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Structured adaptive control for solving LMIs

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Abstract: Numerical problems such as finding eigenvalues, singular value decomposition, linear programming, are traditionally solved with algorithms that can be interpreted as discrete-time processes. One can also find in the literature continuous-time methods for these same problems where solutions are the equilibrium points to which converge stable differential equations. The paper exposes one such continuous-time method for solving linear matrix inequalities. The proposed differential equations are those of an adaptive control feedback loop on an LTI system. The adaptive law is passivity-based with additional structural constraints of two types. The first constraint imposes the gain to be block-diagonal at all times. It can be interpreted as a decentralized control structure. The second constraint is only required asymptotically. It for example reads as requiring the feedback gain to be symmetric when time goes to infinity. Point-wise global stability is proved with quadratic Lyapunov functions. Results are illustrated on LMIs related to an H_∞ norm computation problem. Solutions to the LMIs are obtained by simulations in Simulink.

1. INTRODUCTION

Passivity-based adaptive control is a rich and lively field of research. At the core of all results is the assumption that the system can be made passive via static output feedback. This property is sometimes called "almost passivity". Based on almost passivity assumption, adaptive laws are constructed both for stabilization and for model reference tracking (MRAC: model reference adaptive control). Since almost passive properties do not hold in general, most theoretical efforts have been dedicated to modifications of the adaptive control scheme that relax the assumptions. See for example Kaufman et al. [1994], Barkana et al. [2006], Fradkov [2003], Peaucelle et al. [2011], Hsu et al. [2011], Son and Seo [2002] to cite just a few. But when reading all of these results remains a question: why doing adaptive control if one knows the existence of a static law that does as good a job? The usual answer is that computing the static law needs knowledge of model parameters while adaptation will find these values by itself. The adaptive law hence works as a solver for finding static gains that stabilize linear systems. If a stabilizing static gain exists it is usually not unique, but the adaptive law will converge to one of such.

The contribution of the paper is to take advantage of this feature for solving linear matrix inequalities (LMIs). Finding solutions to LMI constraints, also known as semi-definite programming (SDP), is a major issue in applied mathematics with many applications to control problems. The references Boyd et al. [1994], El Ghaoui and Niculescu [2000], Scherer [2006] give an idea of how important is this issue for the robust control field, and these are only part of all applications of LMIs. The other reason for which LMI-based results have become popular is that SDP is a convex problem for which polynomial-time algorithms have been proposed. LMIlab by Gahinet and Nemirovskii [1993] was the first one and many others have been proposed since. All are interfaced by a unique easy to use parser YALMIP by Löfberg [2001].

Our aim is not to compete with these solvers, but to explore an alternative continuous-time approach. Most optimization tools are algorithms that can be interpreted as discrete-time systems $x_{k+1} = f_d(x_k)$. Convergence to optimal values is from this view point related to uniqueness of asymptotically stable equilibrium points. An interesting related topic is finding continuous-time versions of the algorithms, that is differential equations $\dot{x}(t) = f_c(x(t))$ with the same stable equilibria. Amazing results exist in that field and many are described in Helmke and Moore [1996]. One of such continuous-time algorithm solves by simulation of ODEs the linear programing (LP) problem, see Brockett [1991]. Since LP is a sub case of SDP, our contribution can be seen as an extension of that result. Yet, our result is not obtained by the same approach. Our contribution takes advantage of adaptive control properties for solving LMIs via continuous-time algorithms.

The paper is organized as follows. First, we recall classical passivity-based adaptive control results and state that solving the LMIs for passifying static gain design and simulating the adaptive law are linked problems. In the next section, we show that all LMI problems can be rewritten as static output feedback passification problems provided that some structural constraints are added to the static feedback gains. The fourth section gives a new adaptive law for solving this structured pacification issue. The structural constraints are twofold. One

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is that the gains are block diagonal. This is handled at all times via decentralized-like adaptive control. The second structural constraints are only required asymptotically. It for example reads as requiring the feedback gain to be symmetric when time goes to infinity. Convergence of the LMI solving adaptive law is proved by means of Lyapunov theory. In section 6 the results are illustrated on a simple LMI problem. The adaptive laws are simulated in Simulink. It is shown how the gains converge to the structured (symmetric) solution of the LMI problem. Finally, conclusions are drawn and orientations are given for future work.

Notation: $\mathbf{1}$ stands for the identity matrix. A^T is the transpose of A. $\{A\}^{\mathcal{S}}$ stands for the symmetric matrix $\{A\}^{\mathcal{S}} = A + A^T$. $\mathsf{Tr}(A)$ is the trace of A. $a = \mathsf{vec}(A)$ is a vector composed of all the components of the matrix A taken column-wise. $\mathsf{mat}(a)$ is the reverse operation such that $\mathsf{mat}(\mathsf{vec}(A)) = A$. $\mathsf{diag}[\cdots\ F_i\ \cdots]$ stands for a bloc-diagonal matrix whose diagonal blocs are the F_i matrices.

2. MATRIX INEQUALITIES FOR PASSIVITY BASED ADAPTIVE CONTROL

Consider the following static output-feedback stabilization and passification problems (see Peaucelle and Fradkov [2008] for the passivity-related definitions):

Problem 1. (Stabilization) Given an LTI system:

$$\dot{x}(t) = \mathbb{A}x(t) + \mathbb{B}u(t)\ ,\ y(t) = \mathbb{C}x(t) \tag{1}$$

find a static output feedback gain F_s such that the control law:

$$u(t) = F_s y(t) \tag{2}$$

makes the closed loop system asymptotically stable.

Problem 2. (Passification) Given an LTI system:

$$\dot{x}(t) = \mathbb{A}x(t) + \mathbb{B}u(t)\ ,\ y(t) = \mathbb{C}x(t)\ ,\ z(t) = y(t) \tag{3}$$

find a static output feedback gain F_p such that the control law:

$$u(t) = F_p y(t) + w(t) \tag{4}$$

makes the closed loop system strictly passive with respect to inputs w and outputs z.

A system is said to be strictly passive if it is passive and asymptotically stable. Hence, all solutions to Problem 2 are solutions to Problem 1. The converse is not true.

In terms of matrix inequalities, Problem 1 reads as the existence of Q_s and F_s solution to the Lyapunov inequalities:

$$Q_s > 0,\ \{Q_s(\mathbb{A} + \mathbb{B}F_s\mathbb{C})\}^{\mathcal{S}} < \mathbf{0}, \tag{5}$$

while Problem 2 contains an additional equality constraint:

$$Q_p > 0,\ \{Q_p(\mathbb{A} + \mathbb{B}F_p\mathbb{C})\}^{\mathcal{S}} < \mathbf{0},\ Q_p\mathbb{B} = \mathbb{C}^T. \tag{6}$$

This result corresponds to the positive-real lemma (Boyd et al. [1994]) written for the system (3) connected in closed-loop with the static control law (4). Notice that while (5) is in general a hard non-convex problem, the inequalities for the passification problem happen to reduce to convex LMI conditions:

$$Q_p > 0,\ \{Q_p\mathbb{A}\}^{\mathcal{S}} + \mathbb{C}^T\{F_p\}^{\mathcal{S}}\mathbb{C},\ Q_p\mathbb{B} = \mathbb{C}^T. \tag{7}$$

Moreover, the passification problem has the following important feature in terms of adaptive control (see for example Fradkov and Andrievsky [2007], Barkana [2007]):

Theorem 1. If there exists a solution to Problem 2, then, for any $\Gamma > \mathbf{0}$, the following adaptive control law

$$u(t) = K(t)y(t) + w(t)\ \ ,\ \ \dot{K}(t) = -y(t)y^T(t)\Gamma \tag{8}$$

makes the closed-loop globally x-strictly passive.

The proof of the theorem is only sketched here, a detailed proof of a similar result is given in Section 4. Consider the Lyapunov function: $V(x, K) = x^T Q_p x + \mathsf{Tr}\left((K - F_p)\Gamma^{-1}(K - F_p)^T\right)$ where F_p and Q_p are a solutions to (6). After several computations, the time-derivative of V along the trajectories of the closed-loop system is proved to satisfy: $\dot{V} = -\varepsilon x^T x + wz$ for some positive $\epsilon > 0$. This proves x-strict passivity (passivity of the closed-loop and asymptotic convergence to zero of the systems state x).

This nowadays classical result is the core of adaptive control strategies, for example for adaptive stabilization of systems without knowledge of the $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ matrices. We shall not detail these results. Let us rather focus our attention to the asymptotic behavior of the adapted gain K.

Theorem 2. If there exists a solution to Problem 2, then, for any $\Gamma > \mathbf{0}$, the adaptive control law (8) makes the closed-loop globally x-strictly passive and if there are no disturbances K converges asymptotically to a value F_s solution of the stabilization Problem 1.

Proof: The proof follows the reasoning developed in Ioannou and Sun [1996]. If $w = 0$ and there are no disturbances on the closed-loop system, the condition $\dot{V} = -\varepsilon x^T x < 0$ implies that $x \in L_2$. As $y = \mathbb{C}x$ it also follows that $y \in L_2$ is square integrable. Taking into account the expression (8) of $\dot{K}$, one gets that $K(t)$ is expressed in terms of the integral of the square of y. It therefore converges $K(t) \to K(\infty) = F$. At this stage $(x = 0, K = F)$ is an equilibrium point proved to be asymptotically stable for the non-linear system with adaptive control. The first method of Lyapunov implies that the linearized system at the equilibrium point is asymptotically stable as well, that is $F = F_s$ is a solution to Problem 1. ∎

The value F_s to which $K(t)$ converges is not unique. It depends on the initial conditions $(x(0), K(0))$. Stability of the system with adaptive control is hence a pointwise stability property (see Goebel [2011]) of the set $\{(0, F_s)\ /\ F_s \text{ verifies } (5)\}$. If Problem 2 has a solution then simulating the closed-loop system (3)-(8) gives asymptotically a solution to Problem 1. Assuming Problem 2 is feasible the simulation gives a solution to the matrix inequality constraints (5). This property is used in the following for solving general LMI problems by means of simulation. More precisely, it is used in the following special case of symmetric systems for which Problems 1 and 2 are equivalent.

Lemma 1. Assume $\mathbb{A}$ is symmetric and $\mathbb{B} = \mathbb{C}^T$, then all symmetric solutions $F_s = F_s^T$ to Problem 1 are symmetric solutions $F_s = F_p = F_p^T$ to Problem 2.

Proof: For any symmetric matrix: $S < 0 \Leftrightarrow 2S = \{S\}^{\mathcal{S}} < 0$. Applied to $S = \mathbb{A} + \mathbb{B}F_s\mathbb{C}$ this means that $Q_s = \mathbf{1}$ is a trivial solution to the constraint (5). Moreover, since $Q_s\mathbb{B} = \mathbb{B} = \mathbb{C}^T$, $Q_p = Q_s = \mathbf{1}$ is also solution to (6). ∎

The property of Lemma 1 is essential. It paves the way for adaptive control based algorithms for solving the LMIs. Solutions of Problems 1 and 2 being the same, based on Theorem 2 these solutions may be obtained asymptotically by simulation.

3. LMIS ARE STRUCTURED PASSIFICATION PROBLEMS

This section shows that all LMIs can be seen as passivity problems, provided a structure property is added on the passifying gain. Namely, finding a solution to an LMI problem is equivalent to finding a structured, passifying gain for a fictive LTI system. This result is first proved for a classical LMI example: computation of the H_∞ norm of an LTI system. Then, starting from the canonical LMI expression, it is shown that the same property holds for any LMIs.

3.1 Example of LMIs: the H_∞ norm computation

Consider the example of the H_∞ analysis problem of an LTI system defined by the state space matrices (A, B, C, D), $B \in \mathrm{R}^{n,m}$, $C \in \mathrm{R}^{p,n}$ (see Boyd et al. [1994]): find P and γ^2 such that:

$$\begin{bmatrix} \{PA\}^{\mathcal{S}} + C^TC & PB + C^TD \\ B^TP + D^TC & D^TD - \gamma^2\mathbf{1} \end{bmatrix} < \mathbf{0},\ P = P^T > \mathbf{0}. \quad (9)$$

As any LMI problem it can be compacted into one inequality constraint :

$$\begin{bmatrix} \{PA\}^{\mathcal{S}} + C^TC & PB + C^TD & \mathbf{0} \\ B^TP + D^TC & D^TD - \gamma^2\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -P \end{bmatrix} < \mathbf{0},\ P = P^T. \quad (10)$$

It can further be decomposed in a sum with elementary matrix variables:

$$\mathbb{A} + \mathbb{C}_1^T \begin{bmatrix} \mathbf{0} & P & \mathbf{0} \\ P & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -P \end{bmatrix} \mathbb{C}_1 + \mathbb{C}_2^T(-\gamma^2\mathbf{1})\mathbb{C}_2 < 0\,,\ P = P^T \quad (11)$$

where $\mathbb{A} = \begin{bmatrix} C^T \\ D^T \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} C^T \\ D^T \\ \mathbf{0} \end{bmatrix}^T, \mathbb{C}_1 = \begin{bmatrix} A & B & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}, \mathbb{C}_2 = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}^T$.

Equivalently, by gathering all variables in a block-diagonal matrix one gets:

$$\mathbb{A} + \mathbb{C}^TF\mathbb{C} < 0 \quad (12)$$

with $\mathbb{C} = [\,\mathbb{C}_1\ \mathbb{C}_2\,]$, F block-diagonal $F = \mathsf{diag}[F_1, F_2]$ and blocks F_1 and F_2 structured as follows:

$$F_1 = \begin{bmatrix} \mathbf{0} & P & \mathbf{0} \\ P & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -P \end{bmatrix}, P = P^T, F_2 = -\gamma^2\mathbf{1}. \quad (13)$$

Let $\mathbb{B} = \mathbb{C}^T$. The constraint (12) corresponds to the statement that $\mathbb{A} + \mathbb{B}F\mathbb{C}$ is symmetric and Hurwitz stable. The LMI problem is hence transformed into a stabilization problem. Moreover, due to Lemma 1, in the considered case of symmetric matrices, the stabilization problem is equivalent to the passification problem. Yet, remains the important issue of imposing a structure to F (constraints (13) for the considered H_∞ norm example). For the sake of handling this issue, the structure constraints (13) are re-written as follows:

$$U_i\mathsf{vec}(F_i) = 0\ ,\ \ i \in \{1,\ 2\} \quad (14)$$

To illustrate how the U_i matrices can be constructed, consider the example of F_2, with $m = 2$: $F_2 = \begin{bmatrix} f_{21} & f_{22} \\ f_{23} & f_{24} \end{bmatrix}$. Imposing that F_2 verifies the structure constraint $F_2 = -\gamma^2\mathbf{1}$ amounts to choosing U_2:

$$U_2\mathsf{vec}(F_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} f_{21} \\ f_{23} \\ f_{22} \\ f_{24} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (15)$$

This imposes the off-diagonal elements to be zero (first two lines of U_2) and the two diagonal elements to be equal one to the other (third line). U_1 is constructed similarly.

3.2 Passification problems of canonical LMI representations

Proposition 1. Consider the canonical LMI problem:

$$L_0 + \sum_{j=1}^{M} \hat{x}_jL_j < 0 \quad (16)$$

where $L_j = L_j^T$ and $\hat{x}_j \in \mathrm{R}$ are the scalar decision variables, $j = 1 \dots M$. Finding a solution $\hat{x}_j$ to (16) is equivalent to finding a gain $\hat{F}$ which strictly passifies a system $(\hat{\mathbb{A}} = \hat{\mathbb{A}}^T, \hat{\mathbb{B}}, \hat{\mathbb{C}} = \hat{\mathbb{B}}^T)$ and respects structure constraints of the type:

$$\begin{cases} \hat{F} = \mathsf{diag}\left[\cdots\ \hat{F}_j\ \cdots\right] \\ \hat{U}_j\mathsf{vec}(\hat{F}_j) = 0 \end{cases} \quad (17)$$

that impose (among other constraints) the gain to be symmetric. The system matrices $(\hat{\mathbb{A}}, \hat{\mathbb{B}})$ are build based on L_0, L_j and the constraint matrices $\hat{U}_j$ depend of the number of positive and negative eigenvalues of L_j.

Proof. The matrices L_j are symmetric, they can therefore be factorized as $L_j = \bar{N}_j^{\,T}\bar{H}_j\bar{N}_j$, where $\bar{H}_j = \mathsf{diag}(\cdots\lambda_k\cdots)$ is diagonal and contains the non zero eigenvalues of L_j. Equivalently, taking $N_j = \mathsf{diag}(\cdots\sqrt{|\lambda_k|}\cdots)\bar{N}_j$, one gets $L_j = N_j^TH_jN_j$ where H_j is diagonal with diagonal elements equal to either 1 or -1. Hence, (16) reads as:

$$L_0 + \sum_{j=1}^{M} N_j^T\hat{F}_jN_j = \hat{\mathbb{A}} + \sum_{j=1}^{M} \hat{\mathbb{C}}_j^T\hat{F}_j\hat{\mathbb{C}}_j < 0 \quad (18)$$

where $\hat{\mathbb{A}} = L_0$, $\hat{\mathbb{C}}_j = N_j$ and

$$\hat{F}_j = \begin{bmatrix} \hat{x}_j\mathbf{1}_{r_{j1}} & \mathbf{0} \\ \mathbf{0} & -\hat{x}_j\mathbf{1}_{r_{j2}} \end{bmatrix} \quad (19)$$

depend of each individual variables x_j. r_{j1} and r_{j2} are respectively the number of positive and negative eigenvalues of matrices L_j. Similarly to the construction of U_2 in the previous example, the structure constraint (19) can be written as: $\hat{U}_j\mathsf{vec}(\hat{F}_j) = 0$ where the constraint matrices $\hat{U}_j$ depend on r_{j1} and r_{j2}. The end of the proof is based on Lemma 1. ■

A similar property can be formulated for LMI expressions containing matrix-valued variables.

Proposition 2. Consider the generic LMI problem:

$$\mathbb{A} + \sum_{i=1}^{N} \{R_i(G_i \otimes X_i)M_i\}^{\mathcal{S}} + N_i^T(H_i \otimes X_i)N_i < 0 \quad (20)$$

where X_i are matrix-valued decision variables and matrices G_i and H_i are invertible diagonal. Finding a solution to (20) is equivalent to finding a symmetric gain $F = F^T$ which strictly passifies the system $(\mathbb{A} = \mathbb{A}^T, \mathbb{B}, \mathbb{C} = \mathbb{B}^T)$ and respects structure constraints of the type:

$$\begin{cases} F = \mathsf{diag}\left[\cdots\ \hat{F}_i\ \cdots\right] \\ U_i\mathsf{vec}(F_i) = 0 \end{cases} \quad (21)$$

that impose (among other constraints) the gain to be symmetric. The matrix $\mathbb{B}$ depends only of the R_i, M_i, N_i matrices and the constraint matrices U_i are built out of G_i, H_i.

First note that the inequality in (11) enters the general formulation (20). The proof of the proposition follows the lines of the one given in the H_∞ example, where the variables are matrix valued. For conciseness reasons the proof is thus not reproduced here.

Remark 1. Note that (20) includes all possible LMI problems. Indeed, consider (18) which is equivalent to the canonical form (16). By taking $\mathbb{A} = L_0$, $N_i = N_j$, $H_i = \mathsf{diag}[\mathbf{1}_{r_{j1}}, -\mathbf{1}_{r_{j2}}]$ and $X_i = \hat{x}_j$, $R_i = \mathbf{0}$, $M_i = \mathbf{0}$, $i = j = 1 \dots N$, one can see that (18) is included in the formulation (20).

In comparison with the canonical LMI form (16), the expression (20) where the variables can be matrix-valued allows to keep close to the original formulations encountered in control theory. This is the case in the H_∞ problem, illustrated previously, but also in any other case from Boyd et al. [1994]. The second advantage of the matrix-valued formulation (20) is that the related structured passification problem is of lower dimensions in terms of size of the gain F and number of constraints U_i.

To illustrate the influence of the chosen LMI formula on the size of the structured passification problem let us consider the H_∞ norm analysis problem. The fictive system and the structure constraint matrices are constructed based on the two representations of the LMIs applied to (10). Results are given in Table 1, where n and m are respectively the order and number of inputs of the system to be analysed and $M = n(n+1)/2+1$.

Table 1. Size of the passification problem

	Considered LMI representation	
	(11)	(16)
Nb. of outputs of the fictive system	$3n + m$	$\sum_{j=1}^{M} r_j$
Nb. of coeficients in F	$(3n)^2 + m^2$	$\sum_{j=1}^{M} r_j^2$
Nb. scalar constraints	$(3n)^2 - \frac{n(n+1)}{2} + m - 1$	$\sum_{j=1}^{M} (r_j^2 - 1)$

When considering the canonical LMI representation, the size of the associated passification problem depends on $r_j = \mathsf{rank}(L_j)$. Although these ranks cannot be precisely estimated, some upper bounds can be evaluated. By looking at the initial LMI (10), one can see that diagonal coefficients of P appear three times, off-diagonal coefficients appear six times and γ appears m times. The corresponding ranks thus verify: $r_{p_{ii}} \leq 3$, $r_{p_{ij}} \leq 6$, $r_\gamma = m$. Examples show that these ranks are generally close to these upper bounds. Thus when starting from the compact matrix-valued form (11), much smaller passification problems are obtained. This is illustrated on a numerical example in Section 5.

4. STRUCTURED ADAPTIVE CONTROL

The previous section has shown that searching a solution to a given LMI problem is equivalent to searching a passifying, structured static output feedback gain for a system constructed based on the given LMI. Theorem 1 shows how adaptive control can be used for finding stabilizing gains, but the structure issue is not regarded. The structure constraints are of two types. First, the gains are block diagonal. To handle this constraint, a decentralized control law is considered, $u_i(t) = K_i(t)y_i(t)$, where the gains $K_i(t)$ are adapted independently. The gain $K = \mathsf{diag}[\cdots\ K_i\ \cdots]$ thus satisfies at all times the diagonal constraint. Second, the gains should satisfy (21). A modified adaptive law is proposed in order to constrain the gains K_i to satisfy the constraints asymptotically. The constraints are not satisfied at all times but gains converge to values $F_i = K_i(\infty)$, satisfying (21).

Theorem 3. Assume $\mathbb{A} = \mathbb{A}^T$ and $\mathbb{B} = [\cdots\ \mathbb{B}_i\ \cdots] = \mathbb{C}^T$ then the following two conditions are equivalent:

$i)$ There exists a decentralized static control $u_i(t) = F_i y_i(t)$ satisfying the structural constraints $U_i \mathsf{vec}(F_i) = 0$ (which include symmetry constraints $F_i = F_i^T$) that stabilizes asymptotically the system

$$\dot{x}(t) = \mathbb{A}x(t) + \sum \mathbb{B}_i u_i(t)\ ,\ y_i(t) = \mathbb{C}_i x(t). \tag{22}$$

$ii)$ For any $\Gamma_i > 0$, $\alpha_i > 0$ the following adaptive control

$$\begin{aligned} u_i(t) &= K_i(t)y_i(t) \\ \dot{K}_i(t) &= -y_i(t)y_i^T(t)\Gamma_i - \alpha_i \cdot \mathsf{mat}(U_i^T U_i \cdot \mathsf{vec}(K_i(t)))\Gamma_i \end{aligned} \tag{23}$$

makes the closed-loop system globally asymptotically stable and the adaptive gains converge to constant values $F_i = K_i(\infty)$ solution to condition *i)*.

Proof. First consider the implication *i)* $\Rightarrow$ *ii)*. It's proof is rather standard in adaptive control. The existence of a structured stabilizing static gain implies (see Lemma 1):

$$\mathbb{A} + \mathbb{B}F\mathbb{C} < -\epsilon \mathbf{1} < 0 \tag{24}$$

for some positive ϵ and where $F = \mathsf{diag}[\cdots\ F_i\ \cdots]$. For that values F_i let the following Lyapunov function

$$V = \frac{1}{2}\left(x^T x + \sum_i \mathsf{Tr}\left((K_i - F_i)\Gamma^{-1}(K_i - F_i)^T\right)\right)$$

The derivative of the Lyapunov function is such that

$$\dot{V} = \dot{x}^T x + \sum_i \mathsf{Tr}\left(\dot{K}_i \Gamma^{-1}(K_i - F_i)^T\right)$$

which, when incorporating the dynamics of the system and those of the control system, reads as

$$\begin{aligned} \dot{V} = x^T \mathbb{A}^T x &+ y_i^T K_i^T \mathbb{B}_i^T x - \mathsf{Tr}\left(y_i y_i^T (K_i - F_i)^T\right) \\ &-\alpha_i \mathsf{Tr}\left(\mathsf{mat}\left(U_i^T U_i \cdot \mathsf{vec}(K_i)\right)(K_i - F_i)^T\right). \end{aligned}$$

Equivalently (add and subtract $x^T \mathbb{C}^T F^T \mathbb{B} x = \sum_i y_i^T F_i^T \mathbb{B}_i^T x$) it gives:

$$\begin{aligned} \dot{V} = x^T (\mathbb{A} &+ \mathbb{B}F\mathbb{C})^T x \\ &+ y_i^T (K_i - F_i)^T \mathbb{B}_i^T x_i - \mathsf{Tr}\left(y_i y_i^T (K_i - F_i)^T\right) \\ &-\alpha_i \mathsf{Tr}\left(\mathsf{mat}\left(U_i^T U_i \cdot \mathsf{vec}(K_i)\right)(K_i - F_i)^T\right). \end{aligned}$$

Since $\mathbb{B}_i = \mathbb{C}_i^T$ one gets that $y_i^T (K_i - F_i)^T \mathbb{B}_i^T x = y_i^T (K_i - F_i)^T y_i$. Moreover, the following property holds:

$$\begin{aligned} &\mathsf{Tr}\left(y_i y_i^T (K_i - F_i)^T\right) \\ &\quad = \mathsf{Tr}\left(y_i^T (K_i - F_i)^T y_i\right) = y_i^T (K_i - F_i)^T y_i. \end{aligned}$$

Hence, two terms of the sum cancel by subtraction. Remains in the derivative of the Lyapunov function:

$$\begin{aligned} \dot{V} &= x^T (\mathbb{A} + \mathbb{B}F\mathbb{C})^T x \\ &- \sum_i \alpha_i \mathsf{Tr}\left(\mathsf{mat}\left(U_i^T U_i \cdot \mathsf{vec}(K_i)\right)(K_i - F_i)^T\right). \end{aligned}$$

The properties of the trace function also state that

$$\begin{aligned} &\mathsf{Tr}\left(\mathsf{mat}\left(U_i^T U_i \cdot \mathsf{vec}(K_i)\right)(K_i - F_i)^T\right) \\ &\quad = (\mathsf{vec}(K_i - F_i))^T U_i^T U_i \cdot \mathsf{vec}(K_i) \end{aligned}$$

where $(\mathsf{vec}(K_i - F_i))^T U_i^T = (U_i(\mathsf{vec}(K_i) - \mathsf{vec}(F_i)))^T = (U_i \cdot \mathsf{vec}(K_i))^T$ because the matrices F_i are assumed to satisfy the structural constraints. Finally, one gets

$$\dot{V} \leq -\epsilon x^T x - \sum_i \alpha_i (U_i \cdot \mathsf{vec}(K_i))^T (U_i \cdot \mathsf{vec}(K_i)).$$

The first element of this sum is negative for all non zero x. The last term is strictly negative when $U_i \mathsf{vec}(K_i) \neq 0$, that is, as long as the adaptive gains K_i do not satisfy the structure constraints. Hence, the Lyapunov function is strictly decreasing until $x = 0$ and $U_i\text{vec}(K_i) = 0$. The state of the system thus converges to zero and the gains K_i to a set where the structure constraint is satisfied.

To prove that the gains converge to constant values inside these sets first recall (as for the proof of Theorem 2) that $x \in L_2$ and thus $y_i \in L_2$. Then, apply the vec operator to (23) and denote $\bar{K}_i = \text{vec}(K_i)$, $V_i = -\alpha_i(\Gamma_i \otimes \mathbf{1})U_i^T U_i$:

$$\dot{\bar{K}}_i(t) = V_i \bar{K}_i(t) - \text{vec}(y_i(t) y_i^T(t) \Gamma_i). \tag{25}$$

The general solution of this differential equation is:

$$\bar{K}_i(t) = e^{V_i t}\bar{K}_{i0} + \int_0^t e^{V_i(t-\tau)} \cdot \text{vec}(y_i(\tau) y_i^T(\tau)\Gamma_i)\, \mathrm{d}\tau \tag{26}$$

where $V_i \le 0$ and $y_i \in L_2$. Therefore, the first term, as well as the integral in this expression converge as $t \to \infty$. We thus have $\bar{K}_i \to \bar{K}_i(\infty) = F_i$ a constant, included in the set such that $U_i\text{vec}(K_i(\infty)) = 0$. Applying first method of Lyapunov as in the proof of Theorem 2, the control gain F is proved to stabilize the system.

The proof of the implication *i)* $\Rightarrow$ *ii)* follows directly: if the adaptive controller converges to a stabilizing value $F = K(\infty)$ which respects the structure constraints, it is clear that such a value exists. ■

Theorem 3 applies to the stabilization/passification problems build out of the generic LMI constraints. Hence, finding a solution to the LMIs can be done by simulating the corresponding systems with adaptive control. This feature is illustrated in the next section on a simple example.

5. NUMERICAL EXAMPLE

Consider the stable system: $H(s) = \frac{s^2+s+1}{s^2+s+2}$. We aim at using the results presented in the previous sections in order to compute its H_∞ norm, or at least some upper bound on it. The exact value of the norm computed in Matlab with the `norm` function is $\|H\|_\infty = \gamma_{opt} = 1.3251$.

The problem amounts to solving LMIs (10). Based on the state-space representation of $H(s)$ - (A, B, C, D), the fictive system $(\mathbb{A}, \mathbb{B}, \mathbb{C} = \mathbb{B}^T)$ is constructed, as shown in Section 3.1. Theorem 3 states that by applying the adaptive control law (23) a solution to the LMI is given by the values of the adaptive gains, once they have converged.

As discussed, in the considered case, the adaptive control is decentralized, composed of two gains. $u_1 = K_1 y_1$ is adapted and converges to a static gain F_1 solution to the constraints (13). $u_2 = K_2 y_2$ is adapted and converges to $F_2 = -\gamma^2\mathbf{1}$. The following parameters are chosen for the adaptation law: $\Gamma_1 = 1000 \cdot \mathbf{1}_6$, $\Gamma_2 = 10$ and $\alpha_1 = \alpha_2 = 1$. Time responses of the simulations starting from the initial conditions $x = (\,1\ \dots\ 1\,)$, $K_i = \mathbf{0}$ are given in the following figures. The simulation is done using Simulink. Convergence of control gains is obtained after 1000 seconds of simulation time. The actual computation time is approximately 30 seconds on a personal computer. In comparison, solving this same LMI problem with SeDuMi solver Sturm [1999] takes less than 2 seconds.

Figure 1 illustrates the stability of the system. All outputs y_i converge to zero. The left-hand side figure plots the outputs $y_1 \in \mathrm{R}^6$ involved in the control $u_1 = K_1 y_1$. The right-hand side figure plots the scalar output y_2.

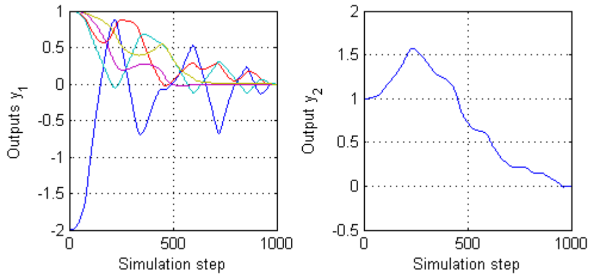

Fig. 1. System outputs

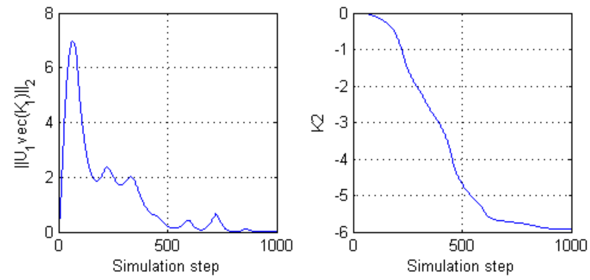

Fig. 2. Controller gains evolution

Figure 2 (left-hand side) is the time history of the norm of $U_1\text{vec}(K_1)$. It shows that the gain K_1 indeed converges to a value such that the structure constraint is satisfied. At simulation time $t = 1000$ the value is:

$$K_1(1000) = F_1 = \begin{bmatrix} 0 & 0 & 4.01 & 0.96 & 0 & 0 \\ 0 & 0 & 0.96 & 10.8 & 0 & 0 \\ 4.01 & 0.96 & 0 & 0 & 0 & 0 \\ 0.96 & 7.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4.01 & -0.96 \\ 0 & 0 & 0 & 0 & -0.96 & -7.25 \end{bmatrix}$$

Figure 2 (right-hand side) shows the time history of the gain K_2. It also converges and $K_2(1000) = F_2 = -5.9263 = -\gamma^2$. (P, γ) solution to LMI (10) are thus directly obtained from F_1 and F_2. The numerical values are:

$$P = \begin{bmatrix} 4.0062 & 0.9584 \\ 0.9584 & 7.2525 \end{bmatrix} \quad , \quad \gamma = 2.4344 \tag{27}$$

P is indeed symmetric positive definite. Its eigenvalues are $\{3.7443, 7.5143\}$. Moreover, the pair (P, γ) is indeed a solution to the LMI (10). The eigenvalues of the associated matrix are all negative: {-9.7975, -0.7571, -3.3010, -3.7443, -7.5143}.

Figure 3 illustrates the time histories of the coefficients $K_{1,13}$, $K_{1,14}$, $K_{1,23}$, $K_{1,24}$ which converge to the coefficients P_{11}, P_{12}, P_{21}, P_{22}. It is interesting to notice that the symmetry constraint $P = P^T$ is satisfied quite fast and not only in steady-state.

It has thus been shown that it is possible in practice to solve an LMI problem through adaptive control simulations when this LMI is feasible. The theory exposed upper also states that if the LMI is not feasible the adaptive law shall not converge.

To illustrate this feature the adaptive law is tested for several, fixed, values of γ. Results are shown in Figure 4: as one would expect, the system converges for all values $\gamma > 1.3251$ ($-\gamma^2 < -1.7559$, represented by the horizontal line) and goes into unstable oscillatory behavior if $\gamma < 1.3251$. When $\gamma > 1.3251$ but close to this limit, the convergence is rather slow which is not surprising since the system is close be unstable.

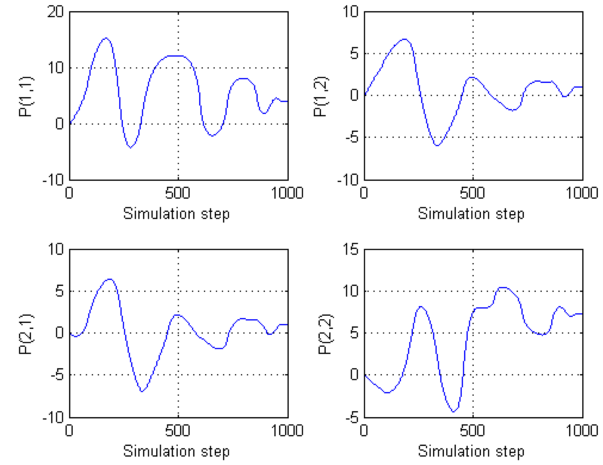

Fig. 3. Evolution of matrix P

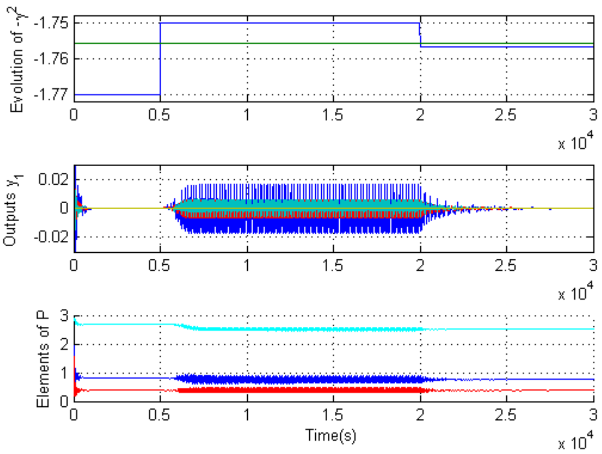

Fig. 4. System response to variation of K_2

Finally, the size of the adaptive problem associated to the numerical example is considered. Matrices L_j corresponding to the canonical representation (16) of the H_∞ LMI (10) are obtained by using the YALMIP *getbase* function (see Löfberg [2001]). The ranks r_j are found to be: $r_1 = 3$, $r_2 = 5$, $r_3 = 3$, $r_4 = 1$. The dimensions of the associated adaptive problems, when considering the representations (11) and (16) respectively, can thus be computed and are given in Table 2. As expected, the matrix-valued variables representation yields a smaller adaptive problem and this confirms its interest.

Table 2. Numerical example - dimensions of the related adaptive control problem

	Considered LMI representation	
	(11)	(16)
Nb. of outputs of the fictive system	7	12
Nb. of coeficients in F	37	45
Nb. scalar constraints	33	40

6. CONCLUSIONS

A new methodology is exposed for solving LMI feasibility problems. It is based on adaptive control methods. Feasible solutions are obtained by simulation (continuous-time algorithm). The methodology is applied successfully on a simple numerical example. In comparison with existing LMI solvers, the computation time is rather long. More numerical tests will be done in the near future to establish a precise evaluation of the numerical properties. Future work will also be devoted to extending the method to optimization problems under LMI constraints.

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