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Synchronization in Networks of Linear Agents with Output Feedbacks

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Abstract—The problem is considered of asymptotic synchronization by states in networks of identical linear agents in the application of the consensual output feedback. For the networks with fixed topology and without delay in the information transmission, on the basis of the passification theorem and the Agaev–Chebotarev theorem, the possibility is established of the provision of synchronization (consensus) of strong feedback under the assumption of the strict passification of agents and the existence of the incoming spanning tree in the information graph. In contrast to the known works, in which only the problems with the number of controls equal to the number of variables of the state of agents are investigated, in this work a substantially more complex case is considered, where the number of controls is less than the number of variables of the state, namely: the control is scalar. The results are illustrated by the example for the ring-shaped network of four dual integrators.

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1. INTRODUCTION

The problems of control of network systems generate considerable interest in the last years. Among numerous examples it is possible to mention multiprocessor systems of the transmission and processing of information, various transport networks, high-suitable production networks, the systems of coordinated control of motion of flying vehicles, submarine apparatus, and mobile robots, distributed systems of control of power networks, complex crystal lattices, and nanostructural objects. The confirmation of actuality and scientific significance of the problem is "boom" in the world scientific literature in the field of complex networks. Review articles are published, [1, 2], monographs [3–5], special issues of journals [6–8], conferences are held [9, 10].

One of the problems of network control is synchronization: provision of the time-consistent behavior of subsystems (agents). In modern works on synchronization, use is essentially made in networks of the graph of connections, which describes the structure of information flows in the network. The simplest and the most widespread control law in synchronization problems is the so-called "consensual control," at which the control signal for each node (agent) is built up as a weighted sum of differences of states or outcomes of adjacent nodes [2, 11, 12]. Here in the case of approach of the states of nodes with time, this points to the achievement in the network of *consensus*. In the investigation of the consensual control, Laplace matrices of graphs play an important role [5, 13, 14]. The necessary and sufficient conditions of the achievement of consensus, in the case when each node of the network is the integrator, are obtained in [15, 16]. The generalization of this result to the networks of dual integrators and the networks of dual integrators with delay is obtained in [17]. In the case of the networks of agents with dynamics of an arbitrary order, in most of the known works, feedbacks in the state of an object are built up. The existing controllers for the problems, where only outputs are accessible for the measurement, are based on the introduction of additional dynamic links into the controllers and the use of observers [12, 18, 19]. This complicates the realization and can increase the effect of noise and uncertainties, particularly at a large number of nodes in the network.

In this work the method is suggested of synthesis of static consensual controllers that afford the synchronization of solutions of dynamic systems at incomplete measurements and controls in the networks of identical linear objects at the arbitrary order of the agent model without the use of observers. For the networks with the fixed topology and without delay in the information transmission, on the basis of the passification theorem and the Agaev–Chebotarev theorem on the outgoing spanning tree, the synchronization conditions are obtained (the achievement of consensus in the network). The results are illustrated by the example for the ring-shaped network of four dual integrators.

2. PRELIMINARY INFORMATION

2.1. Information from the Theory of Graphs

We will present the necessary information from the theory of graphs, in particular, the definition of the Laplace matrix and its some properties (see [5, 13, 16, 20, 21]).

The oriented (directed) graph $\mathcal{G}$ is said to be the pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of vertices, while $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of arcs. Let N be the number of vertices (the power of the set $\mathcal{V}$). If for each arc $(\alpha, \beta) \in \mathcal{E}$, where $\alpha, \beta \in \mathcal{V}$, the $(\beta, \alpha) \in \mathcal{E}$ is fulfilled, then the graph is called the indirected one, while the arc is called the edge. Here and further, the graphs without loops are considered, i.e., for any vertex $\alpha \in \mathcal{V}$ the $(\alpha, \alpha) \notin \mathcal{E}$ is fulfilled. By way of the length j from the vertex α_1 to the vertex α_j, we have the ordered set $\{\alpha_1, \alpha_2, \ldots, \alpha_j\}$, where $(\alpha_{i-1}, \alpha_i) \in \mathcal{E}$ for each $i = 2, \ldots, j$ and all vertices α_j are different. The vertex α is accessible from the vertex β if $\alpha = \beta$ or the way exists in the directed graph from the vertex β to the vertex α. If for each vertex of the graph there exists a way to any other vertex, then the oriented graph is called the strongly connected graph, while the inoriented (indirected) graph is called the connected graph; in these cases the connectivity component in the (di)graph will be one. It is known that the indirected graph is connected if and only if it has the spanning tree—the tree that has the same set of vertices as the graph itself. For the digraph the notion of directed spanning trees is introduced: the incoming and the outgoing. The digraph is called the incoming tree if in each its vertex, apart from one, called the root, exactly one arc enters. The incoming spanning tree of the digraph $\mathcal{G}$ is called the incoming tree composed of the arcs of this digraph, such that in it there exists a way to the root from any other vertex $\mathcal{G}$. In a similar way, a more common notion is introduced: the incoming spanning forest (a set of trees). The incoming spanning forest $\mathcal{F}$ of the digraph $\mathcal{G}$ is called the maximum incoming forest if in $\mathcal{G}$ there is no incoming spanning forest with the number of arcs that is higher than that in $\mathcal{F}$. It is evident that each maximum incoming forest contains a minimum possible number of roots; this number is called the forest dimension of the digraph (by incoming trees) and is denoted by ν. The number of arcs in any maximum incoming forest is obviously equal to $N - \nu$. Let us note that the forest dimension by the outgoing trees can, generally speaking, differ from ν.

The indirected graph is called weighted if we correlate with each pair of vertices $\alpha, \beta \in \mathcal{V}$ the number $w(\alpha, \beta) \geqslant 0$ such that

(1) $w(\alpha, \beta) > 0$, if $(\alpha, \beta) \in \mathcal{E}$ and $w(\alpha, \beta) = 0$, if $(\alpha, \beta) \notin \mathcal{E}$;

(2) $w(\alpha, \beta) = w(\beta, \alpha)$.

The digraph is called weighted if we correlate with each pair of vertices $\alpha, \beta \in \mathcal{V}$ the number $w(\alpha, \beta) \geqslant 0$ such that condition (1) is fulfilled. The adjacency matrix $A(\mathcal{G}) = [a_{ij}]$ represents the $(N \times N)$-matrix, the i, jth element of which is equal to $w(\alpha_i, \alpha_j)$. For the vertex α_i we will introduce the in-degree

$$d_{in}(\alpha_i) = \sum_{j=1}^{N} a_{ji}$$

and the out-degree

$$d_{out}(\alpha_i) = \sum_{j=1}^{N} a_{ij}.$$

If for each vertex of the digraph $\mathcal{G}$ the in-degree is equal to the out-degree, then such a digraph is called the balanced one [5, 20].

We will introduce $(N \times N)$-matrix $D(\mathcal{G}) = \mathrm{diag}\{d_{out}(\alpha_1), d_{out}(\alpha_2), \ldots, d_{out}(\alpha_N)\}$. The Laplace matrix of the digraph $\mathcal{G}$ is said to be the matrix

$$L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G}).$$

We will denote by $\mathbf{1}_N$ the column vector of the dimension N, consisting of units. As is known [5, 13, 14, 16, 20–22], the introduced matrix L displays the following properties:

(1) The matrix $L(\mathcal{G})$ has the zero eigenvalue, to which the right eigenvector corresponds: $\mathbf{1}_N$: $L(\mathcal{G})\mathbf{1}_N = 0$;
(2) The multiplicity of the zero eigenvalue L of the indirected graph is equal to the connected components;
(3) The zero eigenvalue of the Laplace matrix L has the unit multiplicity if the appropriate digraph is strongly connected;
(4) All eigenvalues of the Laplace matrix have nonnegative real numbers;
(5) For the balanced graph, $\mathbf{1}_N$ is the left eigenvector corresponding to the zero eigenvalue:

$$\mathbf{1}_N^{\mathrm{t}} L(\mathcal{G}) = 0.$$

An important result was obtained by R.P. Agaev and P.Yu. Chebotarev in the year 2000 [21], see also [11].

Theorem 1 (Agaev–Chebotarev theorem [21]). *The rank of the Laplace matrix of the graph $\mathcal{G}$ is equal to $N - \nu$, where ν is the forest dimension of the graph by incoming trees. In particular,* rank $L = N - 1$, *i.e., the zero eigenvalue of the matrix L has the unit multiplicity if and only if the digraph $\mathcal{G}$ has the incoming spanning tree.*

2.2. The Method of Passification

We will present the necessary information on passification of linear systems [23, 24].

We will consider the linear system with one input and a few outputs (single-input-multiple-outputs—IMO):

$$\dot{x} = Ax + Bu, \quad z = C^{\mathrm{t}}x, \tag{1}$$

where $x = x(t) \in \mathbb{R}^n$ is the state vector, $u = u(t) \in \mathbb{R}^1$ is the control action (input), $z = z(t) \in \mathbb{R}^l$ is the measurable vector of outputs, A, B, C are the constant real matrices of dimensions $n \times n$, $n \times 1$, $n \times l$, respectively.

The problem of passification for the system (1) is understood as the discovery of $(1 \times l)$-matrix K, such that the system closed by the feedback $u = -Kz + v$, is strictly passive with respect to the auxiliary output $\sigma = Gz$ (G—$(1 \times l)$-matrix): for a certain $\rho > 0$ and any $T > 0$ the inequality $\int_0^T (\sigma v - \rho|x|^2)dt \geqslant 0$ is fulfilled along trajectories of the system (1) with the initial condition $x(0) = 0$. As follows from the Yakubovich–Kalman–Popov lemma and from the properties of passive systems [25–27], the passification of the system is equivalent to the existence of the matrix K affording *the strict positive reality* (SPR) of the closed system: its transfer function[1] $W(\lambda) = GC^{\rm t}(\lambda I_n - A + BKC^{\rm t})^{-1}B$ from the input v to the output $\sigma = Gz$ satisfies the relations:

$$\mathrm{Re}\, W(i\omega) > 0 \quad \forall\, \omega \in \mathbb{R}^1, \quad i^2 = -1, \tag{2}$$

$$\lim_{\omega\to+\infty} \omega^2\, \mathrm{Re}\, W(i\omega) > 0. \tag{3}$$

The importance of the properties of passivity and passification in the theory of control is defined by their close connection with the stability and stabilizability (see [25, 26, 28]).

Definition. The system (1) is called *minimum phase* over the output $\sigma = Gz$ if the polynomial

$$\varphi_0(s) = \det \begin{bmatrix} sI_n - A & -B \\ GC^{\rm t} & 0 \end{bmatrix} \tag{4}$$

is the Hurwitz one (all its roots have negative real parts) and *hyper-minimum phase* (HMP) if it is minimum-phase and $GC^{\rm t}B > 0$.

If the transfer function of the system (1) "from the input u to the output $\sigma = Gz$" has the form $\bar{W}(s) = b(s)/a(s)$, where $b(s), a(s)$ are polynomials of the degrees k, n respectively, $k \leqslant n$, then the system is the hyper-minimum-phase one if and only if $b(s)$ is the Hurwitz polynomial, $k = n - 1$ and $b(0) > 0$.

Theorem 2 (the theorem of passification [23, 24]). *The following assertions are equivalent:*

(A1) *The positive definite $(n \times n)$-matrix H and $(1 \times l)$-matrix K exist, such that the following relations are fulfilled:*

$$H(A + BKC^{\rm t}) + (A + BKC^{\rm t})^{\rm t}H < 0, \quad HB = CG^{\rm t}; \tag{5}$$

(B1) *The system* (1) *is the hyper-minimum-phase one with respect to the output $\sigma = Gz$;*

(C1) *There exists a feedback*

$$u = Kz + v, \tag{6}$$

that makes the closed system (1), (6) *strictly passive with respect to the output $\sigma = Gz$.*

On fulfilling condition (B1), the matrix K in (5) can be found in the form $K = -\varkappa G$, where $\varkappa$ is a sufficiently large positive number. Here the lower bound $\varkappa_0$ for $\varkappa$ has the form [24, 28]:

$$\varkappa > \varkappa_0 = \sup_{\omega\in\mathbb{R}^1} \mathrm{Re}\big(GW(i\omega)\big)^{-1}. \tag{7}$$

The generalization of Theorem 2 for the case of a few inputs (MIMO) can be found in [24]. For a few inputs, in the definition of the hyper-minimum-phasing we include the additional requirement for symmetry $(GC^{\rm t}B)^{\rm t} = GC^{\rm t}B$. The central part of the theorem is the equivalence of (A1) and (B1), which was established for MIMO systems in [23].

[1] I_n denotes the identity $(n \times n)$-matrix.

3. STATEMENT OF THE PROBLEMS

We will consider the network S consisting of N subsystems (agents) $S_i, i = 1, \ldots, N$. Let for each $i = 1, \ldots, N$ the subsystem S_i be described by the following equation:

$$\dot{x}_i = Ax_i + Bu_i, \quad y_i = C^{\mathrm{t}} x_i, \tag{8}$$

where $x_i(t) \in \mathbb{R}^n$ is the state vector, $u_i(t) \in \mathbb{R}^1$ is control, $y_i(t) \in \mathbb{R}^l$ is the measurement vector, the time is $t \in [0, +\infty)$.

We will examine the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of vertices and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of arcs. For each $i = 1, \ldots, N$ the vertex v_i is associated with the subsystem S_i. We will assume that the arc (v_i, v_j) belongs to the set of arcs $\mathcal{E}$ if the information comes from the subsystem S_j to the subsystem S_i. It is assumed that the graph has no loops, i.e., $(v_i, v_i) \notin \mathcal{E}$ for all $i = 1, \ldots, N$. In addition, it is assumed that the unit weight is correlated with each arc.

Let the control law for the agent S_i has the form

$$u_i = K \sum_{j \in \mathcal{N}_i} (y_i - y_j) = KC^{\mathrm{t}} \sum_{j \in \mathcal{N}_i} (x_i - x_j), \tag{9}$$

where $K \in \mathbb{R}^{1 \times l}$ is the row vector of amplification factors, $\mathcal{N}_i = \{k = 1, \ldots, N | (v_i, v_k) \in \mathcal{E}\}$ is the set of indices of vertices accessible from v_i in one step. The control of the form (9) is said to be consensual. In the last few years a great many of the works are devoted to the study of the properties of systems with the consensual control (see the bibliography in [1, 4, 11]). However, in the known works only the problems are investigated in which the number of controls is equal to the number of variables of the state of agents. In this work, a rather more complex case is considered, when the number of controls is less than the number of variables of the state. For definiteness, we will assume that the control is scalar.

As a goal of control, we will consider the asymptotic synchronization in the states (the asymptotic coordinate synchronization in the terminology of [29]) of agents S:

$$\lim_{t \to \infty} (x_i(t) - x_j(t)) = 0, \qquad i, j = 1, \ldots, N. \tag{10}$$

The problem consists in finding such a K from (9) as to fulfil the goal of control (10).

In the case of the fulfilment of (10), the asymptotic behavior of all objects of the network S will be identical. Denoting by $c(t)$ the time function describing the asymptotic behavior of objects of the network in synchronization, it is possible to reformulate the goal of control (10) in the following way:

$$\exists\, c(t) : \lim_{t \to \infty} (x_i(t) - c(t)) = 0, \quad i = 1, \ldots, N. \tag{11}$$

4. CONDITIONS OF ACHIEVEMENT OF THE GOAL OF CONTROL IN THE CASE OF THE BALANCED INFORMATION GRAPH

We denote $\chi(s) = C^{\mathrm{t}}(sI_n - A)^{-1}B, s \in \mathbb{C}$, and make the following suppositions.

(A1) *The digraph $\mathcal{G}$ has the incoming spanning tree.*

By the Agaev–Chebotarev theorem, this supposition affords the unit multiplicity of the zero eigenvalue of the Laplace matrix $L(\mathcal{G})$ (see the Section 2.1).

(A2) *There exists a vector $g \in \mathbb{R}^l$ such that the function $g^{\mathrm{t}}\chi(s)$ is the hyper-minimum-phase one.*

According to Theorem 2 of the Section 2.2, the last supposition affords the existence of the matrix $H = H^{\mathrm{t}} > 0$ and the vector $\theta \in \mathbb{R}^l$, such that the following relations are fulfilled:

$$HA_* + A_*^{\mathrm{t}}H < 0, \quad HB = Cg, \quad A_* = A + B\theta^{\mathrm{t}}C^{\mathrm{t}}. \tag{12}$$

Here the vector θ can be chosen in the form

$$\theta = -\varkappa g, \tag{13}$$

where the number $\varkappa > 0$ is rather large (see Section 2). The row vector of amplification factors K of the control law (9) will be taken in such a form:

$$K = -kg^{\mathrm{t}}, \quad k \in \mathbb{R}^1. \tag{14}$$

We will correlate with the digraph $\mathcal{G}$ the graph $\widehat{\mathcal{G}}$, such that $A(\widehat{\mathcal{G}}) = A(\mathcal{G}) + A(\mathcal{G})^{\mathrm{t}}$. The Laplace matrix $L(\widehat{\mathcal{G}})$ of the so obtained graph $\widehat{\mathcal{G}}$ is symmetric and has the zero eigenvalue of the unit multiplicity. The matrix $L(\widehat{\mathcal{G}})$ has the following eigenvalues: $0 = \lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_N$. Let $x = \mathrm{col}(x_1, \ldots, x_N)$ and $\otimes$ be the Kronecker product.

Theorem 3. *Let the suppositions* (A1), (A2) *be fulfilled and the graph* $\mathcal{G}$ *be balanced. Then for sufficiently large values of* $k > 0$ *the control* (9) *with the vector* (14) *of amplification factors ensures the fulfilment of the goal* (11) *with the function* $c(t) = N^{-1/2}e^{At}(\mathbf{1}_N^{\mathrm{t}} \otimes I_n)x(0)$.

The proof of the Theorem 3 is given in the Appendix.

Remark 1. Theorem 3 can be stated in such a way (see the proof):

Let the suppositions A1, A2 *be fulfilled and the graph* $\mathcal{G}$ *be balanced. Then for* k *values, such that*

$$k \geqslant \frac{2\varkappa}{\lambda_2},$$

the control (9) *with the vector* (14) *of amplification factors affords the fulfilment of the goal* (11) *with the function* $c(t) = N^{-1/2}e^{At}(\mathbf{1}_N^{\mathrm{t}} \otimes I_n)x(0)$.

Remark 2. From the proof of the theorem 3 and from the Agaev–Chebotarev theorem it follows that if the incoming spanning tree in the graph of information links is unavailable, i.e., the forest dimension of the graph in incoming trees $\nu > 1$, then the consensual controllers (9) ensure the convergence of solutions of the system (8), (9) to the subspace of the dimension ν stretched over the vectors corresponding to the root vertices of the forest. Thus, in this case the partial synchronization takes place with the number of leaders corresponding to the number of root vertices of the connected graph.

5. CONDITIONS OF ACHIEVEMENT OF THE CONTROL AIM IN THE CASE OF THE UNBALANCED INFORMATION GRAPH

Let the supposition (A1) be fulfilled, so that the zero eigenvalue of the Laplace matrix L has the unit multiplicity. We will bring the Laplace matrix L to the Jordan form

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_e \end{pmatrix} = P^{-1}LP,$$

considering that the first column of the nonsingular matrix P is equal to $N^{-1/2}\mathbf{1}$. We will denote by l_1 the left eigenvector corresponding to the zero eigenvalue of the Laplace matrix L, such that $l_1^{\mathrm{t}}N^{-1/2}\mathbf{1} = 1$.

The row vector of the amplification factors K of the control law (9) will be taken in the form

$$K = k\,\theta^{\mathrm{t}}, \quad k \in \mathbb{R}^1. \tag{15}$$

Theorem 4. *Let the suppositions* (A1), (A2) *and* $\Lambda_e + \Lambda_e^* > 0$ *be fulfilled. Then for* k *values such that*

$$I_{N-1} + \frac{k}{2}(\Lambda_e + \Lambda_e^*) \leqslant 0,$$

the control (9) *with the vector of amplification factors* (15) *ensures the fulfilment of the goal* (11) *with the function* $c(t) = N^{-1/2} e^{At}(l_1^{\mathrm{t}} \otimes I_n)x(0)$.

The proof is similar to the proof of the theorem 3.

6. CONDITIONS OF ACHIEVEMENT OF THE CONTROL AIM IN THE CASE OF THE INDIRECTED INFORMATION GRAPH

We will state the conditions of achieving the control aim in the case of the indirected information graph $\mathcal{G}$ and fulfilling the following supposition:

(A3) *The indirected graph* $\mathcal{G}$ *has the spanning tree.*

As is known, this supposition is equivalent to the connectivity of the graph $\mathcal{G}$. Thus, in the fulfilment of this supposition the multiplicity of the zero eigenvalue of the Laplace matrix $L = L(\mathcal{G})$ will be equal to unity. We will denote the eigenvalues of the matrix L in such a way: $0 = \lambda_1(L) < \lambda_2(L) \leqslant \ldots \leqslant \lambda_N(L)$.

In the following theorem the conditions are given for the achievement of the control aim in the case of the indirected information graph.

Theorem 5. *Let the suppositions* (A2), (A3) *be fulfilled. Then for* k *values, such that*

$$k \geqslant \frac{\varkappa}{\lambda_2(L)},$$

the control (9) *with the vector of amplification factors* (14) *affords the fulfilment of the goal* (11) *with the function* $c(t) = N^{-1/2} e^{At}(\mathbf{1}_N^{\mathrm{t}} \otimes I_n)x(0)$.

The theorem is the corollary of Remark 1 from Section 4.

7. EXAMPLE. THE RING-SHAPED NETWORK OF DUAL INTEGRATORS

We will consider the network S consisting of four subsystems S_i, $i = 1, \ldots, 4$. Let for each $i = 1, \ldots, 4$ the subsystem S_i be described by the following equation:

$$\dot{x}_i = Ax_i + Bu_i, \quad y_i = C^{\mathrm{t}} x_i,$$

where $x_i(t) \in \mathbb{R}^2$ is the state vector, $u_i(t) \in \mathbb{R}^1$ is the control, $y_i(t) \in \mathbb{R}^1$ is the measurement vector. Let

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then the transfer function of each object has the form

$$\chi(s) = C^{\mathrm{t}}(sI_2 - A)^{-1}B = \frac{s+1}{s^2}. \tag{16}$$

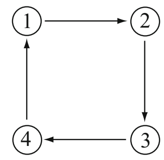

Fig. 1. Digraph $\mathcal{G}$.

It can be said that each object is the dual integrator in which the sum of the output variable and its derivative are accessible for the measurement.

The first component of the state vector of an individual object can be treated as a point speed, and the second component as a position of the point on the straight line. The achievement of the control goal (10) implies the approach of four points on the straight line and their motion at the identical speed depending on the initial conditions.

Let us assume that the digraph $\mathcal{G}$ describing the information connections in the network is balanced and has the form displayed in Fig. 1. This means that the consensual controllers (9) in this case have the form

$$u_1 = -k(y_1 - y_2), \quad u_2 = -k(y_2 - y_3), \quad u_3 = -k(y_3 - y_4), \quad u_4 = -k(y_4 - y_1).$$

The Laplace matrix of the graph $\widehat{\mathcal{G}}$ has the following eigenvalues: 0, 2, 2, 4.

We will apply the Theorem 3. The transfer function (16) is the hyper-minimum-phase one at $g = 1$. It is easy to see that the inequality (13) for the choice of the number $\varkappa$ takes the form $\varkappa > 1$. Thus, by the Theorem 3, at $k \geqslant 1$ the controller (9) with the row vector of the amplification factors (14) affords the achievement of the goal (10).

7.1. Results of Computational Modeling

Let the subsystems have the following initial data:

$$x_1(0) = \mathrm{col}(0.5; 2), \quad x_2(0) = \mathrm{col}(-7; 3), \quad x_3(0) = \mathrm{col}(1; 0), \quad x_4(0) = \mathrm{col}(10; -10).$$

At $k = 1$ the achievement of the control aim is confirmed by the results of the computational 50-second modeling. The trajectories of subsystems on the same phase plane with coordinates $(x_i^{(1)}, x_i^{(2)})$, $i = 1, 2, 3, 4$ at various values of k are displayed in Figs. 2–5.

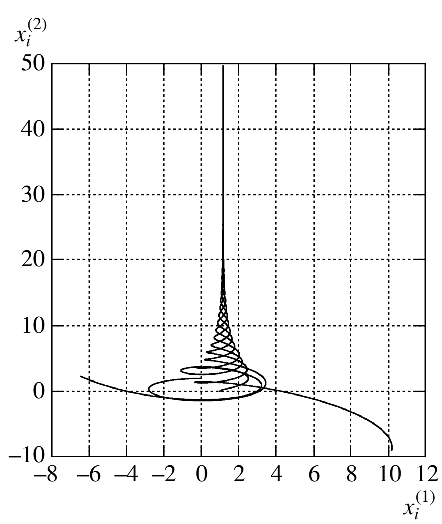

Fig. 2. Phase plane at $k = 1$.

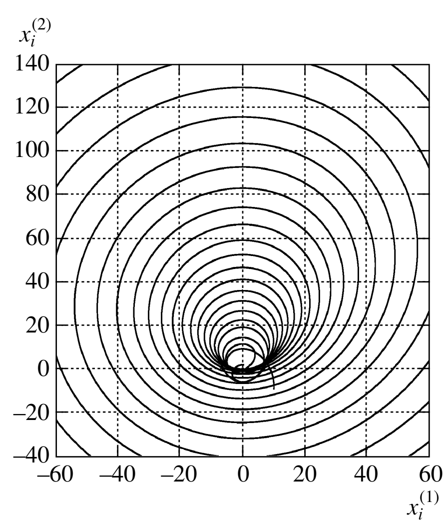

Fig. 3. Phase plane at $k = 0.3$.

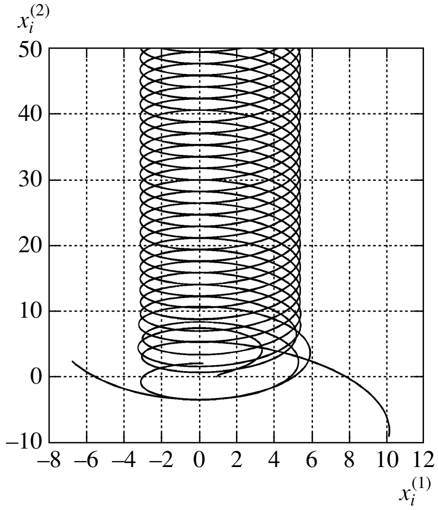

Fig. 4. Phase plane at $k = 0.5$.

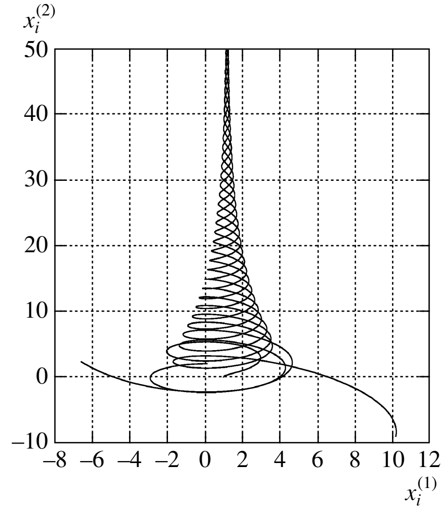

Fig. 5. Phase plane at $k = 0.7$.

It should be noted that for this example from the results of [17] it is possible to derive the exact conditions of asymptotic synchronization: $k > 0.5$.

8. CONCLUSIONS

With the aid of the passification method and the Agaev–Chebotarev theorem the conditions are found for the synchronization achievement at the states in the networks of linear agents at incomplete measurements and controls with the aid of consensual controllers implementing the static output feedback. In contrast to most of the known works in which only the problems are investigated with the number of controls that is equal to the number of variables for the state of agents, in this work a substantially more complex case is dealt with, when the number of controls is less than the number of variables of the state, namely: the control is scalar. As distinct from the result of the work [30] (Theorem 4), in this work there is a need not for passivity, but only for passification, which permits us to synchronize the networks of unstable agents.

The example of synchronization in the ring-shaped networks of four dual integrators shows that the obtained conditions are not too far from the exact ones. Their advantage also lies in the fact that they are easily extended to nonlinear systems with the sector nonlinearity. This extension is assumed to be performed later on.

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APPENDIX

Proof of Theorem 3. In the proof it will be necessary for us some Kronecker products of the matrices, (see, for example, [31, 32]). For brevity, we denote $L = L(\mathcal{G}), \widehat{L} = L(\widehat{\mathcal{G}})$. Let P be the real orthogonal matrix, such that

$$P^{\mathrm{t}}\widehat{L}P = \mathrm{diag}(0, \lambda_2, \dots, \lambda_N)$$

with the first column equal to $N^{-1/2}\mathbf{1}_N$. The first column of the product LP is zero because $\mathbf{1}_N$ is the right eigenvector of the matrix L. Consequently, the first column $P^{\mathrm{t}}LP$ is also zero. By the supposition (A1), the digraph $\mathcal{G}$ is balanced, hence, L has the left eigenvector corresponding to the zero eigenvalue. The latter enables us to conclude that the first row $P^{\mathrm{t}}LP$ is zero. Thus,

$$P^{\mathrm{t}}LP = \begin{pmatrix} 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & \Lambda_e & & \\ \vdots & & & & & \\ 0 & & & & & \end{pmatrix}, \tag{A.1}$$

where $\Lambda_e \in \mathbb{R}^{(N-1)\times(N-1)}$. Let $u = \mathrm{col}(u_1, \dots, u_N)$. Then (9) can be presented in the form

$$u = (L \otimes KC^{\mathrm{t}})x. \tag{A.2}$$

We will rewrite (8) using (A.2):

$$\dot{x} = \Big((I_N \otimes A) + (L \otimes BKC^{\mathrm{t}})\Big)\, x. \tag{A.3}$$

The idea on the coordinate transformation of the composite state vector is taken from [19].

Let $z = (P^{\mathrm{t}} \otimes I_n)x$, $z \in \mathbb{R}^{Nn}$, and $z = \mathrm{col}(z_1, z_e)$, $z_1 \in \mathbb{R}^n$, $z_e \in \mathbb{R}^{(N-1)n}$. Then (A.3) with consideration for (A.1) can be presented in such a way:

$$\dot{z}_1 = Az_1, \tag{A.4}$$

$$\dot{z}_e = \Big((I_{N-1} \otimes A) + (\Lambda_e \otimes BKC^{\mathrm{t}})\Big)\, z_e. \tag{A.5}$$

If the solution $z_e(t) \equiv 0$ of the system of Eqs. (A.5) is asymptotically stable, then the control aim (11) is reached with the function $c(t) = N^{-1/2}e^{At}(\mathbf{1}_N^{\mathrm{t}} \otimes I_n)x(0)$.

We will take the Lyapunov function in the form

$$V_e = z_e^{\mathrm{t}}(I_{N-1} \otimes H)z_e,$$

where the matrix H is defined from (12). We will calculate the derivative V_e in view of (A.5):

$$\begin{aligned} \dot{V}_e &= z_e^{\mathrm{t}}(I_{N-1} \otimes (A^{\mathrm{t}}H + HA) + \Lambda_e^{\mathrm{t}} \otimes CK^{\mathrm{t}}B^{\mathrm{t}}H + \Lambda_e \otimes HBKC^{\mathrm{t}})z_e \\ &= z_e^{\mathrm{t}}(I_{N-1} \otimes (A^{\mathrm{t}}H + HA) + \Lambda_e^{\mathrm{t}} \otimes CK^{\mathrm{t}}g^{\mathrm{t}}C^{\mathrm{t}} + \Lambda_e \otimes CgKC^{\mathrm{t}})z_e. \end{aligned}$$

Here use was made of the fact that $HB = Cg$ (see (12)). We denote

$$\mathcal{P} = I_{N-1} \otimes (A^{\mathrm{t}}H + HA) + \Lambda_e^{\mathrm{t}} \otimes CK^{\mathrm{t}}g^{\mathrm{t}}C^{\mathrm{t}} + \Lambda_e \otimes CgKC^{\mathrm{t}}.$$

We will show that the zero solution of the system (A.5) is asymptotically stable. For this it is sufficient to show that $\mathcal{P} < 0$ because $\dot{V}_e = z_e^{\mathrm{t}}\mathcal{P}z_e$.

We will denote

$$\mathcal{K} = -I_{N-1} \otimes (C\theta g^{\mathrm{t}}C^{\mathrm{t}} + Cg\theta^{\mathrm{t}}C^{\mathrm{t}}) + \Lambda_e^{\mathrm{t}} \otimes CK^{\mathrm{t}}g^{\mathrm{t}}C^{\mathrm{t}} + \Lambda_e \otimes CgKC^{\mathrm{t}}.$$

Noting that

$$A^{\mathrm{t}}H + HA = A_*^{\mathrm{t}}H + HA_* - (C\theta g^{\mathrm{t}}C^{\mathrm{t}} + Cg\theta^{\mathrm{t}}C^{\mathrm{t}}),$$

we will transform the expression for $\mathcal{P}$:

$$\mathcal{P} = I_{N-1} \otimes (A_*^{\mathrm{t}}H + HA_*) + \mathcal{K}.$$

It is sufficient to show that $\mathcal{K} \leqslant 0$. We will rewrite the expression for $\mathcal{K}$ using (13), (14):

$$\begin{aligned}\mathcal{K} &= -I_{N-1} \otimes (-\varkappa Cgg^{\mathrm{t}}C^{\mathrm{t}} - \varkappa Cgg^{\mathrm{t}}C^{\mathrm{t}}) - \Lambda_e^{\mathrm{t}} \otimes Cgkg^{\mathrm{t}}C^{\mathrm{t}} - \Lambda_e \otimes Cgkg^{\mathrm{t}}C^{\mathrm{t}} \\ &= (2\varkappa I_{N-1} - k(\Lambda_e + \Lambda_e^{\mathrm{t}})) \otimes (Cgg^{\mathrm{t}}C^{\mathrm{t}}).\end{aligned}$$

Because $Cgg^{\mathrm{t}}C^{\mathrm{t}} \geqslant 0$, while $2\varkappa I_{N-1} - k(\Lambda_e + \Lambda_e^{\mathrm{t}}) \leqslant 0$ according to the condition of the Theorem 3, we obtain $\mathcal{K} \leqslant 0$, which is what we set out to prove.

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