Event-Triggered Control of Sampled-Data Nonlinear Systems

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Abstract: The paper combines authors’ previous results on extending Emilia Fridman’s method to a class of nonlinear systems with sector bounded nonlinearity with the recent results of A. Selivanov and E. Fridman on a switching approach to event-triggered control. In this paper the sampled-data control under continuous event-trigger is considered. The closed-loop system is represented as the system switching between periodic and event-triggered sampling. Applying Fridman’s method and Yakubovich’s S-procedure the problem is reduced to feasibility analysis of linear matrix inequalities. Particularly it is demonstrated that the event-trigger can reduce the network workload by the example on synchronization of Chua’s circuits.

Keywords: nonlinear control systems, event-trigger, sampled-data control, LMI, S-procedure

1. INTRODUCTION

Digital controllers have become a standard mean for implementation of control systems. Including the computer into the control loop leads naturally to the fact that the resulting system will have a hybrid discrete-continuous dynamics. Sampling effects have been studied intensively in the early stages of research and design of hybrid systems. Later the problem has lost its importance with the growth of computing power. However, nowadays there is a new wave of interest in the study of sampling effects due to active use of communication networks (including the Internet) in control. Due to the fact that these networks are significantly loaded, its use imposes information limitations, and an important problem of the proper choice of the sampling interval providing stability and the desired performance of the control system becomes essential. This problem is by no means trivial even for linear systems if one needs to evaluate nonconservative bounds for maximum admitted value of sampling interval. As for nonlinear systems the problem is not well studied despite its importance.

Recently in the literature an interest has grown up in a novel approach to the sampling time evaluation based on transformation of discrete-continuous system to continuous delayed system with time-varying delay. The origin of the idea can be traced back to Mikheev et al. (1988); Fridman (1992). However being combined with the descriptor method of delayed systems analysis proposed by Fridman (2001) the idea has become equipped with powerful calculation tools based on LMI and has become a powerful design method allowing one to dramatically reduce conservativity of the design Fridman et al. (2004); Fridman (2010).

In the authors’ previous papers Seifullaev and Fradkov (2013, 2016) the extension of Fridman’s method to a class of nonlinear systems with sector bounded nonlinearities was proposed. The aim was to evaluate the upper bound of the sampling interval below which the system is absolutely stable. The effect of sampling was considered as delay followed by the construction and use of Lyapunov–Krasovskii functional Fridman (2010). With S-procedure Yakubovich et al. (2004) the estimation of sampling step was reduced to feasibility analysis of linear matrix inequalities.

To reduce the amount of information transmitted over network one can use continuous event-trigger that allows to avoid the packets when the output signal does not significantly change the control signal. In Selivanov and Fridman (2015) a switching approach to event-triggered control was proposed, where the closed-loop system was represented as switching system between periodic sampling and event-trigger. In the current paper we demonstrate the extension of this approach to a class of nonlinear systems with sector bounded nonlinearities. Two examples (one of them is the problem of Chua’s circuit synchronization), confirming the possibility of reducing the workload of the network with this approach, are given in Section 5.

2. PROBLEM STATEMENT

Consider the nonlinear system

\[ \dot{x}(t) = Ax(t) + q\xi(t) + Bu(t), \]
\[ y(t) = C^T x(t), \quad \sigma(t) = r^T x(t), \]
\[ \xi(t) = \varphi(\sigma(t), t), \]  

(1)

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10.1016/j.ifacol.2016.07.965
where \( x(t) \in \mathbb{R}^n \) is the state vector, \( y(t) \in \mathbb{R}^l \) and \( \sigma(t) \in \mathbb{R} \) are the control input, \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{n \times l} \) are constant matrices, \( q \in \mathbb{R}^n, r \in \mathbb{R}^n \) are constant vectors.

Assume that \( \xi(t) = \varphi(\sigma(t), t) \) is the nonlinear function (see Fig. 1) satisfying sector condition
\[
\mu_1 \sigma^2 \leq \sigma \xi \leq \mu_2 \sigma^2,
\]
for all \( t \geq 0 \) where \( \mu_1 < \mu_2 \) are real numbers.

\[\text{Fig. 1. Sector bounded nonlinearity}\]

**Remark 1.** Let us provide some examples of sector bounded nonlinearities, satisfying (2):
- \( \xi = \sin(\sigma) \): \( \mu_1 \approx -0.2173, \mu_2 = 1 \) (see. Fig. 2),
- \( \xi = \sin(\sigma^2) \): \( \mu_1 \approx -0.855, \mu_2 \approx 0.855 \) (see. Fig. 3),
- \( \xi = \varphi(\sigma, t) \),
- \( \xi = \varphi(\sigma^2, t) \).

\[\text{Fig. 2. } \xi = \sin(\sigma)\]

\[\text{Fig. 3. } \xi = \sin(\sigma^2)\]

Consider a sampled-time feedback control law
\[
u(t) = Ky(t_k), \quad t_k \leq t < t_{k+1},
\]  
where \( K \in \mathbb{R}^{m \times l} \), and \( \{t_k\} \) is an infinite sequence of sampling times \( 0 = t_0 < t_1 < \ldots < t_k < \ldots \).

Based on the input-delay approach Mikheev et al. (1988); Fridman et al. (2004) the closed loop system
\[
\dot{x}(t) = Ax(t) + q\xi(t) + BKC^T x(t_k)
\]
(4) can be rewritten as the system with variable "sawtooth" delay:
\[
\dot{x}(t) = Ax(t) + q\xi(t) + BKC^T x(t - \tau(t)),
\]
(5) where \( \tau(t) = t - t_k, \quad t_k \leq t < t_{k+1} \).

Assume that the sampling time sequence is generated by continuous event-trigger
\[
t_{k+1} = \min \left\{ t \geq t_k + h \mid (y(t) - y(t_k))^T \Omega (y(t) - y(t_k)) \geq \varepsilon y^T(t) \Omega y(t) \right\},
\]
(6) where \( \Omega \in \mathbb{R}^{P \times P} \) is a constant positive semi-defined matrix, \( h, \varepsilon \) are nonnegative scalars. Since event-trigger (6) waits for \( h \) value until checking the switching condition, it can avoid Zeno phenomenon. Such event-trigger was used in, e.g. Tallapragada and Chopra (2012a,b); Selivanov and Fridman (2015).

It is required to establish the exponential stability conditions of the system (1),(3) under event-trigger (6).

3. A SWITCHING APPROACH

In Selivanov and Fridman (2015) an approach to sampling of linear systems consisting in considering the switching between periodic sampling and continuous event-trigger was proposed. According to this approach system (1), (3) can be considered as the following switching system:
\[
\dot{x}(t) = \begin{cases}
    Ax(t) + q\xi(t) + BKC^T x(t - \tau(t)), & \text{if } t \in [t_k, t_k + h), \\
    (A + BKC^T) x(t) + q\xi(t) + BKe(t), & \text{if } t \in [t_k + h, t_{k+1}),
\end{cases}
\]
(7)

where
\[
y(t) = C^T x(t), \quad \sigma(t) = r^T x(t), \quad \xi(t) = \varphi(\sigma(t), t), \quad \tau(t) = t - t_k, \quad t \in [t_k, t_k + h), \quad t_k \leq n, \quad t \\
= y(t_k) - y(t), \quad t \in [t_k + h, t_{k+1}).
\]
(8)

4. STABILITY ANALYSIS BASED ON LYAPUNOV–KRASOVSKII FUNCTIONAL METHOD AND S-PROCEDURE

**Definition 1.** The space of absolutely continuous on \([-h, 0)\) functions \( f : [-h, 0) \to \mathbb{R}^n \) having square integrable first-order derivatives is denoted by \( W \) with the norm
\[
\|f\|_W = \max_{\theta \in [-h, 0]} |f(\theta)| + \left[ \frac{1}{h} \int_{-h}^0 |f(s)|^2 \, ds \right]^\frac{1}{2}.
\]

Denote \( x_k(\theta) : [-h, 0) \to \mathbb{R}^n \) as \( x_k(\theta) = x(t + \theta), \) where \( x(\theta) = 0 \) if \( \theta \in [-h, 0) \).

**Definition 2.** System (7), (8) will be called exponentially stable with the decay rate \( \alpha > 0 \) if there exists \( \beta > 1 \) such that for solution \( x(t) \) of (7), (8) with initial condition \( x_{t_0} \) the following estimate holds
\[
|\dot{x}(t)|^2 \leq \beta e^{-2\alpha(t-t_0)} \|x_{t_0}\|^2_W, \quad \forall t \geq t_0.
\]

The proof of our main result is based on the following auxiliary statement that can be proved along the lines of Lemma 1 in Fridman (2010).
Lemma 1. Let there exist positive numbers $\beta_1, \beta_2$ and a functional $V_1 : \mathbb{R} \times W \times L_2[-h,0] \to \mathbb{R}$ such that
\[ \beta_1 ||\phi||^2 \leq V_1(t,\phi,\dot{\phi}) \leq \beta_2 ||\phi||^2, \tag{9} \]
and a radially unbounded function $V_2 : \mathbb{R}^n \to \mathbb{R}$ such that $V'(0) = 0$, $V(x) > 0$ for $x \neq 0$.

Let the function
\[ \tilde{V}(t) = \begin{cases} V_1(t,x_k + \dot{x}_k), & \text{if } t \in [t_k, t_{k+1}], \\ V_2(x(t)), & \text{if } t \in [t_k + h, t_{k+1}] \end{cases} \]
be continuous from the right for $x(t)$ satisfying (7), (8), absolutely continuous for $t \neq t_k$, $t \neq t_k + h$ and satisfies
\[ \lim_{t \to t_k^-} \tilde{V}(t) \geq \tilde{V}(t_k), \quad \lim_{t \to (t_k + h)^-} \tilde{V}(t) \geq \tilde{V}(t_k + h). \tag{10} \]

Given $\alpha > 0$ if along $x(t)$
\[ \tilde{V}(t) + 2\alpha \tilde{V}(t) \leq 0 \tag{11} \]
after all $t$ then (7), (8) is exponentially stable with the decay rate $\alpha$.

For the second system of (7) consider the quadratic form with $n \times n$ matrix $P = PT > 0$ as the function $V_2$:
\[ V_2(x(t)) = x^T(t)Px(t). \tag{12} \]

For the first system of (7) consider the following function (introduced in Fridman (2010)) on $\mathbb{R} \times W \times L_2[-h,0]$:
\[ V(t,x_k + \dot{x}_k) = V_P(t,x_k) + V_Q(t,x_k) + V_R(t,x_k), \tag{13} \]

where
\[ V_P(x_k) = x_k(0)^TPx_k(0), \quad V_Q(t,x_k) = (h-\tau(t)) \int_{-\tau(t)}^0 e^{2\alpha s} \dot{x}_k^T(s) Q \dot{x}_k(s)ds, \]
\[ V_R(t,x_k) = (h-\tau(t)) \zeta^T(t,x_k)R\zeta(t,x_k), \]
\[ R = \begin{bmatrix} \frac{X + X^T}{2} & -X + X_1 \\ -X_1 & \frac{X + X^T}{2} \end{bmatrix}, \]

and the matrix $P$ is as above, $Q = Q^T > 0$ is $n \times n$ positive definite matrix, the vector $\zeta(t,x_k) = [x_k(0), x_k-\tau(t)(0)]^T$, $X \in \mathbb{R}^{n \times n}$, $X_1 \in \mathbb{R}^{n \times n}$ are some matrices.

To formulate the main result of the paper let us check the conditions of Lemma 1.

For fulfillment of (9) it is sufficient that
\[ \Theta > 0, \tag{14} \]

where
\[ \Theta = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} + hR. \]

Indeed,
\[ x_k(0)^TPx_k(0) + V_X(t,x_k) = \frac{h-\tau(t)}{h} \zeta^T(t,x_k) \Theta \zeta + \frac{\tau(t)}{h} \zeta^T(t,x_k) \Theta \zeta + 2 \beta_1 |x_k(0)|^2, \tag{15} \]

where $\beta_1 = \min(\nu_1, \nu_2)$, $\nu_1$ and $\nu_2$ are minimum eigenvalues of $P$ and $\Theta$ respectively.

Define function $\tilde{V}(t)$ as follows:
\[ \tilde{V}(t) = \begin{cases} \tilde{V}_1(t,x(t),\dot{x}(t)), & \text{if } t \in [t_k, t_k + h], \\ \tilde{V}_2(x(t)), & \text{if } t \in [t_k + h, t_{k+1}] \end{cases} \]
with
\[ \tilde{V}_1(t,x(t),\dot{x}(t)) = \tilde{V}_P(x(t)) + \tilde{V}_Q(t,\dot{x}(t)) + \tilde{V}_R(t,x(t)), \]

where
\[ \tilde{V}_P(x(t)) = V_2(x(t)), \]
\[ \tilde{V}_Q(t,\dot{x}(t)) = (h-\tau(t)) \int_{-\tau(t)}^0 e^{2\alpha s} \dot{x}^T(t+s)Q \dot{x}(t+s)ds, \]
\[ \tilde{V}_R(t,x(t)) = (h-\tau(t)) \zeta^T(t,x(t))R\zeta(t,x(t)), \]

and the vector $\zeta(t,x(t)) = [x(t), x(t-\tau(t))]^T$.

Since $\tau(t) = 0$ at $t \to t_k^+$ and $\tau(t) = h$ at $t \to (t_k + h)^-$,

- $\lim_{t \to t_k^+} \tilde{V}_Q(t,\dot{x}(t)) = \lim_{t \to (t_k + h)^-} \tilde{V}_Q(t,\dot{x}(t)) = 0,$
- $\lim_{t \to t_k^+} \tilde{V}_R(t,x(t)) = \lim_{t \to (t_k + h)^-} \tilde{V}_R(t,x(t)) = 0.$

Therefore, $\tilde{V}(t)$ is continuous and condition (10) holds.

Let us check the condition (11). First consider the second equation of (7). By direct calculations one has
\[ V_2(t) + 2\alpha V_2(t) \leq 2x^T(t)P\dot{x}(t) + 2a^T(t)PX(t). \tag{16} \]
Along the solutions of (7) the following equality holds on $[t_k + h, t_{k+1})$:
\[ 2 \left[ x^T(t)R_2^T + \dot{x}^T(t)R_3^T \right] \times [(A + BKC)^T x(t) + q^T(t) + BK \epsilon(t) - \dot{x}(t)] = 0, \tag{17} \]

where $R_2, R_3$ are some $n \times n$ matrices. From the switching condition of event-trigger (6) on $[t_k + h, t_{k+1})$ one has
\[ \epsilon x^T(t)C^T \Omega Cx(t) - \epsilon^T(t) \Omega \epsilon(t) \geq 0, \tag{18} \]

Add the left-hand sides of (16) and (17) to (15). Then
\[ \tilde{V}_2(t) + 2\alpha \tilde{V}_2(t) \leq v_0(t) \Psi \eta_0(t), \]

where $\eta_0(t) = [x(t), \dot{x}(t), \xi(t), \epsilon(t)]^T$, $\eta_0 \in \mathbb{R}^{2n+i+1}$ and
\[ \Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{12}^T R_2^T R_2 & \Psi_{22} \end{bmatrix}, \]

where "\$" stands for corresponding block of the symmetric matrix
\[ \Psi_{11} = (A + BKC)^T R_2 + R_2^T (A + BKC)^T + 2aP + \varepsilon C^T \Omega C, \]
\[ \Psi_{12} = P - R_2^T + (A + BKC)^T R_3. \]

Therefore, for the fulfillment of
\[ \tilde{V}_2(t) + 2\alpha \tilde{V}_2(t) \leq 0 \tag{19} \]
it is sufficient to require
\[ \Psi < 0. \tag{20} \]

Now evaluate the left hand side of (11) for the first system of (7) on $[t_k, t_k + h)$. Since $\frac{d}{dt}x(t-\tau(t)) = (1-\tau(t))\dot{x}(t-\tau(t)) = 0$, one obtains
\[ \tilde{V}_1(t) + 2\alpha \tilde{V}_1(t) \leq 2x^T(t)P\dot{x}(t) + 2a^T(t)PX(t) + (h-\tau(t))\dot{x}^T(t)Q \dot{x}(t) + e^{-2\alpha h} \int_{-\tau(t)}^0 \dot{x}^T(t+s)Q \dot{x}(t+s)ds \]
\[ - \dot{\zeta}^T(t)R \zeta(t) + 2a \dot{V}_1(t,x(t)) + (h-\tau(t)) \times \left[ \dot{x}^T(t)(X + X^T)x(t) + 2e^T(t)(-X + X_1)x(t-\tau(t)) \right]. \tag{21} \]

Denote
\[ v_1(t) = \frac{1}{\tau(t)} \int_{-\tau(t)}^0 \dot{x}(t+s)ds, \tag{22} \]
where right hand side of (21) for \( \tau(t) = 0 \) is understood as 
\[
\lim_{\tau(t) \to 0} v_1 = \dot{x}(t).
\]

From the Jensen’s inequality Gu et al. (2003) one has
\[
\int_{-r(t)}^{0} \dot{x}^T(t + s) Q \dot{x}(t + s) ds \geq \tau(t) v_1^T Q v_1. \tag{22}
\]

Along the solutions of (7) the following equalities hold on \([t_k, t_k + h)\)
\[
0 = 2 \left[ -x(t) + x(t - \tau(t)) + \tau(t) v_1 \right] \\
\times \left[ x^T(t)Y_1^T + \dot{x}(t)Y_2^T + x^T(t - \tau(t))T^T + \xi(t)G_0x \right],
\]
\[
0 = 2 \left[ x^T(t)P_2^T + \dot{x}(t)P_3^T \right] \\
\times \left[ A x(t) + B(t)K C^T x(t - \tau(t)) + \xi(t) - \dot{x}(t) \right],
\tag{23}
\]
where \(P_2, P_3, Y_1, Y_2, Y_3, T \) are some \( n \times n \) matrices.

Denote \( \eta_1(t) = \left[ x(t), \dot{x}(t), x(t - \tau(t)), \xi(t), v_1(t) \right]^T \).

Adding (23) to the right-hand side of (20) and using (22) one obtains
\[
\dot{V}_1(t) + 2aV_1(t) < \eta_1^T(t) \Psi(t) \eta_1(t),
\tag{24}
\]
where
\[
\Psi(t) = \begin{bmatrix}
\Phi_{11}(t) & \Phi_{12}(t) & \Phi_{13}(t) & \Phi_{14} \\
\Phi_{22}(t) & \Phi_{23}(t) & \Phi_{24} \\
\Phi_{33} & \Phi_{34} & \Phi_{35} \\
* & * & * & \tau(t)Q \xi(t) - \dot{x}(t)
\end{bmatrix}. \tag{25}
\]

Denote \( \eta_0(t) = \text{col} \{ x(t), \dot{x}(t), x(t - \tau(t)), \xi(t) \} \). Then (26) and (27) imply \( \Psi(t) < 0 \) \( \forall t > 0 \) because
\[
\frac{h - \tau(t)}{h} \eta_0^T \Psi_0 \eta_0 + \frac{\tau(t)}{h} \eta_1^T \Psi_1 \eta_1 = \eta_1^T \Psi(t) \eta_1 < 0, \quad \forall \eta_1 \neq 0.
\]

Denote
\[
F_0(\eta_0) = \eta_0^T \Psi_0 \eta_0, \quad F_1(\eta_1) = \eta_1^T \Psi_1 \eta_1. \tag{28}
\]

Thus, if
\[
F_0(\eta_0) < 0, \quad \forall \eta_0 \neq 0, \tag{29}
\]
then condition (11) of Lemma 1 holds.

Introduce quadratic forms
\[
G_0(\eta_0) = (\xi - \mu_1 \tau)^T (\mu_2 \tau T x - \xi),
\]
\[
G_1(\eta_1) = (\xi - \mu_1 \tau)^T (\mu_2 \tau T x - \xi).
\]

Note that the forms \( G_0(\eta_0) \) and \( G_1(\eta_1) \) are defined on different spaces. From (2) the following inequalities holds along trajectories of system (7), (8):
\[
G_0(\eta_0) \geq 0, \quad G_1(\eta_1) \geq 0.
\]

Therefore, we can require that the first inequality of (29) holds in the set \( G_0(\eta_0) \geq 0, \) i.e.
\[
F_0(\eta_0) < 0 \quad \text{if} \quad G_0(\eta_0) \geq 0, \quad \forall \eta_0 \neq 0. \tag{30}
\]

Similarly,
\[
F_1(\eta_1) < 0 \quad \text{if} \quad G_1(\eta_1) \geq 0, \quad \forall \eta_1 \neq 0. \tag{31}
\]

Let us transform (30) and (31) using S-procedure Yakubovich et al. (2004). Consider the following forms:
\[
S_0(\eta_0) = F_0(\eta_0) + \alpha_0 G_0(\eta_0),
\]
\[
S_1(\eta_1) = F_1(\eta_1) + \alpha_1 G_1(\eta_1),
\]
and require them to be negative for some non-negative \( \alpha_0 \) and \( \alpha_1 \) respectively:
\[
\exists \alpha_0 \geq 0 : \quad S_0(\eta_0) < 0, \quad \forall \eta_0 \neq 0, \tag{32}
\]
\[
\exists \alpha_1 \geq 0 : \quad S_1(\eta_1) < 0, \quad \forall \eta_1 \neq 0. \tag{33}
\]

Theorem 1. (S-Procedure, Yakubovich (1971)). Let \( F \in \mathbb{R}^{n \times n}, \) \( G \in \mathbb{R}^{m \times n} \), and there exists \( \eta_0 \in \mathbb{R}^n : \eta_0^T G_0 > 0 \).

The following statements are equivalent:
\bullet \eta^T F \eta > 0 \quad \text{for any} \quad \eta \in \mathbb{R}^n \quad \text{satisfying} \quad \eta^T G \eta > 0.
\bullet \exists \alpha > 0 \quad \text{such that} \quad F - \alpha G > 0.

From Theorem 1 conditions (30) and (32) are equivalent. Similarly, (31) and (33) are equivalent. Therefore, if conditions (32) and (33) hold, then (11) is fulfilled. Using (20), (28) we obtain the following inequalities:
\[
S_0(\eta_0) \leq \eta_0^T \Psi S_0 \eta_0, \quad S_1(\eta_1) \leq \eta_1^T \Psi S_1 \eta_1,
\] where
\[
\Psi_0 = \begin{bmatrix}
\Phi_{11} \tau(t) = 0 & \Phi_{12} \tau(t) = 0 & \Phi_{13} \tau(t) = 0 & \Phi_{14} \\
* & \Phi_{22} \tau(t) = 0 & \Phi_{23} \tau(t) = 0 & \Phi_{24} \\
* & * & \Phi_{33} \tau(t) = 0 & \Phi_{34} \\
* & * & * & 0
\end{bmatrix},
\]
\[
\Psi_1 = \begin{bmatrix}
\Phi_{11} \tau(t) = h & \Phi_{12} \tau(t) = h & \Phi_{13} \tau(t) = h & \Phi_{14} \\
* & \Phi_{22} \tau(t) = h & \Phi_{23} \tau(t) = h & \Phi_{24} \\
* & * & \Phi_{33} \tau(t) = h & \Phi_{34} \\
* & * & * & -hQe^{-2\alpha h}
\end{bmatrix}.
\]

and
\[ \Phi_{S1} = -\kappa_0 \mu_1 \mu_2 r T, \quad \Phi_{S2} = \frac{1}{2} \kappa_0 (\mu_1 + \mu_2) r, \]
\[ \Phi_{S3} = -\kappa_0, \quad \Phi_{S4} = -\kappa_1 \mu_2 r T, \]
\[ \Phi_{S5} = \frac{1}{2} \kappa_1 (\mu_1 + \mu_2) r, \quad \Phi_{S6} = -\kappa_1. \]

Hence, if
\[ \Psi_{S0} < 0, \tag{34} \]
\[ \Psi_{S1} < 0, \tag{35} \]
then (11) holds.

Therefore, we arrive at the main result:

**Theorem 2.** Given \( \alpha > 0 \), let there exist matrices \( P \in \mathbb{R}^{n \times n} (P > 0), Q \in \mathbb{R}^{n \times n} (Q > 0), \Omega \in \mathbb{R}^{p \times p} (\Omega > 0), \)
\( P_2 \in \mathbb{R}^{n \times n}, P_3 \in \mathbb{R}^{n \times n}, R_2 \in \mathbb{R}^{n \times n}, R_3 \in \mathbb{R}^{n \times n}, \)
\( X \in \mathbb{R}^{n \times n}, X_1 \in \mathbb{R}^{n \times n}, T \in \mathbb{R}^{n \times n}, Y_1 \in \mathbb{R}^{n \times n}, \)
\( Y_2 \in \mathbb{R}^{n \times n}, Y_3 \in \mathbb{R}^{n \times n} \) and positive real numbers \( \kappa_0 \) and \( \kappa_1 \), such that LMIs (13), (19), (34) and (35) are feasible.

Then system (7), (8) is exponentially stable with a small enough decay rate. If LMIs (13), (19), (34) and (35) are feasible for \( \alpha = 0 \), then (7), (8) is exponentially stable with a small enough decay rate.

### 5. EXAMPLES

#### 5.1 Example 1

Consider the following system:
\[ \dot{x}_1(t) = -2x_1(t) + \sin x_2(t), \]
\[ \dot{x}_2(t) = x_1(t) - x_2(t) + 2 \sin x_2(t) + u(t), \tag{36} \]
\[ u(t) = -Kx_3(t_k), \]
where time instances \( t_k \) are generated by event-trigger (6).

System (36) can be rewritten as follows:
\[ \dot{x}(t) = Ax(t) - BK C^T x(t_k) + q \xi(t), \]
\[ \sigma(t) = r^T x(t), \quad \xi(t) = \varphi(\sigma(t)), \tag{37} \]
where
\[ x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \]
\[ q = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad r = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \xi(t) = \sin \sigma(t) - \sigma(t). \]

Note that the nonlinearity \( \xi(t) \) satisfies the following sector inequalities (see Fig. 4):
\[ \mu_1 \sigma^2 \leq \sigma \xi \leq \mu_2 \sigma^2 \]
for all \( t \geq 0 \) where \( \mu_1 = -1.2173, \mu_2 = 0 \). Hence, condition (2) is fulfilled.

In Seifullaev and Fradkov (2015) it was found that the system (36) is exponentially stable for \( K > 1.44 \) in the case of continuous control.

Let \( K = 2.9 \), initial conditions \( x(0) = [3; 0] \) and time of simulation \( T_f = 15 \).

For \( \varepsilon = 0 \) (without event-trigger) and \( \alpha = 0 \) from Theorem 2 one obtains that the maximum period \( h \), providing the exponential stability of (36) with a small enough decay rate, is 0.55. In this case the average amount of sent measurements (SM) is \( \left\lceil \frac{T_f}{h} \right\rceil + 1 = 28 \), where \( \lceil \cdot \rceil \) is the integer part.

Next attempt to reduce the average amount of sampling measurements by using the event-trigger (6). From Theorem 2 we found that the minimum of SM occurs for \( \varepsilon = 0.24, h = 0.54 \). In this case SM = 14. The trajectories of closed-loop system are illustrated on Fig. 5.

![Fig. 4. Sector bounded nonlinearity](image)

![Fig. 5. Trajectories of closed-loop system under the event-trigger (6). Green dotted lines denote the time instances \( t_k \).](image)

Table 1. The average amount of sampling measurements.

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>( h )</th>
<th>SM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.55</td>
<td>28</td>
</tr>
<tr>
<td>0.24</td>
<td>0.54</td>
<td>14</td>
</tr>
</tbody>
</table>

#### 5.2 Example 2. Synchronization of Chua's circuits.

Consider the Chua's circuit differential equations for a master system
\[ \dot{x}(t) = Ax(t) + q \xi_1(t), \quad y_1(t) = C^T x(t), \]
\[ \sigma_1(t) = r^T x(t), \quad \xi_1(t) = \varphi(\sigma_1(t)), \]
and for a slave system
\[ \dot{z}(t) = Az(t) + q \xi_2(t) + u(t), \quad y_2(t) = C^T z(t), \]
\[ \sigma_2(t) = r^T z(t), \quad \xi_2(t) = \varphi(\sigma_2(t)), \]
\[ u(t) = K(y_1(t_k) - y_2(t_k)), \quad t_k \leq t < t_{k+1}. \]

Fig. 6 illustrates the "double scroll" chaotic attractor of the master system with following parameters:

![Fig. 6. Double Scroll](image)
where
\[ \kappa \] for all \( P \)

Let \( \kappa > 0 \). \varepsilon\) is the rate, is 0

For \( \varepsilon = 0 \) (without event-trigger) and \( \alpha = 0 \) from Theorem 2 one obtains that the maximum period \( h \), providing the exponential stability of (38) with a small enough decay rate, is 0.22, and SM=70. With the event-trigger (6) the SM is quite reduced for \( \varepsilon = 0.058 \) and \( h = 0.21 \), and is equal to 60. The trajectories of error system (38) are illustrated on Fig. 7.

![Chaotic attractor of the master system](image)

**Fig. 6.** Chaotic attractor of the master system
down the error system
\[ \dot{e}(t) = Ae(t) + \eta(t) + KC^T e(t_k), \] (38)
where \( \eta(t) \) belongs to the sector \([0, 1]\) for all \( t \).

Let \( K = [2.5, 1, -1]^T \), \( z(0) = [-1; 3; 0.9] \) and \( T_f = 15 \).

For \( \varepsilon = 0 \) (without event-trigger) and \( \alpha = 0 \) from Theorem 2 one obtains that the maximum period \( h \), providing the exponential stability of (38) with a small enough decay rate, is 0.22, and SM=70. With the event-trigger (6) the SM is quite reduced for \( \varepsilon = 0.058 \) and \( h = 0.21 \), and is equal to 60. The trajectories of error system (38) are illustrated on Fig. 7.

![Synchronization errors of Chua’s circuits](image)

**Fig. 7.** Synchronization errors of Chua’s circuits. \( K = [2.5, 1, -1]^T, \varepsilon = 0.058, h = 0.21 \)

The results are summarized in Table 2.

<table>
<thead>
<tr>
<th>Periodic sampling</th>
<th>( \varepsilon )</th>
<th>( h )</th>
<th>SM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Event-trigger</td>
<td>0.058</td>
<td>0.21</td>
<td>60</td>
</tr>
</tbody>
</table>

**Table 2.** The average amount of sampling measurements

6. CONCLUSIONS

The paper combines authors’ previous results on extending Fridman’s method to a class of nonlinear systems with sector bounded nonlinearity with the results of Selivanov and Fridman (2015) on a switching approach to event-triggered control. The closed-loop system is represented as switching system between periodic sampling and event-trigger. With Fridman’s method and Yakubovich’s S-procedure the problem is reduced to feasibility analysis of linear matrix inequalities. The obtained results are illustrated by two examples, demonstrating the possibility of reducing the workload of the network with the proposed approach.

REFERENCES


