta,\varphi}\left[\mathcal{D}_{\theta,\varphi}\left[Y_l^k, Y_j^i\right]\right]\,, \tag{28a}$$

[4]The operator $D_r^{(2)}$ is defined through the recurrence relation:

$$D_r^{(n)}[f] = D_r^{(1)}\left[D_r^{(n-1)}[f]\right]\,, \quad n \in \mathbb{N}\,.$$

where the anti-symmetric operator $\mathcal{D}_{\theta,\varphi}$ is defined as:

$$\mathcal{D}_{\theta,\varphi}[f,g] := \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \varphi} - \frac{\partial f}{\partial \varphi}\frac{\partial g}{\partial \theta}\,. \tag{28b}$$

These non-linearities disturb the KRONECKER-δ-property and lead to a coupling of modes. After extensive calculations and application of additional theorems concerning the operators and so-called triple product integrals it can be shown that the coupling structure is governed by the so-called ADAMS-GAUNT- and ELSASSER-integrals K_{jln}^{ikm} and L_{jln}^{ikm}. These integrals are defined as:

$$K_{jln}^{ikm} := \oiint_{\partial B_r} Y_j^i Y_l^k \overline{Y_n^m} \frac{1}{r^2}\,\mathrm{d}A\,, \tag{29a}$$

$$L_{jln}^{ikm} := \oiint_{\partial B_r} \mathcal{D}_{\theta,\varphi}\left[Y_j^i, Y_l^k\right] \overline{Y_n^m} \frac{1}{r^2}\,\mathrm{d}A\,. \tag{29b}$$

In the following, we demonstrate exemplary how the ELSASSER-integral occurs. It may be shown that for arbitrary functions f and g:

$$\boldsymbol{\mathcal{D}}_{\theta,\varphi}[f]\cdot\boldsymbol{\mathcal{D}}_{\theta,\varphi}[g] = \nabla_{\theta,\varphi} f \cdot \nabla_{\theta,\varphi} g\,. \tag{30a}$$

By applying GAUSS' integral theorem analogously to Eqn. (23) and using Eqn. (14), we obtain:

$$\begin{aligned}\oiint_{\partial B_r} \nabla_{\theta,\varphi}\left(\mathcal{D}_{\theta,\varphi}\left[Y_l^k, Y_j^i\right]\right)\cdot\nabla_{\theta,\varphi}\overline{Y_n^m}\frac{1}{r^2}\,\mathrm{d}A &= \oiint_{\partial B_r} \mathcal{D}_{\theta,\varphi}\left[Y_l^k, Y_j^i\right]\Delta_{\theta,\varphi}\overline{Y_n^m}\frac{1}{r^2}\,\mathrm{d}A \\ &= -n\,(n+1)\oiint_{\partial B_r} \mathcal{D}_{\theta,\varphi}\left[Y_l^k, Y_j^i\right]\overline{Y_n^m}\frac{1}{r^2}\,\mathrm{d}A\,.\end{aligned} \tag{30b}$$

Hence:

$$\begin{aligned}&\left\langle \boldsymbol{\mathcal{D}}_{\theta,\varphi}\left[\mathcal{D}_{\theta,\varphi}\left[Y_l^k, Y_j^i\right]\right], \boldsymbol{\mathcal{D}}_{\theta,\varphi}[Y_n^m]\right\rangle_{\partial B_r} \\ &\qquad = -n\,(n+1)\oiint_{\partial B_r} \mathcal{D}_{\theta,\varphi}\left[Y_l^k, Y_j^i\right]\overline{Y_n^m}\frac{1}{r^2}\,\mathrm{d}A = n\,(n+1)\,L_{jln}^{ikm}\,.\end{aligned} \tag{30c}$$

This provides a brief insight into what manipulations are necessary to tackle the velocity dependent terms. The operator notation presented in this paper turned out to be extremely beneficial in context with the projection method. As an example for the components of the velocity dependent terms, we present two expressions:

$$\left\langle \nabla\times\left(\boldsymbol{o}_l^k\times\boldsymbol{m}_j^i\right), \tilde{\boldsymbol{m}}_n^m\right\rangle_{\partial B_r} = -L_{jln}^{ikm}\,n\,(n+1)\,\frac{o_l^k(r)\,m_j^i(r)}{r}\,, \tag{31a}$$

$$\left\langle \nabla\times\left(\boldsymbol{p}_l^k\times\boldsymbol{m}_j^i\right), \tilde{\boldsymbol{n}}_n^m\right\rangle_{\partial B_r} = -L_{jln}^{ikm}\,l\,(l+1)\,n\,(n+1)\,\frac{m_j^i(r)\,p_l^k(r)}{r^3}\,. \tag{31b}$$

Note that the first expression leads to a coupling of the toroidal mode m_j^i and other toroidal modes. The second expression leads to a coupling of the toroidal mode m_j^i with other poloidal modes. The existence of the coupling is governed by the prescribed velocity field. The coupling structure is controlled by the ADAMS-GAUNT- and ELSASSER-integrals, which may be zero or non-zero depending on the index pairs. The values of the ADAMS-GAUNT- and ELSASSER-integrals may be attributed to the so-called WIGNER-3j- symbols, *cf.*, (Arfken, 2005; Winch, 1973). The WIGNER-3j-symbols obey so-called selection rules and may be numerically calculated using recurrence relations, *cf.*, (Luscombe, 1998).

4.2 Boundary condition and projection method

In general, from this point on the kinematic dynamo problem is treated numerically by using finite difference methods, *cf.*, (Bullard, 1954; Dudley, 1989; Gubbins, 2000). This requires consideration of the boundary condition, *i.e.*, Eqn. (8), in terms of the toroidal-poloidal decomposition. The boundary conditions for the modes, *i.e.*, the radial functions m_j^i and n_j^i respectively are obtained by applying the projection method again. This is convenient since the exterior solution in Eqn. (10) is determined up to a constant c_n^m. If the exponential solution *ansatz* is applied to Eqn. (10) the factor c_n^m is no longer time dependent.

5 Conclusion

A rational approach to dynamo theory based on Maxwell's equation was presented. A scale analysis lead to the so-called induction equation, *i.e.*, Eqn. (5). In this context kinematic dynamos with a prescribed velocity were addressed in this paper. In case of an exterior vacuum it was shown that the solution for the external magnetic is determined up to a constant.

By using the toroidal-poloidal decomposition for the magnetic and velocity field and by applying the projection method the induction equation was analytically converted to a coupled system of ordinary differential equations. Since in the case of the kinematic dynamo the velocity is prescribed, the unknowns of this system are given by the radial functions m_j^i and n_j^i. The induction equation was therefore treated in a spectral sense with respect. This paper presented a short insight on how the velocity dependent terms need to be addressed mathematically.

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