ISSN 0005-1179, Automation and Remote Control, 2009, Vol. 70, No. 7, pp. 1190–1205. © Pleiades Publishing, Ltd., 2009.
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Adaptive Synchronization of a Network of Interconnected Nonlinear Lur'e Systems[1]

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Received July 17, 2008

Abstract—The problem of adaptive synchronization is formulated for networks of interconnected dynamical subsystems containing a leading subsystem. Considered are networks of subsystems specified in the Lur'e form (a linear part plus nonlinearity) of the following three types: Subsystems containing Lipschitz nonlinearities, those with non-Lipschitz nonlinearities in a certain class, and networks with structural matching between the leading subsystem and the driven ones. Decentralized algorithms of adaptive control are synthesized using the speed gradient method. The conditions of synchronizability are obtained through use of the method of passification and the Yakubovich–Kalman lemma. In contrast to the existing results, the case of incomplete measurements is considered along with the situation where the control input may not affect the equations of all subsystems. The results are exemplified by applications to Chua's circuits.

PACS numbers: 02.30.Yy, 01.60.+q

DOI: 10.1134/S0005117909070108

1. INTRODUCTION

Since recently, an increasing attention has been paid to controlling networks of interconnected systems. This interest is stipulated not only by the fact that this subject area is relatively uncharted, but rather by its significance in practical applications, since many physical objects can be thought of as interconnected systems. For example, these are telecommunication networks, molecular ensembles, biological objects, trophic chains, embedded systems, formations of robots or transport vehicles, to name just a few. The development of such systems is driven by extremely rapid progress observed in information and communication technologies, in particular those based on wireless communication and wireless sensors. However, the design of synchronizing controllers may be difficult due to complexity and spatial distribution of the subsystems as well as by limitations on the exchange of information between them.

One of the tools for solving the problems of this sort is decentralization; specifically, the controller of every subsystem makes use of only the local measurements and possibly the information about the control objective. Although the decentralized control problems are well-studied, [1–5], more complicated tasks keep on emerging, e.g., such as control through a communication channel with limited capacity.

In spite of the currently sharp interest to network control, so far, just a limited class of such problems has been successfully solved. For example, a large number of works in the current literature mostly deal with linear models of subsystems [6, 7]. On top of that, in many papers on

[1] This work was supported by the Russian Foundation for Basic Research, project no. 08-01-00775, Scientific Program no. 22 of the Presidium of the Russian Academy of Sciences, project no. 1.8, and Council on Grants under the President of the Russian Federation to support young Russian scientists and leading scientific schools, project no. NSh2387.2008.1.

synchronizability and stabilizability of networks [8–11], it is usually assumed that the measurements of the full state vector of an individual subsystem are available and that all the equations of all subsystems contain the control term (i.e., the dimension of control coincides with the dimension of the subsystem states).

In this paper we consider networks of identical plants described by equations in the Lur'e form, i.e., by systems of first-order differential equations whose right-hand sides are decomposed into the linear and nonlinear components. The interconnections between the plants are not assumed to be linear, these can be nonlinear as well. In contrast to the well-known works such as [10, 11], here it is assumed that the measurements of a certain function of the state vector are only available rather than those of the whole state of an individual subsystem, and that the control input may not affect the equations of all subsystems. It is also assumed that the interconnections between the plants depend on the vector of unknown parameters. Moreover, in the case when matching conditions hold, the parameters of the plants in the network are also assumed to depend on the vector of unknown parameters. Among the subsystems, a certain leading (master, driving) one is distinguished, which is isolated, i.e., it has no connections with the other subsystems. The control function of the leading subsystem is known. The problem is to design a decentralized algorithm of adaptive control together with the conditions under which the solutions of every subsystem approach the solutions of the leading subsystem; this is what is referred to as synchronization. This control objective must be attainable for every value of the vector of unknown parameters in a certain class. The connections between the subsystems are assumed to be Lipschitz.

We consider three forms of equations of subsystems; specifically, the case of Lipschitz nonlinearities entering the subsystems, the case of non-Lipschitz nonlinearities satisfying certain monotonicity conditions, and the case characterized by matching between the structure of the plants and the leading subsystem. The solution of the problems stated here is accomplished via use of the results in [3, 4, 12–14]. The adaptation algorithm is synthesized on the basis of the speed gradient method. Estimates of the Lipschitz constants of interconnections are obtained such that they guarantee the attainment of the control objective under certain additional conditions; namely, the hyper minimum phase property of the associated transfer functions in the first two cases, and the strict passivity of the leading subsystem in the third case.

The results obtained are illustrated via the example of synchronization of several interconnected Chua's circuits exhibiting a chaotic behavior. Computer simulations testify to the validity of the theoretical results.

2. FORMAL STATEMENT OF THE PROBLEM

We consider a network S containing d interconnected subsystems S_i, $i = 1, \dots, d$. Let the subsystem S_i be specified by the following equation:

$$\dot{x}_i = Ax_i + Bu_i + \varphi_0(x_i) + \sum_{j=1}^{d} \alpha_{ij}\varphi_{ij}(x_i - x_j), \quad y_i = C^{\mathrm{T}}x_i, \quad i = 1, \dots, d, \tag{1}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^1$, $\alpha_{ij} \in \mathbb{R}^1$, $y_i \in \mathbb{R}^l$. The functions $\varphi_{ij}(\cdot)$, $i = 1, \dots, d$, $j = 1, \dots, d$, describe the interaction between the subsystems, and the function $\varphi_0(\cdot)$ represents nonlinearity in the subsystem S_i, $i = 1, \dots, d$. It is assumed that $\varphi_{ii}(0) = 0$, $\alpha_{ii} = 0$, $i = 1, \dots, d$. Without loss of generality we assume that the condition $\varphi_{ij}(x) \equiv 0$ for some i, j implies $\alpha_{ij} = 0$, and vice versa. Let the matrices A, B, C and the function $\varphi_0(\cdot)$ be known, while the functions $\varphi_{ij}(\cdot)$, $i = 1, \dots, d$, $j = 1, \dots, d$, depend on the vector of unknown parameters $\xi \in \Xi$, where Ξ is a known set.

By *isolated subsystem* we mean the one with removed (idle) connections, i.e., $\dot{x}_i = Ax_i + Bu_i + \varphi_0(x_i)$, $y_i = C^{\mathrm{T}}x_i$, $i = 1, \dots, d$.

We also consider the leading (master, driving) subsystem of the form

$$\dot{\overline{x}} = A\overline{x} + B\overline{u} + \varphi_0(\overline{x}), \quad \overline{y} = C^{\mathrm{T}}\overline{x}, \tag{2}$$

keeping in mind that $\overline{u} \in \mathbb{R}^1$ is a known given control.

The control objective is to force the trajectories of all subsystems tend to the trajectory of the leading subsystem:

$$\lim_{t\to+\infty} (x_i(t) - \overline{x}(t)) = 0, \quad i = 1, \ldots, d. \tag{3}$$

The goal of adaptive synchronization is to determine decentralized controls $u_i = \mathcal{U}_i(y_i, t)$ which guarantee the attainment of the control objective (3) for all values of the uncertain parameters $\xi \in \Xi$.

3. CONTROL DESIGN

By letting $z_i = x_i - \overline{x}$, $\tilde{u}_i = u_i - \overline{u}$, we introduce the equations of auxiliary subsystems (error equations):

$$\begin{gathered}\dot{z}_i = Az_i + B\tilde{u}_i + \varphi_0(x_i) - \varphi_0(\overline{x}) + \sum_{j=1}^{d}\alpha_{ij}\varphi_{ij}(x_i - x_j), \\ \tilde{y}_i = C^{\mathrm{T}}z_i, \quad i = 1, \ldots, d.\end{gathered} \tag{4}$$

For an auxiliary subsystem, we specify linear output feedback controllers with adjustable coefficients in the following form:

$$\tilde{u}_i = \theta_i^{\mathrm{T}}(t)\tilde{y}_i, \quad \theta_i(t) \in \mathbb{R}^l, \quad i = 1, \ldots, d,$$

where $\tilde{y}_i$ is defined in (4), i.e., $\tilde{y}_i = y_i - \overline{y}$, and $\theta_i(t)$ are the adjustable parameters.

In the sequel, the adaptation algorithm $\theta_i(t), i = 1, \ldots, d$, will be found by using the speed gradient method. According to this method, a nonnegative objective function is to be chosen such that its convergence to zero is equivalent to the attainment of the control objective (3). In our case, the objective function can be taken in the form[2]

$$Q(z_i) = \frac{1}{2}\, z_i^{\mathrm{T}} H z_i, \quad H = H^{\mathrm{T}} > 0.$$

Differentiating the function $Q(z_i)$ in t on the trajectories of the isolated subsystem, we obtain

$$\omega_i(x_i, \overline{x}) = z^{\mathrm{T}} H \left(Az + B\theta_i^{\mathrm{T}}\tilde{y}_i + \varphi_0(x_i) - \varphi_0(\overline{x})\right), \quad i = 1, \ldots, d. \tag{5}$$

We next take the gradient with respect to θ_i:

$$\nabla_{\theta_i}\omega_i(x_i, \overline{x}) = z_i^{\mathrm{T}} HB\tilde{y}_i, \quad i = 1, \ldots, d.$$

The scalar $z_i^{\mathrm{T}}HB$ in the last relation is a function of the output vector $\tilde{y}_i(t)$. Since $z_i^{\mathrm{T}}HB$ and $\tilde{y}_i = C^{\mathrm{T}}z_i$ are linear in z_i, the quantity $z_i^{\mathrm{T}}HB$ is representable as a linear combination of the outputs, i.e., $B^{\mathrm{T}}Hz_i = g^{\mathrm{T}}C^{\mathrm{T}}z_i$ for some $g \in \mathbb{R}^l$ and all $z_i \in \mathbb{R}^n$. In other words, $HB = Cg$. We thus arrive at the adaptation algorithm

$$\dot{\theta}_i(t) = -g^{\mathrm{T}}\tilde{y}_i(t)\Gamma_i\tilde{y}_i(t), \quad i = 1, \ldots, d, \tag{6}$$

where $\Gamma_i = \Gamma_i^{\mathrm{T}} > 0$ are positive-definite matrices of size $l \times l$.

[2] In what follows, we use the Euclidean norm, $H > 0$ stands for positive definiteness of the symmetric matrix H, and $\mathrm{col}(a_1, \ldots, a_k)$ denotes $(a_1, \ldots, a_k)^{\mathrm{T}}$, where $^{\mathrm{T}}$ is the transposition sign.

Hence, for every $i = 1, \ldots, d$, the controller for the subsystem S_i has the following form:

$$u_i(t) = \theta_i^{\mathrm{T}}(t)[y_i(t) - \overline{y}(t)] + \overline{u}(t), \tag{7}$$

where $\theta_i(t)$ satisfies Eq. (6). The conditions under which this controller ensures the attainment of the control objective and the boundedness of $\theta_i(t), i = 1, \ldots, d$, will be derived in the next section.

4. SYNCHRONIZABILITY CONDITIONS

4.1. The Case of Lipschitz Nonlinearities

To derive the synchronizability conditions, we need the following definition of the hyper minimum phase property; e.g., see [3, 12, 15].

Definition 1. A proper rational function $W(s) = \beta(s)/\alpha(s)$, where $\beta(s)$ and $\alpha(s)$ are real polynomials, is said to be minimum phase if its numerator $\beta(s)$ is Hurwitz. This function is said to be hyper minimum phase if it is minimum phase and the number $\lim\limits_{s\to+\infty} sW(s)$ is positive.

Consider real matrices $H = H^{\mathrm{T}} > 0, g, \theta_*$ having dimensions $n \times n, l \times 1, l \times 1$, respectively, and a number $\rho > 0$ such that

$$HA_* + A_*^{\mathrm{T}}H < -\rho H, \quad HB = Cg, \quad A_* = (A + LI_n) + B\theta_*^{\mathrm{T}}C^{\mathrm{T}}. \tag{8}$$

Let $\lambda_{\min}(H)$ and $\lambda_{\max}(H)$ denote the minimal and maximal eigenvalues of the matrix H, and let $\lambda_* = \lambda_{\max}(H)/\lambda_{\min}(H)$ be the condition number of H.

The analysis of synchronizability below is based on the following assumption.

(A1) The functions $\varphi_0(\cdot)$ and $\varphi_{ij}(\cdot)$, $i = 1, \ldots, d$, $j = 1, \ldots, d$, are globally Lipschitz:

$$\|\varphi_0(x) - \varphi_0(x')\| \leqslant L\|x - x'\|, \quad L > 0,$$
$$\|\varphi_{ij}(x) - \varphi_{ij}(x')\| \leqslant L_{ij}\|x - x'\|, \quad L_{ij} > 0.$$

Theorem 1. *Let Assumption* (A1) *be valid for every $\xi \in \Xi$, and let there exist $g \in \mathbb{R}^l$ such that the function $g^{\mathrm{T}}\chi(s-L)$ is hyper minimum phase, where the transfer function $\chi(s) = C^{\mathrm{T}}(sI_n - A)^{-1}B$. Then there exist matrices $H = H^{\mathrm{T}} > 0$ and θ_* of dimensions $n \times n$, $l \times 1$, respectively, and a positive number ρ such that relations* (8) *hold. Moreover, if the inequality*

$$\sum_{j=1}^{d} |\alpha_{ij}L_{ij}| < \frac{\rho}{2}\frac{\lambda_{\min}(H)}{\lambda_{\max}(H)} \tag{9}$$

is satisfied for all $i = 1, \ldots, d$, then for every $i = 1, \ldots, d$, the adaptive control (6), (7) *attains the objective*

$$\lim_{t\to+\infty} (x_i(t) - \overline{x}(t)) = 0 \tag{10}$$

and the vector of adjustable parameters θ_i is bounded over $[0, \infty)$ for all solutions of the closed-loop system (1), (2), (7), (6).

Remark 1. Let ρ_* denote the stability degree of the numerator of the function $g^{\mathrm{T}}\chi(s-L)$. From the results in [15], it easily follows that if the function $g^{\mathrm{T}}\chi(s - L)$ is hyper minimum phase, then any $\rho : 0 < \rho < \rho_*$ and any $\theta_* = -\varkappa g$ are feasible in (8), where the number $\varkappa > 0$ is sufficiently large. Therefore, inequality (9) can be replaced with

$$\sum_{j=1}^{d} |\alpha_{ij} L_{ij}| < \gamma, \tag{11}$$

where $\gamma = \rho_*/(2\lambda_*)$.

Remark 2. By the connection graph of the network S we mean a directed graph comprising the set of nodes (vertices) and the set of edges; we define these sets as follows. The set of nodes has cardinality d, where the ith node is associated with the ith subsystem S_i. The edge from node i to node j belongs to the set of edges if $\varphi_{ji} \neq 0$. Define the weighted incoming degree of node i as $\sum_{j=1}^{d} |\alpha_{ij} L_{ij}|$. If every nonzero term in this sum is equal to 1, the definition above coincides with that of the incoming degree of a node in a directed graph; e.g., see [16]. Hence, condition (9) means that the weighted incoming degree of every node is less than γ.

4.2. The Case $\varphi_0(x_i) = B\psi_0(y_i)$

We now consider the case $\varphi_0(x_i) = B\psi_0(y_i)$, $\psi_0 \colon \mathbb{R}^l \to \mathbb{R}^1$. The subsystem (1) writes as follows:

$$\dot{x}_i = Ax_i + B(u_i + \psi_0(y_i)) + \sum_{j=1}^{d} \alpha_{ij}\varphi_{ij}(x_i - x_j), \quad y_i = C^{\mathrm{T}} x_i, \quad i = 1, \ldots, d, \tag{12}$$

and the leading (master) subsystem writes as

$$\dot{\overline{x}} = A\overline{x} + B(\overline{u} + \psi_0(\overline{y})), \quad \overline{y} = C^{\mathrm{T}}\overline{x}. \tag{13}$$

Similarly to the above, the quantity $\overline{u} \in \mathbb{R}^1$ in (13) is thought of as a given control, which is assumed to be known.

We give the following definition.

Definition 2. Let $G \in \mathbb{R}^l$. The function $f \colon \mathbb{R}^l \to \mathbb{R}^1$ is said to be G–monotonically decreasing if the inequality $(x - y)^{\mathrm{T}} G(f(x) - f(y)) \leqslant 0$ holds for all $x, y \in \mathbb{R}^l$.

For $l = 1$ and $G > 0$ this definition reduces to the standard notion of monotonically decreasing functions.

We next make the following assumption.

(A2) The functions φ_{ij}, $i = 1, \ldots, d$, $j = 1, \ldots, d$, are globally Lipschitz, i.e.,

$$\|\varphi_{ij}(x) - \varphi_{ij}(x')\| \leqslant L_{ij}\|x - x'\|, \quad L_{ij} > 0,$$

and the function $\psi_0(\cdot)$ is such that the existence and uniqueness of solutions of the equations of every plant in the network are ensured.

It is seen that, by using the results in Section 3, the control can be taken in the form (6), (7).

Consider real matrices $H = H^{\mathrm{T}} > 0$, g, θ_* of dimensions $n \times n$, $l \times 1$, $l \times 1$, respectively, and a scalar $\rho > 0$ such that

$$HA_* + A_*^{\mathrm{T}} H < -\rho H, \quad HB = Cg, \quad A_* = A + B\theta_*^{\mathrm{T}} C^{\mathrm{T}}. \tag{14}$$

Note that these relations follow from (8) by letting $L = 0$.

Theorem 2. *Let Assumption* (A2) *be valid for all* $\xi \in \Xi$ *and let the function* $g^{\mathrm{T}}\chi(s)$ *be hyper minimum phase for some* $g \in \mathbb{R}^l$*, where the transfer function* $\chi(s) = C^{\mathrm{T}}(sI_n - A)^{-1}B$*. Then there exist matrices* $H = H^{\mathrm{T}} > 0$*,* θ_* *of dimensions* $n \times n$*,* $l \times 1$ *and a positive* ρ *such that relations* (14) *hold. Moreover, assume that the function* $\psi_0(\cdot)$ *is g-monotonically decreasing, and for all* $i = 1, \ldots, d$*, the inequality*

$$\sum_{j=1}^{d} |\alpha_{ij} L_{ij}| < \gamma \tag{15}$$

holds, where $\gamma = \rho_*/(2\lambda_*)$*,* λ_* *is the condition number of the matrix* H*, and* ρ_* *is the degree of stability of the numerator of the function* $g^{\mathrm{T}}\chi(s)$*. Then for every* $i = 1, \ldots, d$*, the adaptive control* (6)*,* (7) *ensures the attainment of the objective*:

$$\lim_{t \to +\infty} (x_i(t) - \overline{x}(t)) = 0, \tag{16}$$

and the vector θ_i *of adjustable parameters remains bounded over* $[0, \infty)$ *for all solutions of the closed-loop system* (6)*,* (7)*,* (12)*,* (13)*.*

4.3. Synchronization under Matching Conditions

Let the leading subsystem be described by the equations below:

$$\dot{\overline{x}} = A_M \overline{x} + B_M(\overline{u} + \psi_0(\overline{y})), \quad \overline{y} = C^{\mathrm{T}}\overline{x}, \tag{17}$$

where $\overline{x} \in \mathbb{R}^n$, $\overline{u} \in \mathbb{R}^1$, $\overline{y} \in \mathbb{R}^l$, and $\psi_0\colon \mathbb{R}^l \to \mathbb{R}^1$. As usual, we assume that $\overline{u}$ is a given known control. It is also assumed that A_M, B_M, C, and $\psi_0(\cdot)$ are known and do not depend on $\xi \in \Xi$, where Ξ is a known set.

Consider the network comprising d interconnected plants, each of them being described by the following equation:

$$\dot{x}_i = Ax_i + Bu_i + B_M\psi_0(y_i) + \sum_{j=1}^{d} \alpha_{ij}\varphi_{ij}(x_i - x_j), \quad y_i = C^{\mathrm{T}}x_i, \ i = 1, \ldots, d, \tag{18}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^1$, $\alpha_{ij} \in \mathbb{R}^1$, and $y_i \in \mathbb{R}^l$. Assume that the matrices A, B and the functions $\varphi_{ij}(\cdot)$, $i = 1, \ldots, d$, $j = 1, \ldots, d$, depend on the vector of unknown parameters $\xi \in \Xi$.

Assume next that the matching conditions [17, Section 7.2, item 2] between the structure of the leading subsystem (17) and those of every plant (18) in the network are satisfied; i.e., we adopt the following assumption:

(A3) For every $\xi \in \Xi$ there exists a vector $\nu_* = \nu_*(\xi) \in \mathbb{R}^l$ and a number $\theta_* = \theta_*(\xi) > 0$ such that the following equalities are valid:

$$A_M = A + B\nu_*^{\mathrm{T}}C^{\mathrm{T}}, \quad B_M = \theta_* B.$$

Denote $\sigma_i(t) = \mathrm{col}(y_i(t), \overline{u}(t))$. Similarly to [17], we use the adaptive regulator

$$u_i(t) = \tau_i(t)^{\mathrm{T}}\sigma_i(t), \quad i = 1, \ldots, d, \tag{19}$$

where $\tau_i(t) \in \mathbb{R}^{l+1}$ is the vector of adjustable parameters. Using the speed gradient method, we arrive at the adaptation algorithm of the form

$$\dot{\tau}_i = -g^{\mathrm{T}}(y_i - \overline{y})\,\Gamma_i\,\sigma_i(t), \quad i = 1, \ldots, d, \tag{20}$$

where $\Gamma_i = \Gamma_i^{\mathrm{T}} > 0$ are matrices of dimensions $(l+1) \times (l+1)$ and $g \in \mathbb{R}^l$.

Consider now real matrices $H = H^{\mathrm{T}} > 0$ and g of dimensions $n \times n$ and $l \times 1$, respectively, and a number $\rho > 0$ such that

$$HA_M + A_M^{\mathrm{T}} H < -\rho H, \quad HB_M = Cg. \tag{21}$$

Denote $\chi(s) = C^{\mathrm{T}}(sI_n - A_M)^{-1}B_M$. The theorem below provides sufficient conditions of adaptive synchronizablity.

Theorem 3. *Assume that* $\operatorname{rank} B_M = 1$, *the matrix* A_M *is Hurwitz, and for some* $g \in \mathbb{R}^l$, *the frequency inequalities*

$$\operatorname{Re} g^{\mathrm{T}} \chi(i\omega) > 0, \quad \lim_{\omega \to \infty} \omega^2 \operatorname{Re} g^{\mathrm{T}} \chi(i\omega) > 0 \tag{22}$$

hold for all $\omega \in \mathbb{R}^1$. *Then there exist* $H = H^{\mathrm{T}} > 0$ *and* $\rho > 0$ *such that relations* (21) *are satisfied. Let Assumptions* (A2) and (A3) *be valid for all* $\xi \in \Xi$, *the function* $\psi_0(\cdot)$ *be g-monotonically decreasing, and let the inequality*

$$\sum_{j=1}^{d} |\alpha_{ij} L_{ij}| < \gamma \tag{23}$$

hold for all $i = 1, \ldots, d$, *where* $\gamma = \rho_*/(2\lambda_*)$, λ_* *is the condition number of the matrix* H, *and* ρ_* *is the degree of stability of the denominator of the function* $g^{\mathrm{T}}\chi(s)$. *Then for every* $i = 1, \ldots, d$, *the adaptive control* (19), (20), *attains the control objective, i.e.,*

$$\lim_{t \to +\infty} (x_i(t) - \overline{x}(t)) = 0,$$

and the vector τ_i *of adjustable parameters remains bounded over* $[0, \infty)$ *for all solutions of the closed-loop system* (17), (18), (19), (20).

Remark 3. Note that the conditions in the first part of Theorem 3 are equivalent to the strict passivity of the transfer function $g^{\mathrm{T}}\chi(s)$; see [12].

5. EXAMPLE: THE NETWORK OF CHUA'S CIRCUITS

5.1. System Description and Analysis

Chua's circuit is a well-known example of a simple nonlinear system exhibiting a complex chaotic behavior [18]. The trajectories of Chua's circuit are unstable yet being bounded, and the system is of Lur'e form. Below, we apply Theorem 2 to the synchronization of the network of interconnected Chua's systems.

The Chua's circuit is modeled in the dimensionless form with exogenous input $\overline{u}$ as follows:

$$\begin{cases} \dot{\overline{x}}_1 = p(\overline{x}_2 - \overline{x}_1 - f(\overline{x}_1)) + b\overline{u} \\ \dot{\overline{x}}_2 = \overline{x}_1 - \overline{x}_2 + \overline{x}_3 \\ \dot{\overline{x}}_3 = -q\overline{x}_2, \end{cases} \tag{24}$$
$$\overline{y}(t) = \overline{x}_1(t).$$

Here, $\overline{x} = \operatorname{col}(\overline{x}_1, \overline{x}_2, \overline{x}_3) \in \mathbb{R}^3$ is the state vector of the subsystem, $\overline{y}$ is the measured output, $\overline{u}(t)$ is scalar control, $f(x) = m_1 x + \dfrac{m_0 - m_1}{2}(|x+1| - |x-1|)$ is nonlinearity, and p, q, m_1, and m_0 are the parameters of the subsystem.

In order to apply the results obtained above, we find the values of the parameters yielding the hyper minimum phase property and the g-monotonic decrease of the nonlinearity. By letting $v(x) = -\dfrac{m_0 - m_1}{2}(|x+1| - |x-1| - 2x)$, re-write the system:

$$\begin{cases} \dot{\overline{x}}_1 = p\left(-(1+m_0)\overline{x}_1 + \overline{x}_2 + v(\overline{x}_1)\right) + b\overline{u} \\ \dot{\overline{x}}_2 = \overline{x}_1 - \overline{x}_2 + \overline{x}_3 \\ \dot{\overline{x}}_3 = -q\overline{x}_2, \end{cases}$$
$$\overline{y}(t) = \overline{x}_1(t).$$

This system is of the form (13) with matrices

$$A = \begin{pmatrix} -(1+m_0)p & p & 0 \\ 1 & -1 & 1 \\ 0 & -q & 0 \end{pmatrix},$$

$B = \mathrm{col}(b, 0, 0)$, $C = \mathrm{col}(1, 0, 0)$ and nonlinearity $\psi_0(\overline{y}) = pv(\overline{y})/b$.

It can be easily shown that with $p/b > 0$, $g > 0$, $m_0 < m_1$, the function $\psi_0(\cdot)$ is g-monotonically decreasing.

The transfer function, whose hyper minimum phase property is required in Theorem 2, has the following form:

$$g^{\mathrm{T}}\chi(s) = gC^{\mathrm{T}}(sI - A)^{-1}B = gb\,\frac{s^2 + s + q}{s^3 + \beta_2 s^2 + \beta_1 s + \beta_0},$$

where β_2, β_1, β_0 are some numbers. To force this function to be hyper minimum phase, we take $g > 0$, $b > 0$. It now remains to ensure the Hurwitz property of the polynomial $(s^2 + s + q)$; this is equivalent to letting $q > 0$.

Now set $x_i = (x_{i1}, x_{i2}, x_{i3})^{\mathrm{T}}$, $i = 1, \ldots, 5$, and let $A, B, C, \varphi_0(\cdot)$ be defined as required above. Consider five interconnected Chua's circuits:

$$\begin{gathered} \dot{x}_i = Ax_i + B(u_i + \psi_0(y_i)) + \sum_{j=1}^{5}\alpha_{ij}\varphi_{ij}(x_i - x_j), \\ y_i = C^{\mathrm{T}}x_i, \quad i = 1, \ldots, 5, \end{gathered} \tag{25}$$

where u_i, $\alpha_{ij} \in \mathbb{R}^1$. Denote $\varphi_{ij} = \varphi_{ij}(x_i - x_j)$, $i = 1, \ldots, 5$, $j = 1, \ldots, 5$, and set $\varphi_{14}, \varphi_{25}, \varphi_{32}, \varphi_{42}, \varphi_{45}, \varphi_{52}, \varphi_{53}$ to be equal to $(0,0,0)^{\mathrm{T}}$. Note that $\varphi_{ii} = (0,0,0)^{\mathrm{T}}$, $i = 1, \ldots, 5$ (see Section 2). Next, let

$$\begin{gathered} \varphi_{12} = (\sin(x_{11} - x_{21}), 0, 0)^{\mathrm{T}}, \quad \varphi_{13} = (0, x_{12} - x_{32}, 0)^{\mathrm{T}}, \\ \varphi_{15} = (0, 0, \sin(x_{13} - x_{53}))^{\mathrm{T}}, \quad \varphi_{21} = (x_{21} - x_{11}, 0, x_{23} - x_{13})^{\mathrm{T}}, \\ \varphi_{23} = (0, \sin(x_{22} - x_{32}), 0)^{\mathrm{T}}, \quad \varphi_{24} = (0, x_{22} - x_{42}, 0)^{\mathrm{T}}, \\ \varphi_{31} = (\sin(x_{31} - x_{11}), 0, 0)^{\mathrm{T}}, \quad \varphi_{34} = (\sin(x_{31} - x_{41}), 0, 0)^{\mathrm{T}}, \\ \varphi_{35} = (x_{31} - x_{51}, x_{32} - x_{52}, x_{33} - x_{53})^{\mathrm{T}}, \\ \varphi_{41} = (0, \sin(x_{42} - x_{12}), 0)^{\mathrm{T}}, \quad \varphi_{43} = (\sin(x_{41} - x_{31}), 0, 0)^{\mathrm{T}}, \\ \varphi_{51} = (x_{51} - x_{11}, 0, x_{53} - x_{13})^{\mathrm{T}}, \quad \varphi_{54} = (0, x_{52} - x_{42}, 0)^{\mathrm{T}}. \end{gathered}$$

Obviously, the Lipschitz constants of all nonzero φ_{ij}'s are equal to unity. The connection graph is depicted in Fig. 1.

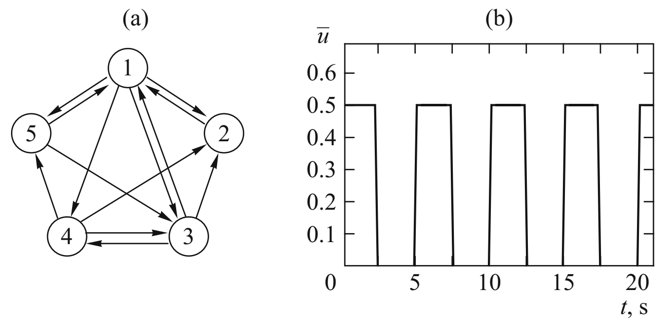

Fig. 1. (a) The connection graph; (b) control of the leading subsystem, $\overline{u}$.

By Theorem 2, for $p > 0$, $q > 0$, $b > 0$, $g > 0$, $m_0 < m_1$ and sufficiently small $\sum_{j=1}^{5} |\alpha_{ij}|$, $i = 1, \ldots, 5$, the adaptive decentralized control

$$u_i = \theta_i[y_i - \overline{y}] + \overline{u},$$
$$\dot{\theta}_i = -g\tilde{y}_i\Gamma_i\tilde{y}_i, \quad i = 1, \ldots, 5,$$

where $\Gamma_i = \Gamma_i^{\mathrm{T}} > 0$ are positive-definite matrices, guarantees the synchronization in the network of Chua's circuits in the sense that the control objective (16) is attained, where $\overline{x}(t)$ is the solution of system (24) subjected to the input $\overline{u}(t)$.

5.2. Simulation Results

Following [19], we take $m_0 = -8/7$, $m_1 = -5/7$, $p = 15.6$, $q = 25.58$, $b = g = 1$, $\Gamma_i = 1$, $i = 1, \ldots, 5$, and

$$\overline{x}_1(0) = 0.5, \quad \overline{x}_2(0) = 0, \quad \overline{x}_3(0) = 0,$$
$$x_1(0) = (7, 14, 0.4)^{\mathrm{T}}, \quad x_2(0) = (0, 4, 4)^{\mathrm{T}}, \quad x_3(0) = (1, -1, 4.5)^{\mathrm{T}},$$
$$x_4(0) = (3, -4, 0.2)^{\mathrm{T}}, \quad x_5(0) = (2, 8, 15).$$

Note that with such a choice of the parameters we have $p > 0$, $q > 0$, $b > 0$, $g > 0$, $m_0 < m_1$.

Let the input $\overline{u}$ be taken in the form of the impulse signal with amplitude 1/2, zero initial phase, and having period $T = 5$ s, see Fig. 1. Denote

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{15} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{25} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{51} & \alpha_{52} & \ldots & \alpha_{55} \end{pmatrix},$$
$$\widehat{\alpha} = \begin{pmatrix} 0 & 0.0051 & 0.1395 & 0 & 0.1676 \\ 0.0662 & 0 & 0.0921 & 0.0065 & 0 \\ 0.2013 & 0 & 0 & 0.2271 & 0.1430 \\ 0.0907 & 0 & 0.0675 & 0 & 0 \\ 0.0663 & 0 & 0 & 0.2773 & 0 \end{pmatrix}.$$

After 50 seconds of simulation with $\alpha = 2.2 \times \widehat{\alpha}$, the control objective is not attained, i.e., $\|z_i\| \nrightarrow 0$; see Fig. 2. If the matrix $\alpha = \widehat{\alpha}$ with smaller entries is considered, simulations result in $\|z_i\| \to 0$ (i.e., the control objective is attained), and the controls $\tilde{u} = u_i - \overline{u} \to 0$, $i = 1, \ldots, 5$.

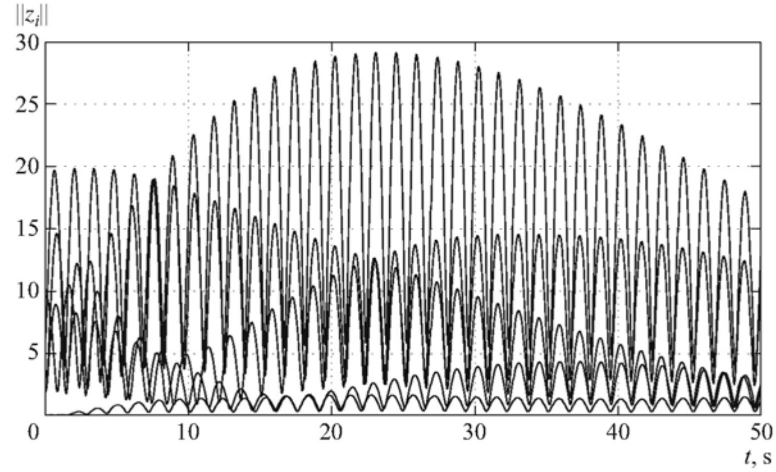

Fig. 2. The plots of $\|z_i\|$, $i = 1, \ldots, 5$, for $\alpha = 2.2 \times \widehat{\alpha}$.

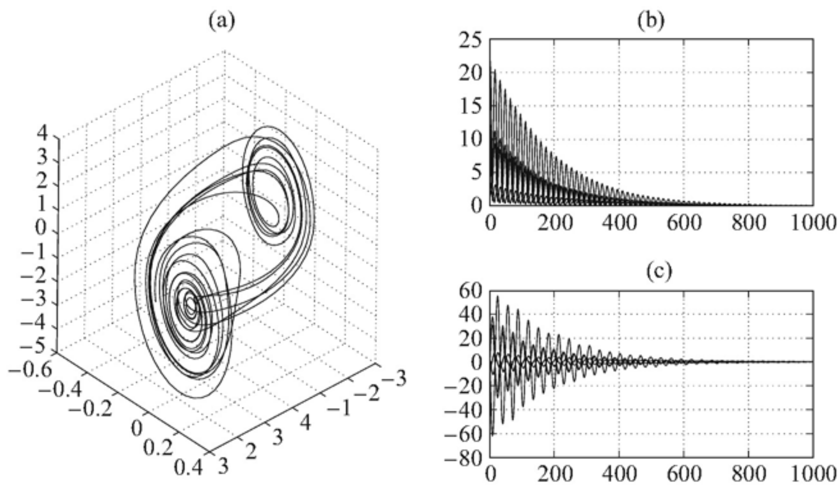

Fig. 3. (a) The phase portrait of the leading subsystem; (b) the plots of $\|z_i\|$; (c) the controls $\widetilde{u}_i = u_i - \overline{u}$, $i = 1, \ldots, 5$, $\alpha = \widehat{\alpha}$.

Figure 3 depicts the phase portrait of the leading subsystem, the objective functions $\|z_i\|$, and the controls $\tilde{u}_i = u_i - \overline{u}$, $i = 1, \ldots, 5$, obtained after 50 seconds of simulation. Notably, both the master and all the slave subsystems exhibit chaotic behavior.

6. CONCLUSIONS

In contrast to the results in the existing literature, this paper presents synchronizability conditions for nonlinear dynamic systems in the situation where observations and information about the parameters of the system and connections, are all incomplete. The design of a decentralized adaptive control algorithm that leads to the synchronization is based on the speed gradient method [3, 12]; the synchronizability conditions are obtained with the use of the passification theorems [3, 15] and the Yakubovich–Kalman lemma [14]. The quantitative conditions on the system parameters ensuring synchronization are obtained for plants with Lipschitz nonlinearities, plants with g-monotonic non-Lipschitz nonlinearities, and plants satisfying the conditions of structural matching between the master and the slave subsystems. It is of apparent interest to extend the results obtained towards problems with partially decentralized control.

APPENDIX

Proof of Theorem 1. The proof exploits the following two auxiliary results. The first one is the passification theorem ([12, Theorem A.3.1; 15]) modified to the case under consideration.

Lemma 1. *For the existence of the real matrices* $H = H^* > 0$ *and* θ_* *such that* $HA_* + A_*^{\mathrm{T}} H < 0$, $HB = Cg$, *where* $A_* = (A+LI_n)+B\theta_*^{\mathrm{T}} C^{\mathrm{T}}$, *it is sufficient that the function* $g^{\mathrm{T}} C^{\mathrm{T}} (sI_n - A - LI_n)^{-1} B$ *be hyper minimum phase.*

The second result establishes the properties of the speed gradient algorithm as applied to the problems of decentralized control; it is a modification of Theorem 2.18 in [3].

Lemma 2. *Consider a system comprising* N *interconnected systems whose individual dynamics is specified by the equations*

$$\dot{x}_i = F_i(x_i, \theta_i, t) + h_i(x, \theta, t), \quad i = 1, \ldots, N, \tag{A.1}$$

$$\dot{\theta}_i = -\gamma_i \nabla_{\theta_i} \omega_i(x_i, \theta_i, t), \quad i = 1, \ldots, N, \tag{A.2}$$

where $x_i \in \mathbb{R}^{n_i}$, $\theta_i \in \mathbb{R}^{m_i}$,

$$\omega_i(x_i, \theta_i, t) = \frac{\partial Q_i}{\partial t} + \nabla Q_i(x_i, t)^{\mathrm{T}} F_i(x_i, \theta_i, t).$$

Here, $Q(\cdot)$ *is an objective function,* $n = \sum n_i$, $m = \sum m_i$, $x = \mathrm{col}(x_1, \ldots, x_l) \in \mathbb{R}^n$. *Assume that* (A.1) *satisfies the following three groups of conditions*:

(1) *The functions* $F_i(\cdot)$ *are continuous in* x_i *and* t, *continuously differentiable in* θ_i *and locally bounded in* $t > 0$; *the functions* $\omega_i(x_i, \theta_i, t)$ *are convex in* θ_i; *there exist vectors* $\theta_i^* \in \mathbb{R}^{m_i}$ *and scalar continuous increasing functions* $\kappa_i(Q)$, $\rho_i(Q)$ *such that* $\kappa_i(0) = \rho_i(0) = 0$, *and* $\kappa_i(Q) \to +\infty$ *as* $Q_i \to +\infty$:

$$\omega_i(x_i, \theta_i^*, t) \leqslant -\rho_i(Q_i(x_i, t)) \tag{A.3}$$

and $Q_i(x_i, t) \geqslant \kappa_i(\|x_i - x_i^*(t)\|)$, *where* $x_i^* = \mathrm{argmin}_{x_i} Q_i(x_i, t)$ *and* $Q_i(x_i^*(t), t) \equiv 0$.

(2) *The functions* $h_i(x, \theta, t)$ *are continuous and satisfy the inequalities*

$$|\nabla_{x_i} Q_i(x_i, t)^{\mathrm{T}} h_i(x, \theta, t)| \leqslant \sum_{j=1}^{l} \mu_{ij} \rho_j(Q_j(x_j, t)), \tag{A.4}$$

where the matrix $M - I$ *is Hurwitz,* $M = \{\mu_{ij}\}$, $\mu_{ij} \geqslant 0$, *and* I *is the identity matrix.*

Then system (A.1), (A.2) *is globally asymptotically stable in the variables* $x_i - x_i^*(t)$, *all its trajectories are bounded, and* $\lim\limits_{t\to\infty} Q_i(x_i, t) = 0$, $i = 1, \ldots, N$.

We now turn to the proof of Theorem 1 itself. Consider the first group of conditions in Lemma 2. The condition of local boundedness in $t > 0$ is satisfied, since for every $i = 1, \ldots, d$, both the right-hand side of (4) and the function $Q(z_i)$ are smooth t-independent functions. The convexity condition is assured by the linearity of the right-hand side of (5) in θ_i. As regards the functions $\rho_i(\cdot)$, $i = 1, \ldots, d$, which appear in Lemma 2, we take them in the form of the same linear function $Q \to \rho \times Q$ and show that the existence condition for $\theta_* \in \mathbb{R}^l$ and ρ such that $\omega_i(z_i, \theta_*) \leqslant -\rho Q(z_i)$, is guaranteed by the hyper minimum phase property of the function $g^{\mathrm{T}}\chi(s)$. Indeed, by Lemma 1, the hyper minimum phase property of the function $g^{\mathrm{T}}\chi(s)$ ensures the existence of $H = H^{\mathrm{T}} > 0$ and θ_* such that

$$HA_* + A_*^{\mathrm{T}}H < 0,\ HB = Cg,$$

where $A_* = (A + LI_n) + B\theta_*^{\mathrm{T}}C^{\mathrm{T}}$. Then we have

$$\omega_i(z_i, \theta_*) \leqslant z_i^{\mathrm{T}}H[(A + LI_n) + B\theta_*{}^{\mathrm{T}}C^{\mathrm{T}}]z_i = \frac{1}{2}z_i^{\mathrm{T}}[HA_* + A_*^{\mathrm{T}}H]z_i,\ i = 1, \ldots, d.$$

Since $HA_* + A_*^{\mathrm{T}}H$ is negative definite, there exists a positive number ρ such that $HA_* + A_*^{\mathrm{T}}H \leqslant -\rho H$, which ensures the condition

$$\omega_i(z_i, \theta_*) \leqslant -\rho Q(z_i), \quad i = 1, \ldots, d.$$

We now turn to the conditions on the interconnections between the subsystems (the second group of conditions in Lemma 2). In the case considered, they take the following form:

$$\left|\nabla_{z_i} Q(z_i)^{\mathrm{T}} \sum_{j=1}^{d} \alpha_{ij}\varphi_{ij}(z_i - z_j)\right| \leqslant \sum_{j=1}^{d} \mu_{ij}\rho \times Q(z_j), \quad i = 1, \ldots, d,$$

where the matrix $M - I$ is Hurwitz, $M = \{\mu_{ij}\}$, $\mu_{ij} \geqslant 0$, and I is the identity matrix. Re-write the last inequality to obtain

$$\left|z_i^{\mathrm{T}}H \sum_{j=1}^{d} \alpha_{ij}\varphi_{ij}(z_i - z_j)\right| \leqslant \frac{\rho}{2}\sum_{j=1}^{d} \mu_{ij}\, z_j^{\mathrm{T}}Hz_j, \quad i = 1, \ldots, d. \tag{A.5}$$

We now evaluate the quantity on the left-hand side of (A.5):

$$\left|z_i^{\mathrm{T}}H \sum_{j=1}^{d} \alpha_{ij}\varphi_{ij}(z_i - z_j)\right| \leqslant \sum_{j=1}^{d} \left|z_i^{\mathrm{T}}H\alpha_{ij}\varphi_{ij}(z_i - z_j)\right|$$

$$= \sum_{j=1}^{d} |\alpha_{ij}| \times \left|z_i^{\mathrm{T}}H\varphi_{ij}(z_i - z_j)\right| \leqslant \sum_{j=1}^{d} |\alpha_{ij}| \times \|z_i^{\mathrm{T}}H\| \times \|L_{ij}(z_i - z_j)\|$$

$$= \sum_{j=1}^{d} |\alpha_{ij}L_{ij}| \times \|z_i^{\mathrm{T}}H\| \times \|z_i - z_j\| \leqslant \sum_{j=1}^{d} |\alpha_{ij}L_{ij}| \times \|z_i\| \times \|H\| \times \|z_i - z_j\|$$

$$\leqslant \sum_{j=1}^{d} |\alpha_{ij}L_{ij}| \times \|H\| \times (\|z_i\|^2 + \|z_i\| \times \|z_j\|)$$

$$\leqslant \sum_{j=1}^{d} |\alpha_{ij}L_{ij}| \times \lambda_{\max}(H) \times (\|z_i\|^2 + \|z_i\| \times \|z_j\|), \quad i = 1, \ldots, d.$$

Next, we estimate from below the right-hand side of (A.5):

$$\frac{\rho}{2}\sum_{j=1}^{d} \mu_{ij}z_j^{\mathrm{T}}Hz_j \geqslant \frac{\rho}{2}\sum_{j=1}^{d} \mu_{ij}\lambda_{\min}(H)\|z_j\|^2, \quad i = 1, \ldots, d.$$

Hence, it suffices to satisfy the inequality

$$\sum_{j=1}^{d} |\alpha_{ij} L_{ij}| \times \lambda_{\max}(H) \times (\|z_i\|^2 + \|z_i\| \times \|z_j\|) \leqslant \frac{\rho}{2} \sum_{j=1}^{d} \mu_{ij} \lambda_{\min}(H) \|z_j\|^2, \quad i = 1, \ldots, d,$$

or, equivalently,

$$\sum_{j=1}^{d} |\alpha_{ij} L_{ij}| \times (\|z_i\|^2 + \|z_i\| \times \|z_j\|) \leqslant \frac{\rho}{2} \frac{\lambda_{\min}(H)}{\lambda_{\max}(H)} \sum_{j=1}^{d} \mu_{ij} \|z_j\|^2, \quad i = 1, \ldots, d.$$

Let $\mathbf{z} = \mathrm{col}(\|z_1\|, \|z_2\|, \ldots, \|z_d\|)$. For every $i = 1, \ldots, d$, introduce the matrices $\nu_i^{(1)}$, $\nu_i^{(2)}$, η_i, of dimension $n \times n$ and let

$$\nu_i^{(1)} = \begin{pmatrix} 0 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \ldots & \sum\limits_{j=1}^{d} |\alpha_{ij} L_{ij}| & \ldots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & 0 \end{pmatrix}.$$

Here, the nonzero element $\sum\limits_{j=1}^{d} |\alpha_{ij} L_{ij}|$ is located on the principal diagonal in the ith row. Admitting $\alpha_{i,i} = 0$, let

$$\nu_i^{(2)} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ |\alpha_{i1} L_{i1}| & |\alpha_{i2} L_{i2}| & \ldots & |\alpha_{i,i-1} L_{i,i-1}| & 0 & |\alpha_{i,i+1} L_{i,i+1}| & \ldots & |\alpha_{id} L_{id}| \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Here, the nonzero elements are located in the ith row. Next, let

$$\eta_i = \frac{\rho}{2} \frac{\lambda_{\min}(H)}{\lambda_{\max}(H)} \begin{pmatrix} \mu_{i1} & 0 & \ldots & 0 \\ 0 & \mu_{i2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_{id} \end{pmatrix}.$$

With this notation, in order to satisfy (A.5), it is sufficient to require the fulfilment of the following inequality for all $i = 1, \ldots, d$:

$$\mathbf{z}^{\mathrm{T}} \left(\nu_i^{(1)} + \nu_i^{(2)} \right) \mathbf{z} \leqslant \mathbf{z}^{\mathrm{T}} \eta_i \mathbf{z},$$

i.e., for all $i = 1, \ldots, d$, the matrix $\eta_i - \nu_i^{(1)} - \nu_i^{(2)}$ should be nonnegative definite.

We then take the matrix $M = \{\mu_{ij}\}$ in the diagonal form as follows:

$$0 < \mu_{ii} < 1, \quad \mu_{ij} = 0, \quad i \neq j, \quad i = 1, \ldots, d, \quad j = 1, \ldots, d.$$

Then the matrix $M - I$ is Hurwitz.

Nonnegative-definiteness of the matrix $\eta_i - \nu_i^{(1)} - \nu_i^{(2)}$ implies the following condition:

$$\mu_{ii}\frac{\rho}{2}\frac{\lambda_{\min}(H)}{\lambda_{\max}(H)} - \sum_{j=1}^{d}|\alpha_{ij}L_{ij}| \geqslant 0, \quad i = 1, \dots, d,$$

or,

$$\mu_{ii} \geqslant \left(\frac{\rho}{2}\frac{\lambda_{\min}(H)}{\lambda_{\max}(H)}\right)^{-1}\sum_{j=1}^{d}|\alpha_{ij}L_{ij}|, \quad i = 1, \dots, d.$$

Noting that $\mu_{ii} < 1$, we arrive at condition (9). Therefore, the application of Lemma 2 is well justified, which proves Theorem 1.

Proof of Theorem 2. We mimic the proof of Theorem 1 and apply Lemmas 1 and 2 with L being changed for 0.

Let us show that $\omega_i(z_i, \theta_*) \leqslant -\rho Q(z_i)$ for $i = 1, \dots, d$. Indeed, we have

$$\begin{aligned}
\omega_i(z_i, \theta_*) &= z_i^{\mathrm{T}}H(A + B\theta_*^{\mathrm{T}}C^{\mathrm{T}})z_i + z_i^{\mathrm{T}}HB(\psi_0(y_i) - \psi_0(\overline{y})) \\
&= z_i^{\mathrm{T}}H(A + B\theta_*^{\mathrm{T}}C^{\mathrm{T}})z_i + z_i^{\mathrm{T}}Cg(\psi_0(y_i) - \psi_0(\overline{y})) \\
&= z_i^{\mathrm{T}}H(A + B\theta_*^{\mathrm{T}}C^{\mathrm{T}})z_i + (\psi_0(y_i) - \psi_0(\overline{y}))^{\mathrm{T}}g^{\mathrm{T}}(y_i - \overline{y}) \\
&= z_i^{\mathrm{T}}H(A + B\theta_*^{\mathrm{T}}C^{\mathrm{T}})z_i + (y_i - \overline{y})^{\mathrm{T}}g(\psi_0(y_i) - \psi_0(\overline{y})) \\
&\leqslant z_i^{\mathrm{T}}H(A + B\theta_*C^{\mathrm{T}})z_i, \quad i = 1, \dots, d.
\end{aligned}$$

The last inequality is valid since $\psi(\cdot)$ is g-monotonically decreasing. Next, we have

$$\omega_i(z_i, \theta_*) \leqslant z_i^{\mathrm{T}}H(A + B\theta_*^{\mathrm{T}}C^{\mathrm{T}})z_i = \frac{1}{2}z_i^{\mathrm{T}}\left[HA_* + A_*^{\mathrm{T}}H\right]z_i, \quad i = 1, \dots, d,$$

where $A_* = A + B\theta_*^{\mathrm{T}}C^{\mathrm{T}}$. Since $HA_* + A_*^{\mathrm{T}}H$ is negative definite, there exists a positive number ρ such that $HA_* + A_*^{\mathrm{T}}H \leqslant -\rho H$, which ensures the condition

$$\omega_i(z_i, \theta_*) \leqslant -\rho Q(z_i)$$

for every $i = 1, \dots, d$. By repeating the proof of Theorem 1 and taking Remark 1 into account, we arrive at the assertion of the theorem.

Proof of Theorem 3. To prove the theorem, we need the Yakubovich–Kalman lemma (the frequency theorem) [14] in the following form.

Lemma 3. *Let* $u \in \mathbb{R}^m$, $\chi(s) = C^{\mathrm{T}}(sI_n - A)^{-1}B$, $\operatorname{rank} B = m$. *Then the following two conditions are equivalent*:

(1) *There exists a positive-definite matrix* $H = H^{\mathrm{T}} > 0$ *such that*

$$HA + A^{\mathrm{T}}H < 0, \ HB = C;$$

(2) *The polynomial* $\det(sI_n - A)$ *is Hurwitz, and the frequency inequality*

$$\operatorname{Re} u^{\mathrm{T}}\chi(i\omega)u > 0, \quad \lim_{\omega\to\infty}\omega^2 \operatorname{Re} u^{\mathrm{T}}\chi(i\omega)u > 0$$

holds for all $\omega \in \mathbb{R}^1$, $u \in \mathbb{R}^m$, $u \neq 0$.

Note that in our case, $m = 1$, i.e., u is scalar, so that we take Cg instead of C. Then the conditions of Lemma 3 and Theorem 3 imply the existence of a matrix $H = H^{\mathrm{T}} > 0$ such that

$$HA_M + A_M^{\mathrm{T}}H < 0,\ HB = Cg.$$

Let z_i and $Q(z_i)$, $i = 1,\ldots,d$, denote the same quantities as in Section 3. For every $i = 1,\ldots,d$, by $\omega_i(x_i,\overline{x},\tau_i)$ we mean the derivative along the trajectories of the isolated subsystem:

$$\omega_i(x_i,\overline{x},\tau_i) = z_i^{\mathrm{T}}H(Ax_i + B\tau_i^{\mathrm{T}}(t)\sigma_i(t) + B_M\psi_0(y_i) - A_M\overline{x} - B_M(\overline{u} + \psi_0(\overline{y}))).$$

Denote $\tau_* = \mathrm{col}(\nu_*,\theta_*)$. For $\tau = \tau_*$ we obtain

$$\begin{aligned}\omega_i(x_i,\overline{x},\tau_*) &= z_i^{\mathrm{T}}H(Ax_i + B(\nu_*C^{\mathrm{T}}x_i + \theta_*\overline{u}) + B_M\psi_0(y_i) - A_M\overline{x} - B_M(\overline{u} + \psi_0(\overline{y})))\\ &= z_i^{\mathrm{T}}H(A_Mz_i + B_M\psi_0(y_i) - B_M\psi(\overline{y})) \leqslant z_i^{\mathrm{T}}HA_Mz_i, \quad i = 1,\ldots,d.\end{aligned}$$

The last inequality is valid since $\psi(\cdot)$ is g-monotonically decreasing (see proof of Theorem 2). Hence,

$$\omega_i(z_i,\tau_*) \leqslant z_i^{\mathrm{T}}HA_Mz_i = \frac{1}{2}z_i^{\mathrm{T}}\left[HA_M + A_M^{\mathrm{T}}H\right]z_i, \quad i = 1,\ldots,d,$$

i.e., there exists a number $\rho > 0$ such that $\omega_i(z_i,\tau_*) \leqslant -\rho Q(z_i)$. By repeating the rest of the proof of Theorem 1, we arrive at the desired result.

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This paper was recommended for publication by B. T. Polyak, a member of the Editorial Board