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Event-Triggered Control of Sampled-Data Nonlinear Systems ⋆

Ruslan E.Seifullaev *,**,***, **Alexander L.Fradkov** *,**,***

* *Institute for Problems in Mechanical Engineering, St.Petersburg, Russia (e-mail: ruslan.seifullaev@yandex.ru, fradkov@mail.ru)*
** *ITMO University, Saint Petersburg, Russia*
*** *Department of Theoretical Cybernetics, Saint-Petersburg State University, St.Petersburg, Russia*

Abstract: The paper combines authors' previous results on extending Emilia Fridman's method to a class of nonlinear systems with sector bounded nonlinearity with the recent results of A.Selivanov and E.Fridman on a switching approach to event-triggered control. In this paper the sampled-data control under continuous event-trigger is considered. The closed-loop system is represented as the system switching between periodic and event-triggered sampling. Applying Fridman's method and Yakubovich's S-procedure the problem is reduced to feasibility analysis of linear matrix inequalities. Particularly it is demonstrated that the event-trigger can reduce the network workload by the example on synchronization of Chua's circuits.

Keywords: nonlinear control systems, event-trigger, sampled-data control, LMI, S-procedure

1. INTRODUCTION

Digital controllers have become a standard mean for implementation of control systems. Including the computer into the control loop leads naturally to the fact that the resulting system will have a hybrid discrete-continuous dynamics. Sampling effects have been studied intensively in the early stages of research and design of hybrid systems. Later the problem has lost its importance with the growth of computing power. However, nowadays there is a new wave of interest in the study of sampling effects due to active use of communication networks (including the Internet) in control. Due to the fact that these networks are significantly loaded, its use imposes information limitations, and an important problem of the proper choice of the sampling interval providing stability and the desired performance of the control system becomes essential. This problem is by no means trivial even for linear systems if one needs to evaluate nonconservative bounds for maximum admitted value of sampling interval. As for nonlinear systems the problem is not well studied despite its importance.

Recently in the literature an interest has grown up in a novel approach to the sampling time evaluation based on transformation of discrete-continuous system to continuous delayed system with time-varying delay. The origin of the idea can be traced back to Mikheev et al. (1988); Fridman (1992). However being combined with the descriptor method of delayed systems analysis proposed by Fridman (2001) the idea has become equipped with powerful calculation tools based on LMI and has become a powerful design method allowing one to dramatically reduce conservativity of the design Fridman et al. (2004); Fridman (2010).

In the authors' previous papers Seifullaev and Fradkov (2013, 2016) the extension of Fridman's method to a class of nonlinear systems with sector bounded nonlinearities was proposed. The aim was to evaluate the upper bound of the sampling interval below which the system is absolutely stable. The effect of sampling was considered as delay followed by the construction and use of Lyapunov–Krasovskii functional Fridman (2010). With S-procedure Yakubovich et al. (2004) the estimation of sampling step was reduced to feasibility analysis of linear matrix inequalities.

To reduce the amount of information transmitted over network one can use continuous event-trigger that allows to avoid the packets when the output signal does not significantly change the control signal. In Selivanov and Fridman (2015) a switching approach to event-triggered control was proposed, where the closed-loop system was represented as switching system between periodic sampling and event-trigger. In the current paper we demonstrate the extension of this approach to a class of nonlinear systems with sector bounded nonlinearity. Two examples (one of them is the problem of Chua's circuit synchronization), confirming the possibility of reducing the workload of the network with this approach, are given in Section 5.

2. PROBLEM STATEMENT

Consider the nonlinear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + q\xi(t) + Bu(t), \\ y(t) &= C^{\mathrm{T}}x(t), \quad \sigma(t) = r^{\mathrm{T}}x(t), \\ \xi(t) &= \varphi(\sigma(t), t), \end{aligned} \tag{1}$$

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where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^l$ and $\sigma(t) \in \mathbb{R}$ are the outputs, $u(t) \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{n\times l}$ are constant matrices, $q \in \mathbb{R}^n, r \in \mathbb{R}^n$ are constant vectors.

Assume that $\xi(t) = \varphi(\sigma(t), t)$ is the nonlinear function (see Fig.1) satisfying sector condition

$$\mu_1 \sigma^2 \leqslant \sigma\,\xi \leqslant \mu_2\,\sigma^2, \tag{2}$$

for all $t \geqslant 0$ where $\mu_1 < \mu_2$ are real numbers.

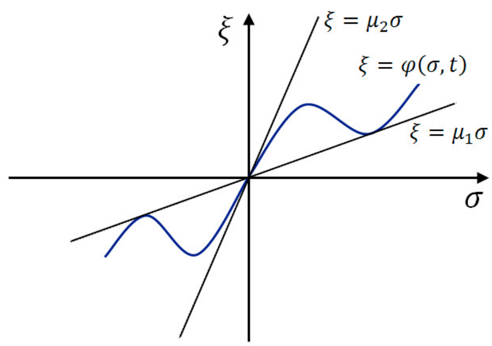

Fig. 1. Sector bounded nonlinearity

Remark 1. Let us provide some examples of sector bounded nonlinearities, satisfying (2):

- $\xi = sin(\sigma)$: $\mu_1 \approx -0.2173$, $\mu_2 = 1$ (see. Fig. 2),

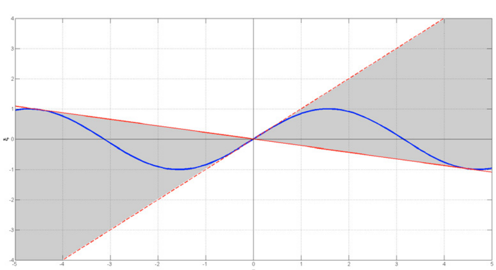

Fig. 2. $\xi = sin(\sigma)$

- $\xi = sin(\sigma^2)$: $\mu_1 \approx -0.855$, $\mu_2 \approx 0.855$ (see. Fig. 3),

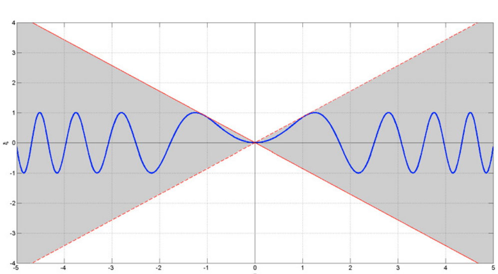

Fig. 3. $\xi = sin(\sigma^2)$

- relay with dead zone, saturation, piecewise-linear function etc. (see Khalil (2002)).

Consider a sampled-time feedback control law

$$u(t) = Ky(t_k), \;\; t_k \leqslant t < t_{k+1}, \tag{3}$$

where $K \in \mathbb{R}^{m\times l}$, and $\{t_k\}$ is an infinite sequence of sampling times $0 = t_0 < t_1 < \ldots < t_k < \;\ldots$.

Based on the input-delay approach Mikheev et al. (1988); Fridman et al. (2004) the closed loop system

$$\dot{x}(t) = Ax(t) + q\xi(t) + BKC^{\mathrm{T}}x(t_k) \tag{4}$$

can be rewritten as the system with variable "sawtooth" delay:

$$\dot{x}(t) = Ax(t) + q\xi(t) + BKC^{\mathrm{T}}x(t - \tau(t)), \tag{5}$$

where $\tau(t) = t - t_k$, $t_k \leqslant t < t_{k+1}$.

Assume that the sampling time sequence is generated by continuous event-trigger

$$t_{k+1} = \min\left\{t \geqslant t_k + h \,|\, (y(t) - y(t_k))^{\mathrm{T}}\Omega\,(y(t) - y(t_k)) \geqslant \varepsilon\, y^{\mathrm{T}}(t)\,\Omega\, y(t)\right\}, \tag{6}$$

where $\Omega \in \mathbb{R}^{p\times p}$ is a constant positive semi-defined matrix, h, ε are nonnegative scalars. Since event-trigger (6) waits for h value until checking the switching condition, it can avoid Zeno phenomenon. Such event-trigger was used in, e.g. Tallapragada and Chopra (2012a,b); Selivanov and Fridman (2015).

It is required to establish the exponential stability conditions of the system (1),(3) under event-trigger (6).

3. A SWITCHING APPROACH

In Selivanov and Fridman (2015) an approach to sampling of linear systems consisting in considering the switching between periodic sampling and continuous event-trigger was proposed. According to this approach system (1), (3) can be considered as the following switching system:

$$\dot{x}(t) = \begin{cases} Ax(t) + q\xi(t) + BKC^{\mathrm{T}}x(t-\tau(t)), \\ \qquad\qquad \text{if } t \in [t_k, t_k + h), \\ (A + BKC^{\mathrm{T}})x(t) + q\xi(t) + BKe(t), \\ \qquad\qquad \text{if } t \in [t_k + h, t_{k+1}), \end{cases} \tag{7}$$

where

$$\begin{aligned} &y(t) = C^{\mathrm{T}}x(t), \quad \sigma(t) = r^{\mathrm{T}}x(t), \quad \xi(t) = \varphi(\sigma(t), t), \\ &\tau(t) = t - t_k \leqslant h, \quad t \in [t_k, t_k + h), \\ &e(t) = y(t_k) - y(t), \quad t \in [t_k + h, t_{k+1}). \end{aligned} \tag{8}$$

4. STABILITY ANALYSIS BASED ON LYAPUNOV–KRASOVSKII FUNCTIONAL METHOD AND S-PROCEDURE

Definition 1. The space of absolutely continuous on $[-h, 0)$ functions $f : [-h, 0] \to \mathbb{R}^n$ having square integrable first-order derivatives is denoted by W with the norm

$$\|f\|_W = \max_{\theta\in[-h,0]} |f(\theta)| + \left[\int_{-h}^{0} |\dot{f}(s)|^2 ds\right]^{\frac{1}{2}}.$$

Denote $x_t(\theta) : [-h, 0] \to \mathbb{R}^n$ as $x_t(\theta) = x(t + \theta)$, where $x(\theta) \equiv 0$ if $\theta \in [-h, 0)$.

Definition 2. System (7), (8) will be called exponentially stable with the decay rate $\alpha > 0$ if there exists $\beta > 1$ such that for solution $x(t)$ of (7), (8) with initial condition x_{t_0} the following estimate holds

$$|x(t)|^2 \leqslant \beta e^{-2\alpha(t-t_0)}\,\|x_{t_0}\|_W^2, \;\; \forall t \geqslant t_0.$$

The proof of our main result is based on the following auxiliary statement that can be proved along the lines of Lemma 1 in Fridman (2010).

Lemma 1. Let there exist positive numbers β_1, β_2 and a functional $V_1 : \mathbb{R} \times W \times L_2[-h, 0] \to \mathbb{R}$ such that

$$\beta_1 \, |\phi(0)|^2 \leqslant V_1(t, \phi, \dot{\phi}) \leqslant \beta_2 \, \|\phi\|_W^2, \tag{9}$$

and a radially unbounded function $V_2 : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$, $V(x) > 0$ for $x \neq 0$.
Let the function

$$\bar{V}(t) = \begin{cases} V_1(t, x_t, \dot{x}_t), & \text{if } t \in [t_k, t_k + h), \\ V_2(x(t)), & \text{if } t \in [t_k + h, t_{k+1}) \end{cases}$$

be continuous from the right for $x(t)$ satisfying (7), (8) , absolutely continuous for $t \neq t_k$, $t \neq t_k + h$ and satisfies

$$\lim_{t \to t_k^-} \bar{V}(t) \geqslant \bar{V}(t_k), \qquad \lim_{t \to (t_k+h)^-} \bar{V}(t) \geqslant \bar{V}(t_k + h). \tag{10}$$

Given $\alpha > 0$ if along $x(t)$

$$\dot{\bar{V}}(t) + 2\alpha \bar{V}(t) \leqslant 0 \tag{11}$$

almost for all t then (7), (8) is exponentially stable with the decay rate α.

For the second system of (7) consider the quadratic form with $n \times n$ matrix $P = P^{\mathrm{T}} > 0$ as the function V_2:

$$V_2(x(t)) = x^{\mathrm{T}}(t) P x(t).$$

For the first system of (7) consider the following functional (introduced in Fridman (2010)) on $\mathbb{R} \times W \times L_2[-h, 0]$:

$$V(t, x_t, \dot{x}_t) \;=\; V_P(x_t) \;+\; V_Q(t, \dot{x}_t) \;+\; V_R(t, x_t), \tag{12}$$

where

$$V_P(x_t) = x_t(0)^T P x_t(0),$$
$$V_Q(t, \dot{x}_t) = (h - \tau(t)) \int_{-\tau(t)}^{0} e^{2\alpha s} \, \dot{x}_t^{\mathrm{T}}(s) \, Q \, \dot{x}_t(s) ds,$$
$$V_R(t, x_t) = (h - \tau(t)) \; \zeta^{\mathrm{T}}(t, x_t) R \, \zeta(t, x_t),$$
$$R = \begin{bmatrix} \dfrac{X + X^{\mathrm{T}}}{2} & -X + X_1 \\ * & -X_1 - X_1^{\mathrm{T}} + \dfrac{X + X^{\mathrm{T}}}{2} \end{bmatrix},$$

and the matrix P is as above, $Q = Q^{\mathrm{T}} > 0$ is $n \times n$ positive definite matrix, the vector $\zeta(t, x_t) = \left[x_t(0), x_{t-\tau(t)}(0)\right]^{\mathrm{T}}$, $X \in \mathbb{R}^{n \times n}$, $X_1 \in \mathbb{R}^{n \times n}$ are some matrices.

To formulate the main result of the paper let us check the conditions of Lemma 1.

For fulfillment of (9) it is sufficient that

$$\Theta > 0, \tag{13}$$

where

$$\Theta = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} + hR.$$

Indeed,

$$x_t(0)^{\mathrm{T}} P x_t(0) + V_X(t, x_t) = \frac{h - \tau(t)}{h} \; \zeta^{\mathrm{T}} \, \Theta \, \zeta + \frac{\tau(t)}{h} \; \zeta^{\mathrm{T}} \, \Theta_{|h=0} \, \zeta \geqslant \beta_1 \, |x_t(0)|^2, \tag{14}$$

where $\beta_1 = \min(\nu_1, \nu_2)$, ν_1 and ν_2 are minimum eigenvalues of P and Θ respectively.

Define function $\bar{V}(t)$ as follows:

$$\bar{V}(t) = \begin{cases} \bar{V}_1(t, x(t), \dot{x}(t)), & \text{if } t \in [t_k, t_k + h), \\ V_2(x(t)), & \text{if } t \in [t_k + h, t_{k+1}) \end{cases}$$

with

$$\bar{V}_1(t, x(t), \dot{x}(t)) = \bar{V}_P(x(t)) + \bar{V}_Q(t, \dot{x}(t)) + \bar{V}_R(t, x(t)),$$

where

$$\bar{V}_P(x(t)) = V_2(x(t)),$$
$$\bar{V}_Q(t, \dot{x}(t)) = (h - \tau(t)) \int_{-\tau(t)}^{0} e^{2\alpha s} \, \dot{x}^{\mathrm{T}}(t + s) \, Q \, \dot{x}(t + s) ds,$$
$$\bar{V}_R(t, x(t)) = (h - \tau(t)) \; \bar{\zeta}^{\mathrm{T}}(t, x(t)) R \, \bar{\zeta}(t, x(t)),$$

and the vector $\bar{\zeta}(t, x(t)) = [x(t), x(t - \tau(t))]^{\mathrm{T}}$.

Since $\tau(t) = 0$ at $t \to t_k^+$ and $\tau(t) = h$ at $t \to (t_k + h)^-$,

- $\lim_{t \to t_k^+} \bar{V}_Q(t, \dot{x}(t)) = \lim_{t \to (t_k+h)^-} \bar{V}_Q(t, \dot{x}(t)) = 0,$
- $\lim_{t \to t_k^+} \bar{V}_R(t, x(t)) = \lim_{t \to (t_k+h)^-} \bar{V}_R(t, x(t)) = 0.$

Therefore, $\bar{V}(t)$ is continuous and condition (10) holds.

Let us check the condition (11). First consider the second equation of (7). By direct calculations one has

$$\dot{V}_2(t) + 2\alpha V_2(t) \leqslant 2x^{\mathrm{T}}(t) P \dot{x}(t) + 2\alpha x^{\mathrm{T}}(t) P x(t). \tag{15}$$

Along the solutions of (7) the following equality holds on $[t_k + h, t_{k+1})$:

$$2\left[x^{\mathrm{T}}(t) R_2^{\mathrm{T}} + \dot{x}^{\mathrm{T}}(t) R_3^{\mathrm{T}}\right] \times \left[(A + BKC^{\mathrm{T}}) x(t) + q\xi(t) + BKe(t) - \dot{x}(t)\right] = 0, \tag{16}$$

where R_2, R_3 are some $n \times n$ matrices. From the switching condition of event-trigger (6) on $[t_k + h, t_{k+1})$ one has

$$\varepsilon x^{\mathrm{T}}(t) \, C^{\mathrm{T}} \Omega \, C \, x(t) - e^{\mathrm{T}}(t) \, \Omega \, e(t) \geqslant 0. \tag{17}$$

Add the left-hand sides of (16) and (17) to (15). Then

$$\dot{V}_2(t) + 2\alpha V_2(t) \leqslant \eta_0^{\mathrm{T}}(t) \, \Psi_0 \, \eta_0(t),$$

where $\eta_0(t) = [x(t), \dot{x}(t), \xi(t), e(t)]^{\mathrm{T}}$, $\eta_0 \in \mathbb{R}^{2n+l+1}$ and

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & R_2^{\mathrm{T}} q & R_2^{\mathrm{T}} BK \\ * & -R_3 - R_3^{\mathrm{T}} & R_3^{\mathrm{T}} q & R_3^{\mathrm{T}} BK \\ * & * & 0 & 0 \\ * & * & * & -\Omega \end{bmatrix},$$

where "$*$" stands for corresponding block of the symmetric matrix and

$$\Psi_{11} = (A + BKC^{\mathrm{T}})^{\mathrm{T}} R_2 + R_2^{\mathrm{T}} (A + BKC^{\mathrm{T}}) + 2\alpha P + \varepsilon \, C^{\mathrm{T}} \Omega \, C,$$
$$\Psi_{12} = P - R_2^{\mathrm{T}} + (A + BKC^{\mathrm{T}})^{\mathrm{T}} R_3.$$

Therefore, for the fulfillment of

$$\dot{V}_2(t) + 2\alpha V_2(t) \leqslant 0 \tag{18}$$

it is sufficient to require

$$\Psi < 0. \tag{19}$$

Now evaluate the left hand side of (11) for the first system of (7) on $[t_k, t_k + h)$. Since $\frac{d}{dt} x(t - \tau(t)) = (1 - \dot{\tau}(t)) \dot{x}(t - \tau(t)) = 0$, one obtains

$$\dot{\bar{V}}_1(t) + 2\alpha \bar{V}_1(t) \leqslant 2x^{\mathrm{T}}(t) P \dot{x}(t) + 2\alpha x^{\mathrm{T}}(t) P x(t) + (h - \tau(t)) \dot{x}^{\mathrm{T}}(t) Q \dot{x}(t) - e^{-2\alpha h} \int_{-\tau(t)}^{0} \dot{x}^{\mathrm{T}}(t+s) Q \dot{x}(t+s) ds - \zeta^T(t) R \, \zeta(t) + 2\alpha \bar{V}_R(t, x(t)) + (h - \tau(t)) \times \left[\dot{x}^{\mathrm{T}}(t) (X + X^{\mathrm{T}}) x(t) + 2\dot{x}^{\mathrm{T}}(t) (-X + X_1) x(t - \tau(t))\right]. \tag{20}$$

Denote

$$v_1(t) = \frac{1}{\tau(t)} \int_{-\tau(t)}^{0} \dot{x}(t + s) ds, \tag{21}$$

where right hand side of (21) for $\tau(t)=0$ is understood as $\lim_{\tau(t)\to 0} v_1 = \dot{x}(t)$.

From the Jensen's inequality Gu et al. (2003) one has

$$\int_{-\tau(t)}^{0} \dot{x}^{\mathrm{T}}(t+s)\, Q\, \dot{x}(t+s)ds \geqslant \tau(t)\, v_1^{\mathrm{T}} Q v_1. \qquad (22)$$

Along the solutions of (7) the following equalities hold on $[t_k, t_k+h)$

$$\begin{aligned} 0 &= 2\left[-x(t)+x(t-\tau(t))+\tau(t)v_1\right] \\ &\quad\times\left[x^{\mathrm{T}}(t)Y_1^{\mathrm{T}}+\dot{x}^{\mathrm{T}}(t)Y_2^{\mathrm{T}}+x^{\mathrm{T}}(t-\tau(t))T^{\mathrm{T}}+\xi q^{\mathrm{T}}Y_3^{\mathrm{T}}\right], \\ 0 &= 2\left[x^{\mathrm{T}}(t)P_2^{\mathrm{T}}+\dot{x}^{\mathrm{T}}(t)P_3^{\mathrm{T}}\right] \\ &\quad\times\left[Ax(t)+\mathcal{B}(t)KC^{\mathrm{T}}x(t-\tau(t))+q\xi(t)-\dot{x}(t)\right], \end{aligned} \qquad (23)$$

where $P_2, P_3, Y_1, Y_2, Y_3, T$ are some $n\times n$ matrices.

Denote $\eta_1(t) = [x(t), \dot{x}(t), x(t-\tau(t)), \xi(t), v_1(t)]^{\mathrm{T}}$.

Adding (23) to the right-hand side of (20) and using (22) one obtains

$$\dot{\bar{V}}_1(t) + 2\alpha\bar{V}_1(t) \leqslant \eta_1^{\mathrm{T}}(t)\Psi(t)\,\eta_1(t), \qquad (24)$$

where

$$\Psi(t) = \begin{bmatrix} \Phi_{11}(t) & \Phi_{12}(t) & \Phi_{13}(t) & \Phi_{14} & \tau(t)Y_1^{\mathrm{T}} \\ * & \Phi_{22}(t) & \Phi_{23}(t) & \Phi_{24} & \tau(t)Y_2^{\mathrm{T}} \\ * & * & \Phi_{33}(t) & \Phi_{34} & \tau(t)T^{\mathrm{T}} \\ * & * & * & 0 & \tau(t)q^{\mathrm{T}}Y_3^{\mathrm{T}} \\ * & * & * & * & -\tau(t)Qe^{-2\alpha h} \end{bmatrix}. \qquad (25)$$

where

$$\begin{aligned} \Phi_{11}(t) &= A^{\mathrm{T}}P_2 + P_2^{\mathrm{T}}A + 2\alpha P - Y_1 - Y_1^{\mathrm{T}} \\ &\quad - (1-2\alpha(h-\tau(t)))\frac{X+X^{\mathrm{T}}}{2}, \\ \Phi_{12}(t) &= P - P_2^{\mathrm{T}} + A^{\mathrm{T}}P_3 - Y_2 + (h-\tau(t))\frac{X+X^{\mathrm{T}}}{2}, \\ \Phi_{13}(t) &= Y_1^{\mathrm{T}} + P_2^{\mathrm{T}}BKC^{\mathrm{T}} - T \\ &\quad + (1-2\alpha(h-\tau(t)))(X-X_1), \\ \Phi_{22}(t) &= -P_3 - P_3^{\mathrm{T}} + (h-\tau(t))Q, \\ \Phi_{23}(t) &= Y_2^{\mathrm{T}} + P_3^{\mathrm{T}}BKC^{\mathrm{T}} - (h-\tau(t))(X-X_1), \\ \Phi_{33}(t) &= T + T^{\mathrm{T}} \\ &\quad - (1-2\alpha(h-\tau(t)))\frac{X+X^{\mathrm{T}}-2X_1-2X_1^{\mathrm{T}}}{2}, \\ \Phi_{14} &= P_2^{\mathrm{T}}q - Y_3 q, \quad \Phi_{24} = P_3^{\mathrm{T}}q, \quad \Phi_{34} = Y_3 q. \end{aligned}$$

Thus, to check condition (11) it is sufficient to verify that the matrix $\Psi(t)$ is nonpositive for all $t \geqslant 0$. Consider the following linear matrix inequalities:

$$\Psi_0 < 0, \qquad (26)$$
$$\Psi_1 < 0, \qquad (27)$$

where

$$\Psi_0 = \begin{bmatrix} \Phi_{11|\tau(t)=0} & \Phi_{12|\tau(t)=0} & \Phi_{13|\tau(t)=0} & \Phi_{14} \\ * & \Phi_{22|\tau(t)=0} & \Phi_{23|\tau(t)=0} & \Phi_{24} \\ * & * & \Phi_{33|\tau(t)=0} & \Phi_{34} \\ * & * & * & 0 \end{bmatrix},$$

$$\Psi_1 = \begin{bmatrix} \Phi_{11|\tau(t)=h} & \Phi_{12|\tau(t)=h} & \Phi_{13|\tau(t)=h} & \Phi_{14} & hY_1^{\mathrm{T}} \\ * & \Phi_{22|\tau(t)=h} & \Phi_{23|\tau(t)=h} & \Phi_{24} & hY_2^{\mathrm{T}} \\ * & * & \Phi_{33|\tau(t)=h} & \Phi_{34} & hT^{\mathrm{T}} \\ * & * & * & 0 & hq_1^T Y_3^{\mathrm{T}} \\ * & * & * & * & -hQe^{-2\alpha h} \end{bmatrix}.$$

Denote $\eta_0(t) = col\,\{x(t), \dot{x}(t), x(t-\tau(t)), \xi(t)\}$. Then (26) and (27) imply $\Psi(t) < 0 \quad \forall t > 0$ because

$$\begin{aligned} \frac{h-\tau(t)}{h}\,\eta_0^T\,\Psi_0\,\eta_0 + \frac{\tau(t)}{h}\,\eta_1^T\,\Psi_1\,\eta_1 & \\ = \eta_1^T\Psi(t)\,\eta_1 < 0, \ \ \forall \eta_1 \neq 0. & \end{aligned}$$

Denote

$$F_0(\eta_0) = \eta_0^T\,\Psi_0\,\eta_0, \quad F_1(\eta_1) = \eta_1^T\,\Psi_1\,\eta_1. \qquad (28)$$

Thus, if

$$\begin{aligned} F_0(\eta_0) < 0, \ \ \forall \eta_0 \neq 0, \\ F_1(\eta_1) < 0, \ \ \forall \eta_1 \neq 0, \end{aligned} \qquad (29)$$

then condition (11) of Lemma 1 holds.

Introduce quadratic forms

$$\begin{aligned} G_0(\eta_0) &= (\xi - \mu_1 r^{\mathrm{T}}x)(\mu_2 r^{\mathrm{T}}x - \xi), \\ G_1(\eta_1) &= (\xi - \mu_1 r^{\mathrm{T}}x)(\mu_2 r^{\mathrm{T}}x - \xi). \end{aligned}$$

Note that the forms $G_0(\eta_0)$ and $G_1(\eta_1)$ are defined on different spaces. From (2) the following inequalities holds along trajectories of system (7), (8):

$$G_0(\eta_0) \geqslant 0, \quad G_1(\eta_1) \geqslant 0.$$

Therefore, we can require that the first inequality of (29) holds in the set $G_0(\eta_0) \geqslant 0$, i. e.

$$F_0(\eta_0) < 0 \text{ if } G_0(\eta_0) \geqslant 0 \ \ \forall \eta_0 \neq 0. \qquad (30)$$

Similarly,

$$F_1(\eta_1) < 0 \text{ if } G_1(\eta_1) \geqslant 0 \ \ \forall \eta_1 \neq 0. \qquad (31)$$

Let us transform (30) and (31) using S-procedure Yakubovich et al. (2004). Consider the following forms:

$$\begin{aligned} S_0(\eta_0) &= F_0(\eta_0) + \varkappa_0 G_0(\eta_0), \\ S_1(\eta_1) &= F_1(\eta_1) + \varkappa_1 G_1(\eta_1), \end{aligned}$$

and require them to be negative for some non-negative $\varkappa_0$ and $\varkappa_1$ respectively:

$$\exists \varkappa_0 \geqslant 0: \ S_0(\eta_0) < 0, \ \ \forall \eta_0 \neq 0, \qquad (32)$$

$$\exists \varkappa_1 \geqslant 0: \ S_1(\eta_1) < 0, \ \ \forall \eta_1 \neq 0. \qquad (33)$$

Theorem 1. (S-Procedure, Yakubovich (1971)). Let $F \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times n}$, and there exists $\eta_0 \in \mathbb{R}^n : \eta_0^{\mathrm{T}} G \eta_0 > 0$. The following statements are equivalent:

- $\eta^{\mathrm{T}} F \eta > 0$ for any $\eta \neq 0$, $\eta \in \mathbb{R}^n$ satisfying $\eta^{\mathrm{T}} G \eta \geqslant 0$
- there exists a real scalar $\varkappa \geqslant 0$, such that $F - \varkappa G > 0$.

From Theorem 1 conditions (30) and (32) are equivalent. Similarly, (31) and (33) are equivalent. Therefore, if conditions (32) and (33) hold, then (11) is fulfilled. Using (20), (28) we obtain the following inequalities:

$$S_0(\eta_0) \leqslant \eta_0^{\mathrm{T}}\,\Psi_{S0}\,\eta_0, \quad S_1(\eta_1) \leqslant \eta_1^{\mathrm{T}}\,\Psi_{S1}\,\eta_1,$$

where

$$\Psi_{S0} = \Psi_0 + \begin{bmatrix} \Phi_{S1} & 0 & 0 & \Phi_{S2} \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & \Phi_{S3} \end{bmatrix},$$

$$\Psi_{S1} = \Psi_1 + \begin{bmatrix} \Phi_{S4} & 0 & 0 & \Phi_{S5} & 0 \\ * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & \Phi_{S6} & 0 \\ * & * & * & * & 0 \end{bmatrix},$$

and

$$\Phi_{S1} = -\varkappa_0\mu_1\mu_2 rr^T, \quad \Phi_{S2} = \frac{1}{2}\varkappa_0(\mu_1+\mu_2)r,$$
$$\Phi_{S3} = -\varkappa_0, \quad \Phi_{S4} = -\varkappa_1\mu_1\mu_2 rr^T,$$
$$\Phi_{S5} = \frac{1}{2}\varkappa_1(\mu_1+\mu_2)r, \quad \Phi_{S6} = -\varkappa_1.$$

Hence, if

$$\Psi_{S0} < 0, \tag{34}$$
$$\Psi_{S1} < 0, \tag{35}$$

then (11) holds.

Therefore, we arrive at the main result:

Theorem 2. Given $\alpha > 0$, let there exist matrices $P \in \mathbb{R}^{n\times n}$ $(P > 0)$, $Q \in \mathbb{R}^{n\times n}$ $(Q > 0)$, $\Omega \in \mathbb{R}^{p\times p}$ $(\Omega > 0)$, $P_2 \in \mathbb{R}^{n\times n}$, $P_3 \in \mathbb{R}^{n\times n}$, $R_2 \in \mathbb{R}^{n\times n}$, $R_3 \in \mathbb{R}^{n\times n}$, $X \in \mathbb{R}^{n\times n}$, $X_1 \in \mathbb{R}^{n\times n}$, $T \in \mathbb{R}^{n\times n}$, $Y_1 \in \mathbb{R}^{n\times n}$, $Y_2 \in \mathbb{R}^{n\times n}$, $Y_3 \in \mathbb{R}^{n\times n}$ and positive real numbers $\varkappa_0$ and $\varkappa_1$, such that LMIs (13), (19), (34) and (35) are feasible. Then system (7), (8) is exponentially stable with decay rate α. If LMIs (13), (19), (34) and (35) are feasible for $\alpha = 0$, then (7), (8) is exponentially stable with a small enough decay rate.

5. EXAMPLES

5.1 Example 1

Consider the following system:

$$\begin{aligned} \dot{x}_1(t) &= -2x_1(t) + \sin x_2(t), \\ \dot{x}_2(t) &= x_1(t) - x_2(t) + 2\sin x_2(t) + u(t) \\ u(t) &= -Kx_2(t_k), \end{aligned} \tag{36}$$

where time instances t_k are generated by event-trigger (6).

System (36) can be rewritten as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t) - BKC^{\mathrm{T}}x(t_k) + q\xi(t), \\ \sigma(t) &= r^{\mathrm{T}}x(t), \quad \xi(t) = \varphi(\sigma(t)), \end{aligned} \tag{37}$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \; A = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$q = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad r = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \xi(t) = \sin\sigma(t) - \sigma(t).$$

Note that the nonlinearity $\xi(t)$ satisfies the following sector inequalities (see Fig. 4):

$$\mu_1\,\sigma^2 \leqslant \sigma\,\xi \leqslant \mu_2\,\sigma^2$$

for all $t \geqslant 0$ where $\mu_1 = -1.2173, \mu_2 = 0$. Hence, condition (2) is fulfilled.

In Seifullaev and Fradkov (2015) it was found that the system (36) is exponentially stable for $K > 1.44$ in the case of continuous control.

Let $K = 2.9$, initial conditions $x(0) = [3;\ 0]$ and time of simulation $T_f = 15$.

For $\varepsilon = 0$ (without event-trigger) and $\alpha = 0$ from Theorem 2 one obtains that the maximum period h, providing the exponential stability of (36) with a small enough decay rate, is 0.55. In this case the average amount of sent measurements (SM) is $\left[\dfrac{T_f}{h}\right] + 1 = 28$, where $[\cdot]$ is the integer part.

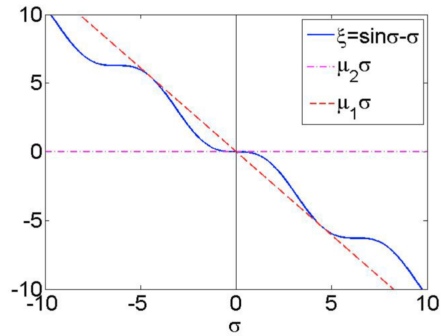

Fig. 4. Sector bounded nonlinearity

Next attempt to reduce the average amount of sampling measurements by using the event-trigger (6). From Theorem 2 we found that the minimum of SM occurs for $\varepsilon = 0.24, h = 0.54$. In this case SM = 14. The trajectories of closed-loop system are illustrated on Fig. 5.

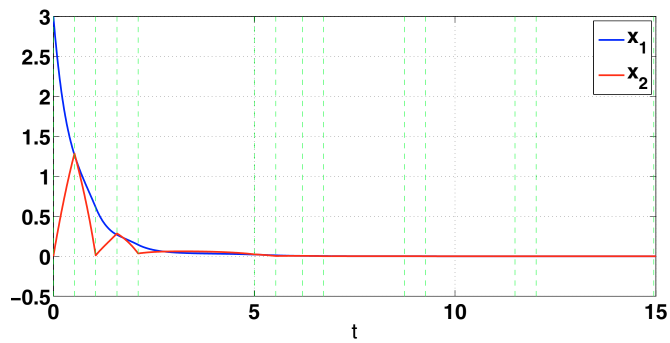

Fig. 5. Trajectories of closed-loop system under the event-trigger (6). Green dotted lines denote the time instants t_k. $K = 2.9, \varepsilon = 0.24, h = 0.54, x(0) = [3;\ 0]$

Therefore, using the event-trigger allows to twice reduce the average amount of sampling measurements compared with periodic sampling. The results are summarized in Table 1.

	ε	h	SM
Periodic sampling	0	0.55	28
Event-trigger	0.24	0.54	14

Table 1. The average amount of sampling measurements

5.2 Example 2. Synchronization of Chua's circuits.

Consider the Chua's circuit differential equations for a master system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + q\xi_1(t), \quad y_1(t) = C^{\mathrm{T}}x(t), \\ \sigma_1(t) &= r^{\mathrm{T}}x(t), \quad \xi_1(t) = \varphi(\sigma_1(t)), \end{aligned}$$

and for a slave system

$$\begin{aligned} \dot{z}(t) &= Az(t) + q\xi_2(t) + u(t), \quad y_2(t) = C^{\mathrm{T}}z(t), \\ \sigma_2(t) &= r^{\mathrm{T}}z(t), \quad \xi_2(t) = \varphi(\sigma_2(t)), \\ u(t) &= K(y_1(t_k) - y_2(t_k)), \quad t_k \leqslant t < t_{k+1}. \end{aligned}$$

Fig. 6 illustrates the "double scroll" chaotic attractor of the master system with following parameters:

$$A = \begin{bmatrix} -2.57 & 9 & 0 \\ 1 & -1 & 1 \\ 0 & -14.28 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 3.86 \\ 0 \\ 0 \end{bmatrix}, \quad C = r = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\varphi(\omega) = \frac{1}{2}(|\omega + 1| - |\omega - 1|), \quad x(0) = [0.6;\ -0.9;\ -1.1].$$

Denote $e(t) = x(t) - z(t)$, $\eta(t) = \xi_1(t) - \xi_2(t)$ and write

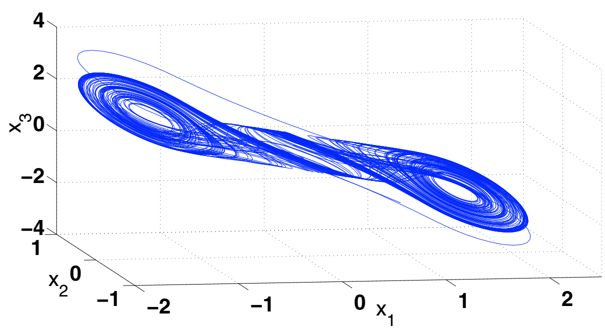

Fig. 6. Chaotic attractor of the master system

down the error system

$$\dot{e}(t) = Ae(t) + q\eta(t) + KC^{\mathrm{T}} e(t_k), \qquad (38)$$

where $\eta(t)$ belongs to the sector $[0, 1]$ for all t.

Let $K = [2.5,\ 1,\ -1]^{\mathrm{T}}$, $z(0) = [-1;\ 3;\ 0.9]$ and $T_f = 15$.

For $\varepsilon = 0$ (without event-trigger) and $\alpha = 0$ from Theorem 2 one obtains that the maximum period h, providing the exponential stability of (38) with a small enough decay rate, is 0.22, and SM=70. With the event-trigger (6) the SM is quite reduced for $\varepsilon = 0.058$ and $h = 0.21$, and is equal to 60. The trajectories of error system (38) are illustrated on Fig. 7.

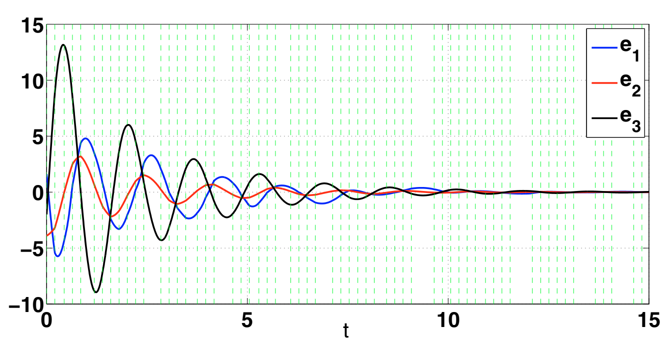

Fig. 7. Synchronization errors of Chua's circuits. $K = [2.5,\ 1,\ -1]^{\mathrm{T}}$, $\varepsilon = 0.058, h = 0.21$

The results are summarized in Table 2.

	ε	h	SM
Periodic sampling	0	0.22	70
Event-trigger	0.058	0.21	60

Table 2. The average amount of sampling measurements

6. CONCLUSIONS

The paper combines authors' previous results on extending Fridman's method to a class of nonlinear systems with sector bounded nonlinearity with the results of Selivanov and Fridman (2015) on a switching approach to event-triggered control. The closed-loop system is represented as switching system between periodic sampling and event-trigger. With Fridman's method and Yakubovich's S-procedure the problem is reduced to feasibility analysis of linear matrix inequalities. The obtained results are illustrated by two examples, demonstrating the possibility of reducing the workload of the network with the proposed approach.

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