Multidimensional pendulum in a nonconservative force field

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Abstract

In this activity, we systematize the results on the study of the equations of motion of dynamically symmetric multidimensional rigid bodies in nonconservative force fields. The form of these equations is taken from the dynamics of real lower-dimensional rigid bodies interacting with resisting medium by laws of jet flows where a body is influenced by a nonconservative tracing force; under action of this force, the velocity of some characteristic point of the body remains constant, which means that the system possesses a nonintegrable servo constraint.

1 Introduction

In the earlier activities, the author has already proved the complete integrability of the equations of a plane-parallel motion of a body in a resisting medium under the jet flow conditions when the system of dynamical equations possesses a first integral, which is a transcendental (in the sense of the theory of functions of a complex variable) function of quasi-velocities having essential singularities. It was assumed that the interaction of the medium with the body is concentrated on a part of the surface of the body that has the form of a (one-dimensional) plate. In the sequel, the planar problem was generalized to the spatial (three-dimensional) case, where the system of dynamical equations possesses a complete set of transcendental first integrals. In this case, it was assumed that the interaction of the medium with the body is concentrated on the part of the surface of the body that has the form of a planar (two-dimensional) disk.

Moreover, we study the dynamic part of equations of motion of a different four-dimensional dynamically symmetric rigid body where a nonconservative force field is concentrated on a part of the surface of the body, which has the form of a two-dimensional (three-dimensional) disk, and the action of the force is concentrated in the two-dimensional plane (one-dimensional line) perpendicular to this disk.

In this work, we discuss results, both new and obtained earlier, concerning the case where the interaction of the medium with the body is concentrated on the part of the surface of the body that has the form of a $(n-1)$-dimensional disk and the force acts in the direction perpendicular to the disk. We systematize these results and formulate them in the invariant form.

2 Certain General Discourse

First of all for n-dimensional rigid body, we will be interested the case $(1—(n-1))$, i. e., when in some coordinate system $Dx_1 \dots x_n$ attached to the body, the operator of inertia has the form

$$\mathrm{diag}\{I_1, I_2, \dots, I_2\}, \tag{1}$$

i. e., the body is dynamically symmetric in the hyperplane $Dx_2 \dots x_n$ (Dx_1 is the axe of dynamical symmetry).

The configuration space of a free, n-dimensional rigid body is the direct product

$$\mathbf{R}^n \times \mathrm{SO}(n) \tag{2}$$

of the space $\mathbf{R}^n$, which defines the coordinates of the center of mass of the body, and the rotation group $\mathrm{SO}(n)$, which defined rotations of the body about its center of mass and has dimension $n(n+1)/2$.

Therefore, the dynamical part of equations of motion has the same dimension, whereas the dimension of the phase space is equal to $n(n+1)$.

In particular, if Ω is the tensor of angular velocity of a n-dimensional rigid body (it is a second-rank tensor, see [1, 2, 3]), $\Omega \in \mathrm{so}(n)$, then the part of dynamical equations of motion corresponding to the Lie algebra $\mathrm{so}(n)$ has the following form (see [2, 3, 4]):

$$\dot{\Omega}\Lambda + \Lambda\dot{\Omega} + [\Omega,\ \Omega\Lambda + \Lambda\Omega] = M, \tag{3}$$

$$\Lambda = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}, \tag{4}$$

$$\lambda_1 = \frac{-I_1 + I_2 + \dots + I_n}{2},\ \lambda_2 = \frac{I_1 - I_2 + I_3 + \dots + I_n}{2}, \dots,$$

$$\lambda_{n-1} = \frac{I_1 + \dots + I_{n-2} - I_{n-1} + I_n}{2},\ \lambda_n = \frac{I_1 + \dots + I_{n-1} - I_n}{2},$$

$M = M_F$ is the natural projection of the moment of external forces $\mathbf{F}$ acting to the body in $\mathbf{R}^n$ on the natural coordinates of the Lie algebra $\mathrm{so}(n)$ and $[.,.]$ is the commutator in $\mathrm{so}(n)$.

Obviously, the following relations hold: $\lambda_i - \lambda_j = I_j - I_i$ for any $i, j = 1, \dots, n$.

For the calculation of the moment of an external force acting to the body, we need to construct the mapping $\mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathrm{so}(n)$, that maps a pair of vectors $(\mathbf{DN}, \mathbf{F}) \in \mathbf{R}^n \times \mathbf{R}^n$ from $\mathbf{R}^n \times \mathbf{R}^n$ to an element of the Lie algebra $\mathrm{so}(n)$, where $\mathbf{DN} = \{\delta_1, \delta_2, \dots, \delta_n\}$, $\mathbf{F} = \{F_1, F_2, \dots, F_n\}$, and $\mathbf{F}$ is an external force acting to the body. Here $\mathbf{DN}$ is the vector directing from the point D of the coordinate system $Dx_1 \dots x_n$ to the point N of force acting). For this end, we construct the following auxiliary matrix

$$\begin{pmatrix} \delta_1 & \delta_2 & \dots & \delta_n \\ F_1 & F_2 & \dots & F_n \end{pmatrix}. \tag{5}$$

Dynamical systems studied in this activity, are dynamical systems with variable dissipation with zero mean (see [4, 5]). We need to examine by direct methods a part of the main system of dynamical equations, namely, the Newton equation, which plays the role of the equation of motion of the center of mass, i.e., the part of the dynamical equations corresponding to the space $\mathbf{R}^n$:

$$m\mathbf{w}_C = \mathbf{F}, \tag{6}$$

where $\mathbf{w}_C$ is the acceleration of the center of mass C of the body and m is its mass. Moreover, due to the higher-dimensional Rivals formula (it can be obtained by the operator method) we have the following relations:

$$\mathbf{w}_C = \mathbf{w}_D + \Omega^2\mathbf{DC} + E\mathbf{DC},\ \mathbf{w}_D = \dot{\mathbf{v}}_D + \Omega\mathbf{v}_D,\ E = \dot{\Omega}, \tag{7}$$

where $\mathbf{w}_D$ is the acceleration of the point D, $\mathbf{F}$ is the external force acting on the body (in our case, $\mathbf{F} = \mathbf{S}$), and E is the tensor of angular acceleration (second-rank tensor).

Let the position of the body Θ in Euclidean space $\mathbf{E}^n$ is defined by the functions which are the cyclic in the following sense: the generalized force $\mathbf{F}$ and its moment $(\mathbf{DN}, \mathbf{F})$ depend on generalized velocities only (quasi-velocities), and do not depend on the position of the body in the space. Then, the system of equations (3) and (6) on the manifold $\mathbf{R}^n \times \mathrm{so}(n)$ is a *closed* system of dynamical equations of the motion of a free n-dimensional rigid body under the action of an external force $\mathbf{F}$. This system have been separated from the kinematic part of the equations of motion on the manifold (2) and can be examined independently.

3 General Problem on the Motion Under a Tracing Force

Consider a motion of a homogeneous, dynamically symmetric (case (1)), rigid body with front end face (a $(n-1)$-dimensional disk interacting with a medium that fills the n-dimensional space) in the field of a resistance force $\mathbf{S}$ under the quasi-stationarity conditions (see [6, 7]).

Let $(v, \alpha, \beta_1, \ldots, \beta_{n-2})$ be the (generalized) spherical coordinates of the velocity vector of the center of the $(n-1)$-dimensional disk lying on the axis of symmetry of the body, Ω be the tensor of angular velocity of the body, $Dx_1 \ldots x_n$ be the coordinate system attached to the body such that the axis of symmetry CD coincides with the axis Dx_1 (recall that C is the center of mass), and the axes $Dx_2, Dx_3, \ldots, Dx_n$ lie in the hyperplane of the disk, and $I_1,\ I_2,\ I_3 = I_2,\ \ldots,\ I_n = I_2,\ m$ are characteristics of inertia and mass.

We adopt the following expansions in the projections to the axes of the coordinate system $Dx_1 \ldots x_n$: $\mathbf{DC} = \{-\sigma, 0, \ldots, 0\}$, $\mathbf{v}_D = v\mathbf{i}_v(\alpha, \beta_1, \ldots, \beta_{n-2})$, where

$$\mathbf{i}_v(\alpha, \beta_1, \ldots, \beta_{n-2}) = \begin{pmatrix} \cos\alpha \\ \sin\alpha\cos\beta_1 \\ \sin\alpha\sin\beta_1\cos\beta_2 \\ \ldots \\ \sin\alpha\sin\beta_1\ldots\sin\beta_{n-3}\cos\beta_{n-2} \\ \sin\alpha\sin\beta_1\ldots\sin\beta_{n-2} \end{pmatrix} \tag{8}$$

is the single vector on the axe of vector $\mathbf{v}$.

In the case (1) we additionally have the expansion for the function of the influence of the medium on the n-dimensional body: $\mathbf{S} = \{-S, 0, \dots, 0\}$, i.e., in this case $\mathbf{F} = \mathbf{S}$. Further, the auxiliary matrix (5) for the calculation of the moment of the resistance force has the form

$$\begin{pmatrix} 0 & x_{2N} & \dots & x_{nN} \\ -S & 0 & \dots & 0 \end{pmatrix}, \tag{9}$$

then the part of the dynamical equations of motion that describes the motion of the body about the center of mass and corresponds to the Lie algebra so(n), can be obtained. We note that system (3), due to the existing dynamical symmetry

$$I_2 = \dots = I_n, \tag{10}$$

possesses cyclic first integrals

$$\omega_{k_1} \equiv \omega^0_{k_1} = \text{const}, \ \dots, \ \omega_{k_s} \equiv \omega^0_{k_s} = \text{const}, \ s = \frac{(n-1)(n-2)}{2}. \tag{11}$$

Here $k_1 = 1, \dots, k_s$ are the certain s nonrecurrent numbers from the set $W_1 = \{1, 2, \dots, n(n-1)/2\}$.

In the sequel, we consider the first integrals (11) of the system on its zero levels:

$$\omega^0_{k_1} = \dots = \omega^0_{k_s} = 0. \tag{12}$$

The choice of nonzero components $\omega_{r_1}, \dots, \omega_{r_p}$ of tensor Ω consists of $p = n(n-1)/2 - (n-1)(n-2)/2 = n-1$ ones (here $r_1, \dots, r_p$ are the rest p of numbers from the set W_1, not equal to $k_1, \dots, k_s$).

If one considers *a more general problem* on the motion of a body under a tracing force $\mathbf{T}$ that lies on the straight line $CD = Dx_1$ and provides the fulfillment of the relation

$$v \equiv \text{const}, \tag{13}$$

throughout the motion, then instead of F_1 system (3), (6) contains $T - s(\alpha)v^2$, $\sigma = DC$.

Choosing the value T of the tracing force appropriately, one can achieve the equality (13) throughout the motion. Indeed, expressing T due to system (3), (6), we obtain for $\cos\alpha \neq 0$, $n > 2$ the relation

$$\begin{gathered} T = T_v(\alpha, \beta_1, \dots, \beta_{n-2}, \Omega) = m\sigma(\omega^2_{r_1} + \dots + \omega^2_{r_p}) + \\ + s(\alpha)v^2 \left[1 - \frac{m\sigma}{(n-2)I_2} \frac{\sin\alpha}{\cos\alpha} \Gamma_v\left(\alpha, \beta_1, \dots, \beta_{n-2}, \frac{\Omega}{v}\right)\right], \end{gathered} \tag{14}$$

$$\begin{gathered} \Gamma_v\left(\alpha, \beta_1, \dots, \beta_{n-2}, \frac{\Omega}{v}\right) = |\mathbf{r}_N| = (\mathbf{r}_N, \mathbf{i}_N(\beta_1, \dots, \beta_{n-2})) = \\ = 0 \cdot \cos\frac{\pi}{2} + \sum_{s=2}^{n} x_{sN}\left(\alpha, \beta_1, \dots, \beta_{n-2}, \frac{\Omega}{v}\right) i_{sN}(\beta_1, \dots, \beta_{n-2}). \end{gathered} \tag{15}$$

Here $i_{sN}(\beta_1,\ldots,\beta_{n-2})$, $s=1,\ldots,n$, $(i_{1N}(\beta_1,\ldots,\beta_{n-2})\equiv 0)$ are the components of single vector on the axe of vector $\mathbf{r}_N=\{0,x_{2N},\ldots,x_{nN}\}$ on $(n-2)$-dimensional sphere $\mathbf{S}^{n-2}\{\beta_1,\ldots,\beta_{n-2}\}$, definied by the equality $\alpha=\pi/2$ as equatorial section of corresponding $(n-1)$-dimensional sphere $\mathbf{S}^{n-1}\{\alpha,\beta_1,\ldots,\beta_{n-2}\}$ (defined by the equality (13)), i. e.,

$$\mathbf{i}_N(\beta_1,\ldots,\beta_{n-2})=$$
$$=\begin{pmatrix} 0\\ \cos\beta_1\\ \sin\beta_1\cos\beta_2\\ \ldots\\ \sin\beta_1\ldots\sin\beta_{n-3}\cos\beta_{n-2}\\ \sin\beta_1\ldots\sin\beta_{n-2}\end{pmatrix}=\mathbf{i}_v\left(\frac{\pi}{2},\beta_1,\ldots,\beta_{n-2}\right) \tag{16}$$

(see Eq. (8)). This procedure can be interpreted in two ways. First, we have transformed the system using the tracing force (control) that provides the consideration of the class (13) of motions interesting for us. Second, we can treat this as an order-reduction procedure. Indeed, system (3), (6) generates the following independent system of following order (due to Eqs. (13), (11), (12)): $n(n+1)/2-(n-1)(n-2)/2-1=2(n-1)$.

Let introduce the new quasi-velocities in system (3), (6). For this, we transform the values $\omega_{r_1},\ldots,\omega_{r_{n-1}}$ by the composition of following $(n-2)$ rotations:

$$\begin{pmatrix} z_1\\ z_2\\ \ldots\\ z_{n-1}\end{pmatrix}=T_{n-2,n-1}(-\beta_1)\circ T_{n-3,n-2}(-\beta_2)\circ\ldots\circ T_{1,2}(-\beta_{n-2})\begin{pmatrix}\omega_{r_1}\\ \omega_{r_2}\\ \ldots\\ \omega_{r_{n-1}}\end{pmatrix}, \tag{17}$$

where the matrix $T_{k,k+1}(\beta)$, $k=1,\ldots,n-2$, is obtained from the unit one by the presence of second order minor $M_{k,k+1}$:

$$T_{k,k+1}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & \ddots & 0 & 0 & 0\\ 0 & 0 & M_{k,k+1} & 0 & 0\\ 0 & 0 & 0 & \ddots & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix},\ M_{k,k+1}=\begin{pmatrix} m_{k,k} & m_{k,k+1}\\ m_{k+1,k} & m_{k+1,k+1}\end{pmatrix}, \tag{18}$$

$$m_{k,k}=m_{k+1,k+1}=\cos\beta,\ m_{k+1,k}=-m_{k,k+1}=\sin\beta.$$

As we see, we cannot solve the system with respect to $\dot\alpha$, $\dot\beta_1$, $\ldots$, $\dot\beta_{n-2}$ on the manifold

$$O_1'=\{(\alpha,\beta_1,\ldots,\beta_{n-2},\omega_{r_1},\ldots,\omega_{r_{n-1}})\in\mathbf{R}^{2(n-1)}:$$
$$\alpha=\frac{\pi}{2}k,\ \beta_1=\pi l_1,\ldots,\beta_{n-3}=\pi l_{n-3},\ k,l_1,\ldots,l_{n-3}\in\mathbf{Z}\}. \tag{19}$$

Therefore, on the manifold (19) the uniqueness theorem formally is violated. Moreover, for even k and any $l_1,\ldots,l_{n-3}$, an indeterminate form appears due to the degeneration of the spherical coordinates $(v,\alpha,\beta_1,\ldots,\beta_{n-2})$. For odd k, the uniqueness theorem is obviously violated since one of the equation degenerates.

This implies that system (3), (6) outside (and only outside) the manifold (19) can be reduced to the following form ($n > 2$):

$$\dot{\alpha} = -z_{n-1} + \frac{\sigma v}{(n-2)I_2}\frac{s(\alpha)}{\cos\alpha}\Gamma_v\left(\alpha, \beta_1, \ldots, \beta_{n-2}, \frac{\Omega}{v}\right), \tag{20}$$

$$\begin{aligned}\dot{z}_{n-1} = {}& \frac{v^2}{(n-2)I_2}s(\alpha)\Gamma_v\left(\alpha, \beta_1, \ldots, \beta_{n-2}, \frac{\Omega}{v}\right) - (z_1^2 + \ldots + z_{n-2}^2)\frac{\cos\alpha}{\sin\alpha} + \\ &+ \frac{\sigma v}{(n-2)I_2}\frac{s(\alpha)}{\sin\alpha}\left\{\sum_{s=1}^{n-2}(-1)^s z_{n-1-s}\Delta_{v,s}\left(\alpha, \beta_1, \ldots, \beta_{n-2}, \frac{\Omega}{v}\right)\right\},\end{aligned} \tag{21}$$

$$\begin{aligned}\dot{z}_{n-2} = {}& z_{n-2}z_{n-1}\frac{\cos\alpha}{\sin\alpha} + (z_1^2 + \ldots + z_{n-3}^2)\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} + \\ &+ \frac{\sigma v}{(n-2)I_2}\frac{s(\alpha)}{\sin\alpha}\{z_{n-1}\Delta_{v,1}\left(\alpha, \beta_1, \ldots, \beta_{n-2}, \frac{\Omega}{v}\right) + \\ &+ \sum_{s=2}^{n-2}(-1)^{s+1}z_{n-1-s}\Delta_{v,s}\left(\alpha, \beta_1, \ldots, \beta_{n-2}, \frac{\Omega}{v}\right)\frac{\cos\beta_1}{\sin\beta_1}\} - \\ &- \frac{v^2}{(n-2)I_2}s(\alpha)\Delta_{v,1}\left(\alpha, \beta_1, \ldots, \beta_{n-2}, \frac{\Omega}{v}\right),\end{aligned} \tag{22}$$

$$\begin{aligned}\dot{z}_{n-3} = {}& z_{n-3}z_{n-1}\frac{\cos\alpha}{\sin\alpha} - z_{n-3}z_{n-2}\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} - \\ &-(z_1^2 + \ldots + z_{n-4}^2)\frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2} + \\ &+ \frac{\sigma v}{(n-2)I_2}\frac{s(\alpha)}{\sin\alpha}\{\Delta_{v,2}\left(\alpha, \beta_1, \ldots, \beta_{n-2}, \frac{\Omega}{v}\right)\left[-z_{n-1} + z_{n-2}\frac{\cos\beta_1}{\sin\beta_1}\right] + \\ &+ \sum_{s=3}^{n-2}(-1)^s z_{n-1-s}\Delta_{v,s}\left(\alpha, \beta_1, \ldots, \beta_{n-2}, \frac{\Omega}{v}\right)\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}\} + \\ &+ \frac{v^2}{(n-2)I_2}s(\alpha)\Delta_{v,2}\left(\alpha, \beta_1, \ldots, \beta_{n-2}, \frac{\Omega}{v}\right),\end{aligned} \tag{23}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{aligned}\dot{z}_1 = {}& \dot{\beta}_{n-2}(-\omega_{r_1}\sin\beta_{n-2} + \omega_{r_2}\cos\beta_{n-2}) + \\ &+ (-1)^n\frac{v^2}{(n-2)I_2}s(\alpha)\Delta_{v,n-2}\left(\alpha, \beta_1, \ldots, \beta_{n-2}, \frac{\Omega}{v}\right) = \\ &= z_1\frac{\cos\alpha}{\sin\alpha}\left\{\sum_{s=1}^{n-2}(-1)^{s+1}z_{n-s}\frac{\cos\beta_{s-1}}{\sin\beta_1\ldots\sin\beta_{s-1}}\right\} + \\ &+ \frac{\sigma v}{(n-2)I_2}\frac{s(\alpha)}{\sin\alpha}(-1)^{n+1}\Delta_{v,n-2}\left(\alpha, \beta_1, \ldots, \beta_{n-2}, \frac{\Omega}{v}\right)\times \\ &\times\left\{\sum_{s=2}^{n-1}(-1)^s z_{n+1-s}\frac{\cos\beta_{s-1}}{\sin\beta_1\ldots\sin\beta_{s-1}}\right\} +\end{aligned}$$

$$+(-1)^n \frac{v^2}{(n-2)I_2} s(\alpha)\Delta_{v,n-2}\left(\alpha, \beta_1, \dots, \beta_{n-2}, \frac{\Omega}{v}\right), \tag{24}$$

$$\dot{\beta}_1 = z_{n-2}\frac{\cos\alpha}{\sin\alpha} + \frac{\sigma v}{(n-2)I_2}\frac{s(\alpha)}{\sin\alpha}\Delta_{v,1}\left(\alpha, \beta_1, \dots, \beta_{n-2}, \frac{\Omega}{v}\right), \tag{25}$$

$$\dot{\beta}_2 = -z_{n-3}\frac{\cos\alpha}{\sin\alpha\sin\beta_1} + \frac{\sigma v}{(n-2)I_2}\frac{s(\alpha)}{\sin\alpha\sin\beta_1}\Delta_{v,2}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right), \tag{26}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dot{\beta}_{n-2} = (-1)^{n+1} z_1 \frac{\cos\alpha}{\sin\alpha\sin\beta_1\dots\sin\beta_{n-3}} +$$

$$+\frac{\sigma v}{(n-2)I_2}\frac{s(\alpha)}{\sin\alpha\sin\beta_1\dots\sin\beta_{n-2}}\Delta_{v,n-2}\left(\alpha, \beta_1, \dots, \beta_{n-2}, \frac{\Omega}{v}\right), \tag{27}$$

$$\Delta_{v,1}\left(\alpha, \beta_1, \dots, \beta_{n-2}, \frac{\Omega}{v}\right) = (\mathbf{r}_N, \mathbf{i}_N\left(\beta_1 + \frac{\pi}{2}, \beta_2, \dots, \beta_{n-2}\right)),$$

$$\Delta_{v,2}\left(\alpha, \beta_1, \dots, \beta_{n-2}, \frac{\Omega}{v}\right) = (\mathbf{r}_N, \mathbf{i}_N\left(\frac{\pi}{2}, \beta_2 + \frac{\pi}{2}, \beta_3, \dots, \beta_{n-2}\right)),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \tag{28}$$

$$\Delta_{v,n-3}\left(\alpha, \beta_1, \dots, \beta_{n-2}, \frac{\Omega}{v}\right) = (\mathbf{r}_N, \mathbf{i}_N\left(\frac{\pi}{2}, \dots, \frac{\pi}{2}, \beta_{n-3} + \frac{\pi}{2}, \beta_{n-2}\right)),$$

$$\Delta_{v,n-2}\left(\alpha, \beta_1, \dots, \beta_{n-2}, \frac{\Omega}{v}\right) = (\mathbf{r}_N, \mathbf{i}_N\left(\frac{\pi}{2}, \dots, \frac{\pi}{2}, \beta_{n-2} + \frac{\pi}{2}\right)),$$

and function $\Gamma_v(\alpha, \beta_1, \dots, \beta_{n-2}, \Omega/v)$ can be represented in the form (15).

In right-hand side of the system (20)–(27) after common multiplier

$$\frac{\sigma v}{(n-2)I_2}\frac{s(\alpha)}{\cos\alpha},$$

the values $\Delta_{v,s}(\alpha, \beta_1, \dots, \beta_{n-2}, \Omega/v)$, $s = 1, \dots, n-2$, are represented in linear form (and always $(n-2)$ coefficients precisely). For instance, in Eq. (21) (with left-hand side $\dot{z}_{n-1}$), the functions (28) are represented with all indices s from 1 to $n-2$ (every index per one time), i. e.,

$$1 \quad 2 \quad 3 \quad 4 \quad \dots \quad n-2. \tag{29}$$

But furthermore, in Eqs. (22)–(24), the set of functions (28) is formed in another way. For instance, in equation with left-hand side $\dot{z}_{n-2}$, the set of functions (28) is formed with corresponding indices (29). But in equation with left-hand side $\dot{z}_{n-3}$, it is already formed with the following indices:

$$2 \quad 2 \quad 3 \quad 4 \quad \dots \quad n-2 \tag{30}$$

i. e., the function $\Delta_{v,2}(\alpha, \beta_1, \dots, \beta_{n-2}, \Omega/v)$ is already repeated twice.

What is the general distribution of indices? It can be represented by table 1.

Table 1: General Distribution of Indices in Set of Functions (28)

Left-hand Side of (20)–(27)	Distribution of Indices s in Set of Functions (28)					
$\dot{z}_{n-2}$	1	2	3	4	...	$n-2$
$\dot{z}_{n-3}$	2	2	3	4	...	$n-2$
$\dot{z}_{n-4}$	3	3	3	4	...	$n-2$
$\dot{z}_{n-5}$	4	4	4	4	...	$n-2$
...	...	...	...	...	...	...
$\dot{z}_1$	$n-2$	$n-2$	$n-2$	$n-2$	...	$n-2$

4 Case Where the Moment of a Nonconservative Force Is Independent of the Angular Velocity

Similarly to the choice of Chaplygin analytic functions, we take the dynamical functions s, $x_{2N}, \ldots, x_{nN}$ in the following form (using (16)):

$$s(\alpha) = B\cos\alpha,\ \mathbf{r}_N = R(\alpha)\mathbf{i}_N,\ R(\alpha) = A\sin\alpha,\ A, B > 0. \tag{31}$$

Herewith, the functions $\Gamma_v(\alpha, \beta_1, \ldots, \beta_{n-2}, \Omega/v)$, $\Delta_{v,s}(\alpha, \beta_1, \ldots, \beta_{n-2}, \Omega/v)$, $s = 1, \ldots, n-2$, in system (20)–(27), take the following form:

$$\Gamma_v\left(\alpha, \beta_1, \ldots, \beta_{n-2}, \frac{\Omega}{v}\right) = R(\alpha) = A\sin\alpha,\ \Delta_{v,s}\left(\alpha, \beta_1, \ldots, \beta_{n-2}, \frac{\Omega}{v}\right) \equiv 0. \tag{32}$$

Then, due to the nonintegrable constraint (13), outside and only outside the manifold (19), system (20)–(27)) has the analytic form

$$\alpha' = -z_{n-1} + b\sin\alpha, \tag{33}$$

$$z'_{n-1} = \sin\alpha\cos\alpha - (z_1^2 + \ldots + z_{n-2}^2)\frac{\cos\alpha}{\sin\alpha}, \tag{34}$$

$$z'_{n-2} = z_{n-2}z_{n-1}\frac{\cos\alpha}{\sin\alpha} + (z_1^2 + \ldots + z_{n-3}^2)\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}, \tag{35}$$

$$z'_{n-3} = z_{n-3}z_{n-1}\frac{\cos\alpha}{\sin\alpha} - z_{n-3}z_{n-2}\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} -$$

$$-(z_1^2 + \ldots + z_{n-4}^2)\frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}, \tag{36}$$

....

$$z'_1 = z_1\frac{\cos\alpha}{\sin\alpha}\left\{\sum_{s=1}^{n-2}(-1)^{s+1}z_{n-s}\frac{\cos\beta_{s-1}}{\sin\beta_1\ldots\sin\beta_{s-1}}\right\}, \tag{37}$$

$$\beta_1' = z_{n-2}\frac{\cos\alpha}{\sin\alpha}, \tag{38}$$

$$\beta_2' = -z_{n-3}\frac{\cos\alpha}{\sin\alpha\sin\beta_1}, \tag{39}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\beta_{n-3}' = (-1)^n z_2\frac{\cos\alpha}{\sin\alpha\sin\beta_1\dots\sin\beta_{n-4}}, \tag{40}$$

$$\beta_{n-2}' = (-1)^{n+1} z_1\frac{\cos\alpha}{\sin\alpha\sin\beta_1\dots\sin\beta_{n-3}}, \tag{41}$$

introducing the dimensionless variables, parameters, and the differentiation as follows:

$$\begin{gathered} z_k \mapsto n_0 v z_k,\ k = 1,\dots,n-1,\ n_0^2 = \frac{AB}{(n-2)I_2}(n>2), \\ b = \sigma n_0,\ <\cdot>= n_0 v <'> . \end{gathered} \tag{42}$$

We see that the $2(n-1)$th-order system (33)–(41) (which can be considered as a system on the tangent bundle $T_*\mathbf{S}^{n-1}\{z_{n-1},\dots,z_1;\alpha,\beta_1,\dots,\beta_{n-2}\}$ of the $(n-1)$-dimensional sphere $\mathbf{S}^{n-1}\{\alpha,\beta_1,\dots,\beta_{n-2}\}$, see below) contains the independent $(2n-3)$th-order system (33)–(40) on its own $(2n-3)$-dimensional manifold.

Theorem 1. The system (3), (6) under conditions (13), (11), (12), is reduced to dynamic system (20)–(27) on the tangent bundle $T_*\mathbf{S}^{n-1}\{z_{n-1},\dots,z_1;\alpha,\beta_1,\dots,\beta_{n-2}\}$ of $(n-1)$-dimensional sphere $\mathbf{S}^{n-1}\{\alpha,\beta_1,\dots,\beta_{n-2}\}$.

For the complete integration of $2(n-1)$th-order system (33)–(41), in general, we need $2n-3$ independent first integrals. However, after the change of variables

$$\begin{gathered} w_{n-1} = z_{n-1},\ w_{n-2} = \sqrt{z_1^2+\dots+z_{n-2}^2},\ w_{n-3} = \frac{z_2}{z_1}, \\ w_{n-4} = \frac{z_3}{\sqrt{z_1^2+z_2^2}},\ \dots,\ w_2 = \frac{z_{n-3}}{\sqrt{z_1^2+\dots+z_{n-4}^2}},\ w_1 = \frac{z_{n-2}}{\sqrt{z_1^2+\dots+z_{n-3}^2}}, \end{gathered} \tag{43}$$

system (33)–(41) splits as follows:

$$\alpha' = -w_{n-1} + b\sin\alpha, \tag{44}$$

$$w_{n-1}' = \sin\alpha\cos\alpha - w_{n-2}^2\frac{\cos\alpha}{\sin\alpha}, \tag{45}$$

$$w_{n-2}' = w_{n-2}w_{n-1}\frac{\cos\alpha}{\sin\alpha}, \tag{46}$$

$$\begin{gathered} w_s' = d_s(w_{n-1},\dots,w_1;\alpha,\beta_1,\dots,\beta_{n-2})\frac{1+w_s^2}{w_s}\frac{\cos\beta_s}{\sin\beta_s}, \\ \beta_s' = d_s(w_{n-1},\dots,w_1;\alpha,\beta_1,\dots,\beta_{n-2}),\ s = 1,\dots,n-3, \end{gathered} \tag{47}$$

$$\beta_{n-2}' = d_{n-2}(w_{n-1},\dots,w_1;\alpha,\beta_1,\dots,\beta_{n-2}), \tag{48}$$

$$\begin{gathered} d_1 = Z_{n-2}(w_{n-1},\dots,w_1)\frac{\cos\alpha}{\sin\alpha}, \\ d_2 = -Z_{n-3}(w_{n-1},\dots,w_1)\frac{\cos\alpha}{\sin\alpha\sin\beta_1}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ d_{n-2} = (-1)^{n+1}Z_1(w_{n-1},\dots,w_1)\frac{\cos\alpha}{\sin\alpha\sin\beta_1\dots\sin\beta_{n-3}}, \end{gathered} \tag{49}$$

herewith, $z_k = Z_k(w_{n-1}, \ldots, w_1)$, $k = 1, \ldots, n-2$, are the functions due to the change (43).

We see that for the complete integration of system (44)–(48) it suffices to specify two independent first integrals of system (44)–(46), on one first integral of systems (47), and an additional first integral that attaches Eq. (48) (*i. e.,* n *in all*).

We have the following transcendental first integral:

$$\Theta_1(w_{n-1}, w_{n-2}; \alpha) = \frac{w_{n-1}^2 + w_{n-2}^2 - bw_{n-1}\sin\alpha + \sin^2\alpha}{w_{n-2}\sin\alpha} = C_1 = \text{const}. \quad (50)$$

Then the additional first integral obtained has the following structural form:

$$\Theta_2(w_{n-1}, w_{n-2}; \alpha) = G\left(\sin\alpha, \frac{w_{n-1}}{\sin\alpha}, \frac{w_{n-2}}{\sin\alpha}\right) = C_2 = \text{const}. \quad (51)$$

For the complete integration, as was mentioned above, it suffices to find on one first integral for (potentially separated) systems (47), and an additional first integral that attaches Eq. (48).

Indeed, we have the desired first integrals as follows:

$$\Theta_{s+2}(w_s; \beta_s) = \frac{\sqrt{1 + w_s^2}}{\sin\beta_s} = C''_{s+2} = \text{const}, \; s = 1, \ldots, n-3, \quad (52)$$

$$\Theta_n(w_{n-3}, \ldots, w_1; \alpha, \beta_1, \ldots, \beta_{n-2}) = C''_n = \text{const}, \quad (53)$$

herewith, we must substitute the left-hand sides of the first integrals (52) for $s = n-4, n-3$, in the expressions of first integral (53) instead C_{n-2}, C_{n-1}.

Theorem 2. The $2(n-1)$th-order system (44)–(48) possesses the sufficient quantity (n) of independent first integrals (50), (51), (52), and (53).

Thus, in the case considered, the system of dynamical equations (3), (6) under condition (31) has $(n^2 - n + 4)/2$, $n > 2$, invariant relations: the nonintegrable analytic constraint of the form (13), the cyclic first integrals of the form (11), (12), the first integral of the form (50), the first integral (51), which is a transcendental function of the phase variables (in the sense of complex analysis) expressed through a finite combination of elementary functions, and, finally, the transcendental first integrals of the form (52), (53).

Theorem 3. System (3), (6) under conditions (13), (31), (11), (12) possesses $(n^2 - n + 4)/2$, $n > 2$, invariant relations (complete set), n of which transcendental functions from the point of view of complex analysis. Herewith, all relations are expressed through finite combinations of elementary functions.

Consider the following $(2n-3)$th-order system:

$$
\begin{aligned}
&\ddot\xi + b_*\dot\xi\cos\xi + \sin\xi\cos\xi -\\
&-\left[\dot\eta_1^{\,2} + \dot\eta_2^{\,2}\sin^2\eta_1 + \dot\eta_3^{\,2}\sin^2\eta_1\sin^2\eta_2 + \ldots + \dot\eta_{n-2}^2\sin^2\eta_1\ldots\sin^2\eta_{n-3}\right]\frac{\sin\xi}{\cos\xi} = 0,\\
&\ddot\eta_1 + b_*\dot\eta_1\cos\xi + \dot\xi\dot\eta_1\frac{1+\cos^2\xi}{\cos\xi\sin\xi} -\\
&-\left[\dot\eta_2^{\,2} + \dot\eta_3^{\,2}\sin^2\eta_2 + \dot\eta_4^{\,2}\sin^2\eta_2\sin^2\eta_3 + \ldots + \dot\eta_{n-2}^2\sin^2\eta_2\ldots\sin^2\eta_{n-3}\right]\times\\
&\times\sin\eta_1\cos\eta_1 = 0,\\
&\ddot\eta_2 + b_*\dot\eta_2\cos\xi + \dot\xi\dot\eta_2\frac{1+\cos^2\xi}{\cos\xi\sin\xi} +\\
&+2\dot\eta_1\dot\eta_2\frac{\cos\eta_1}{\sin\eta_1} -\\
&-\left[\dot\eta_3^{\,2} + \dot\eta_4^{\,2}\sin^2\eta_3 + \dot\eta_5^{\,2}\sin^2\eta_3\sin^2\eta_4 + \ldots + \dot\eta_{n-2}^2\sin^2\eta_3\ldots\sin^2\eta_{n-3}\right]\times\\
&\times\sin\eta_2\cos\eta_2 = 0,\\
&\ddot\eta_3 + b_*\dot\eta_3\cos\xi + \dot\xi\dot\eta_3\frac{1+\cos^2\xi}{\cos\xi\sin\xi} +\\
&+2\dot\eta_1\dot\eta_3\frac{\cos\eta_1}{\sin\eta_1} + 2\dot\eta_2\dot\eta_3\frac{\cos\eta_2}{\sin\eta_2} -\\
&-\left[\dot\eta_4^{\,2} + \dot\eta_5^{\,2}\sin^2\eta_4 + \dot\eta_6^{\,2}\sin^2\eta_4\sin^2\eta_5 + \ldots + \dot\eta_{n-2}^2\sin^2\eta_4\ldots\sin^2\eta_{n-3}\right]\times\\
&\times\sin\eta_3\cos\eta_3 = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ddot\eta_{n-4} + b_*\dot\eta_{n-4}\cos\xi + \dot\xi\dot\eta_{n-4}\frac{1+\cos^2\xi}{\cos\xi\sin\xi} +\\
&+2\dot\eta_1\dot\eta_{n-4}\frac{\cos\eta_1}{\sin\eta_1} + \ldots + 2\dot\eta_{n-5}\dot\eta_{n-4}\frac{\cos\eta_{n-5}}{\sin\eta_{n-5}} -\\
&-\left[\dot\eta_{n-3}^2 + \dot\eta_{n-2}^2\sin^2\eta_{n-3}\right]\sin\eta_{n-4}\cos\eta_{n-4} = 0,\\
&\ddot\eta_{n-3} + b_*\dot\eta_{n-3}\cos\xi + \dot\xi\dot\eta_{n-3}\frac{1+\cos^2\xi}{\cos\xi\sin\xi} +\\
&+2\dot\eta_1\dot\eta_{n-3}\frac{\cos\eta_1}{\sin\eta_1} + \ldots + 2\dot\eta_{n-4}\dot\eta_{n-3}\frac{\cos\eta_{n-4}}{\sin\eta_{n-4}} -\\
&-\dot\eta_{n-2}^2\sin\eta_{n-3}\cos\eta_{n-3} = 0,\\
&\ddot\eta_{n-2} + b_*\dot\eta_{n-2}\cos\xi + \dot\xi\dot\eta_{n-2}\frac{1+\cos^2\xi}{\cos\xi\sin\xi} +\\
&+2\dot\eta_1\dot\eta_{n-2}\frac{\cos\eta_1}{\sin\eta_1} + \ldots + 2\dot\eta_{n-3}\dot\eta_{n-2}\frac{\cos\eta_{n-3}}{\sin\eta_{n-3}} = 0,\ \ b_* > 0,
\end{aligned}
\tag{54}
$$

which describes a fixed n-dimensional pendulum in a flow of a running medium for which the moment of forces is independent of the angular velocity, i.e., a mechanical system in a nonconservative field (see [1, 2]). In general, the order of such a system is equal to $2(n-1)$, but the phase variable η_{n-2} is a cyclic variable, which leads to the stratification of the phase space and reduces the order of the system.

The phase space of this system is the tangent bundle

$$TS^{n-1}\{\dot{\xi}, \dot{\eta}_1, \ldots, \dot{\eta}_{n-2}, \xi, \eta_1, \ldots, \eta_{n-2}\} \tag{55}$$

of the $(n-1)$-dimensional sphere $\mathbf{S}^{n-1}\{\xi, \eta_1, \ldots, \eta_{n-2}\}$. The equations that transform system (54) into the system on the tangent bundle of the two-dimensional sphere $\dot{\eta}_2 \equiv \dot{\eta}_3 \equiv \ldots \equiv \dot{\eta}_{n-2} \equiv 0$, and the equations of great circles $\dot{\eta}_1 \equiv 0$, $\dot{\eta}_2 \equiv 0, \ldots, \dot{\eta}_{n-2} \equiv 0$ define families of integral manifolds.

It is easy to verify that system (54) is equivalent to the dynamical system with variable dissipation with zero mean on the tangent bundle (55) of the $(n-1)$-dimensional sphere. Moreover, the following theorem holds.

Theorem 4. System (3), (6) under conditions (13), (31), (11), and (12), is equivalent to the dynamical system (54).

Indeed, it suffices to set $\alpha = \xi$, $\beta_1 = \eta_1$, $\ldots$, $\beta_{n-2} = \eta_{n-2}$, $b = -b_*$.

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