ISSN 0005-1179, Automation and Remote Control, 2012, Vol. 73, No. 11, pp. 1765–1783. © Pleiades Publishing, Ltd., 2012.
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TOPICAL ISSUE

Approximate Consensus in the Dynamic Stochastic Network with Incomplete Information and Measurement Delays

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Received January 19, 2012

Abstract—Consideration was given to the problem of achieving an approximate consensus in the decentralized stochastic dynamic network under incomplete information about the current states of the nodes, measurement delay, and variable structure of links. Solution was based on the protocol of local voting with nonvanishing steps. It was proposed to analyze dynamics of the closed network with the use of the method of averaged models which was extended to the systems with measurement delays. This method enables one to establish good analytical estimates of the permissible length of the step providing the desired accuracy of consensus and appreciably reduce the computational burden of simulation. The results obtained were applied to the analysis of the dynamics of the system of balancing the computer network loading.

DOI: 10.1134/S000511791211001X

1. INTRODUCTION

Distributed interaction in the networks of dynamic controllable agents attracted recently attention of numerous researchers. In many respects this is due to the popularity of the multiagent systems in diverse fields including control of formations [1] and swarms [2], distributed sensor networks [3], control of overload in the communication networks [4], maneuvering of drone groups [5], relative alignment of the satellite groups, and so on. Many of such problems may be reformulated in terms of achieving consensus in the multiagent systems [6–8].

Practical solution of such problems is complicated by incompleteness of the transmitted information and presence of the measurement noise, on the one hand, and the effects of quantization (digitizing) that are characteristic of the digital systems, on the other hand. For a group of interacting agents interchanging incomplete information with delay at discrete time instants using the local voting protocol under varying link topology, it was proposed [9] to use the approach of stochastic approximation to derive the consensus conditions [10, 11]. Algorithms like stochastic gradient were used in such problems even earlier [12–15]. The stochastic approximation with decreasing step enables each agent to acquire information about the states of its neighbors with simultaneous reduction of the noise impact. The authors of [13, 16] considered the problem of achieving consensus in noisy information about the states of neighbors. They used an algorithm like vanishing-step stochastic approximation.

The algorithms of vanishing-step stochastic approximation are inapplicable under external actions and agents' nonstationarity (reception of new tasks and so on). Operability of the constant-step stochastic approximation algorithms under nonstationary performance functionals (mean risk) was studied in [17–20]. Their validity for balancing the node loads of the centralized computer network under available current noisy information about the queue length and node performance was considered in [21, 22].

To study the discrete stochastic network systems, it was suggested [23–25] to use the method of averaged continuous models [26, 27] which is sometimes called the Derevitskii–Fradkov–Ljung (DFL) scheme [28, 29]). Application of the method of averaged models to the analysis and design of the stochastic discrete-time systems was started in the 1970s. Many papers and some monographs concerning both application and substantiation of the method were published since then [20, 27, 30–33].

The present paper extends and specifies the results established in [25]. Its main result lies in the conditions for achieving an approximate root-mean-square consensus with the use of the local voting protocol. To derive them, established were the conditions for proximity in the root-mean-square trajectory of the initial dynamic network and its averaged model. The results obtained are applied to the problem of balancing the load of the nodes of the distributed computer network upon arrival to each node only of the noisy information about the queue length and performance of the neighbors. The topology of the network links is assumed to vary in time, and the information from the neighbors is assumed to come with a time delay.

The problems of balancing the node loads are often encountered in the literature (for example, [34–41]) which indicates to their topicality. The authors of [34] suggested to use a rule with uniform random distribution and an arbitrary rule for balancing the network load. A load balancing algorithm for the heterogeneous P2P systems based on the Mobile Agent is considered in [35]. A dynamic algorithm of order distribution based on a mixed decentralized and centralized policy was proposed in [36]. The present paper makes use of the local voting protocol to balance the network load and the method of averaged models to study the system dynamics.

The following section introduces the main notions used below. The consensus protocol to be used for balancing the load of the network nodes is defined in Section 3. The method of averaged models is described in Section 4 where the basic results are formulated. The problem of balancing the node load of the decentralized computer network is considered in Section 5, and Section 6 analyzes the described system. An example of the simulation study of the algorithm and its corresponding averaged model is described in Section 7.

2. PROBLEM OF CONSENSUS IN THE DYNAMIC NETWORKS

The paper uses the following notation and terms of the graph theory [10, 42]. The superscript of a node refers to its number and not to the exponent. For the column vectors $Z_1, \ldots, Z_l$, the column vector obtained by their vertical connection is denoted by $[Z_1, \ldots, Z_l]$. The norms of the vectors or matrices M are understood in the sense of the *Frobenius norm*: $||M|| = \sqrt{Tr(M^{\mathrm{T}}M)}$, where $Tr(\cdot)$ is the matrix track (sum of the diagonal elements).

By the dynamic network is meant a set of dynamic systems (agents) interacting according to the graph of information links. Let us consider a dynamic network of n agents. Let i, $i = 1, \ldots, n$, be the agent's number, and $N = \{1, \ldots, n\}$, the set of network nodes (vertices of the link graph). The graph (N, E) is defined by the sets of vertices, N, and arcs, E. We assume that the graph is *simple*, that is, $(i, i) \notin E$, there are no loops for all i, and at most one arc can be between the vertices. By the *neighbor set* of the node i is meant $N^i = \{j : (j, i) \in E\}$, that is, the set of nodes with arcs entering i. We assign the weight $a^{i,j} > 0$ to each arc $(j, i) \in E$. The graph is representable by the *adjacency (or connectedness) matrix* $A = [a^{i,j}]$ with the weights $a^{i,j} > 0$, if $(j, i) \in E$, and $a^{i,j} = 0$, otherwise. We notice that $a^{i,i} = 0$. The graph represented by the adjacency matrix A is denoted by $\mathcal{G}_A$.

We define the *weighted in-degree of the vertex* i as the sum of the ith row of the matrix A: $d^i(A) = \sum_{j=1}^{n} a^{i,j}$. For the graph $\mathcal{G}_A$, we define the *diagonal matrix of degrees of graph vertices* $D(A) = \mathrm{diag}\{d^i(A)\}$ from the in-degrees and the graph *Laplacian* $\mathcal{L}(A) = D(A) - A$. We note that the sums along the rows of the Laplacian elements are zero. The maximal in-degree of the graph $\mathcal{G}_A$ is denoted by $d_{\max}(A)$.

The *directed path* from the node i_1 to the node i_s consists of a sequence of nodes $i_1, \ldots, i_s,\ s \geqslant 2$ such that $(i_k, i_{k+1}) \in E,\ k \in \{1, \ldots, s-1\}$. The notions of the incoming and outgoing directed spanning trees are introduced for the digraph. A digraph is called the incoming tree if each of its vertices, except for one called the root, is entered precisely by a single arc. By the incoming spanning tree of a digraph (or simply the spanning tree) is meant the incoming tree consisting of the arcs of this digraph such that there exists in it a path to the root from any other vertex.

A time-variable state $x_t^i \in \mathbb{R}$ is assigned to each agent $i \in N$ at the time instant $t = 0, 1 \ldots, T$. Let $u_t^i \in \mathbb{R}$ be the control action on the node i at the time instant t, and $y_t^{i,i}$ be the measured output of the agent. We assume that the initial conditions $x_0^i \in \mathbb{R}$ are given and the variations of the agent states obey the difference equations

$$x_{t+1}^i = x_t^i + f^i(x_t^i, u_t^i) \tag{1}$$

with the control u_t^i whose action on the state x_t^i is defined by some function $f^i(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ depending on the current state of the agent x_t^i and the prescribed control u_t^i.

We assume that the structure of links in the dynamic network is described by a sequence of directed graphs $\{(N, E_t)\}_{t \geqslant 0}$, where $E_t \subseteq E$ is time-varying. We denote by A_t the corresponding adjacency matrices and by $E_{\max} = \{(j, i) : \sup_{t \geqslant 0} a_t^{i,j} > 0\}$, the maximal set of link channels. If $(j, i) \in E_t$, the node i is said to receive information at the time instant t from the node j for feedback control.

To generate control, each node $i \in N$ has (possibly noisy) information about its state

$$y_t^{i,i} = x_t^i + w_t^{i,i}, \tag{2}$$

and if the set N_t^i is nonempty, noisy observations about the neighbor states

$$y_t^{i,j} = x_{t-d_t^{i,j}}^j + w_t^{i,j}, \quad j \in N_t^i, \tag{3}$$

where $w_t^{i,i}, w_t^{i,j}$ is the interference (noise), $0 \leqslant d_t^{i,j} \leqslant \bar{d}$ is an integer delay, and $\bar{d}$ is the greatest possible delay. For correctness of the introduced notions, we assume that $x_{-k}^i = 0,\ i \in N,\ k = -d, -d-1, \ldots, -1$. And for convenience we assume that $w_t^{i,j}$ and $d_t^{i,j}$ are defined for all $(j, i) \in E_{\max}$. If (j, i) does not occur in E_t, then (3) is not used and $w_t^{i,j}$ and $d_t^{i,j}$ may be regarded as zeros.

3. CONTROL PROTOCOL AND CONSENSUS

Definition 1. In the dynamic network with the topology (N, E_t), by the *control protocol* is meant the state observation feedback

$$u_t^i = K_t^i\left(y_t^{i,j_1}, \ldots, y_t^{i,j_{m_i}}\right), \tag{4}$$

where the set $\{j_1, \ldots, j_{m_i}\} \subseteq \{i\} \bigcup \bar{N}_t^i,\ \bar{N}_t^i \subseteq N_t^i$, and K_t^i is some smooth function of its arguments for each $t = 0, 1, 2, \ldots$.

The so-called "local voting protocol"

$$u_t^i = \alpha_t \sum_{j \in \bar{N}_t^i} b_t^{i,j}(y_t^{i,j} - y_t^{i,i}), \tag{5}$$

where the input of each node depends on the weighted sum of the differences between its state and the information about the states of its neighbors finds wide use. Here, $\alpha_t > 0$ is the parameter of a step of the control protocol (5), $b_t^{i,j} > 0\ \ \forall j \in \bar{N}_t^i$. We assume that $b_t^{i,j} = 0$ for the rest of the pairs (i, j).

Definition 2. We can assert that *consensus* is achieved in the network at the time instant t if for all $i \in N$ there exists $x^\star$ (*value of consensus*) such that $x^\star = x_t^i$.

The problem of achieving consensus lies in generating a control protocol that provides consensus.

4. ANALYSIS OF CLOSED-LOOP SYSTEM DYNAMICS

Let $(\Omega, \mathcal{F}, \mathrm{Prob})$ be the main probabilistic space. We denote by E the expectation and by E_x the conditional expectation for x. We assume that the following condition is satisfied for the functions $f^i(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ from the network dynamics Eq. (1):

(**A1**) The functions $f^i(x, u)$ are Lipschitzian in x and u for all $\forall i \in N$:

$$|f^i(x, u) - f^i(x', u')| \leqslant L_1(L_x|x - x'| + |u - u'|);$$

for any fixed x, the functions $f^i(x, \cdot)$ are such that $\mathrm{E}_x f^i(x, u) = f^i(x, \mathrm{E}_x\, u)$;

The condition for constrained rate of growth $|f^i(x, u)|^2 \leqslant L_2(L_c + L_x|x|^2 + |u|^2)$ follows from Lipschitzianity.

We denote by $\bar{w}_t \in \mathbb{R}^{n\times n}$ the vector composed of the elements $w_t^{i,j}$, $i, j \in N$ (the $n \times n$ matrix of interferences of observations or zeros represented by rows as a vector), $\bar{x}_t = [x_t^1, \ldots, x_t^n]$ and $\bar{u}_t = [u_t^1, \ldots, u_t^n]$.

If α_t does not tend to zero, then under nondegenerate interferences in the observations $\bar{w}_t$ the precise consensus usually is not achieved. Therefore, we consider the problem of achieving an approximate root-mean-square consensus.

Definition 3. We assert that at the time instant t a root-mean-square ε-consensus is achieved in the network if $\mathrm{E}||x_t^i||^2 < \infty$, $i \in N$ and there exists a random variable $x^\star$ such that $\mathrm{E}||x_t^i - x^\star||^2 \leqslant \varepsilon$ for all $i \in N$.

We formulate the sufficient conditions for the local voting protocol (5) to ensure satisfaction of the ε-consensus.

(**A2**) (**a**) For all $i \in N$, $j \in N_t^i \cup \{i\}$, the observation interferences $w_t^{i,j}$ are centered, independent, identically distributed random variables with bounded variances: $\mathrm{E}(w_t^{i,j})^2 \leqslant \sigma_w^2$.

(**b**) For all $i \in N$, $j \in N_{\max}^i$, the "variable" arcs (j, i) in the graph $\mathcal{G}_{A_t}$ represent independent random events with the probabilities $p_a^{i,j}$, that is, A_t are independent identically distributed random matrices.

(**c**) For all $i \in N$, $j \in \bar{N}_t^i$, the weights $b_t^{i,j}$ in the control protocol are bounded random variables: $\underline{b} \leqslant b_t^{i,j} \leqslant \bar{b}$ with the probability 1, and there are $b^{i,j} = \mathrm{E} b_t^{i,j}$.

(**d**) For all $i, j \in N$, there exists a finite variable $\bar{d} \in \mathbb{N}$: $d_t^{i,j} \leqslant \bar{d}$ with the probability 1, and the integer delays $d_t^{i,j}$ are independent identically distributed random variables taking on values $k = 0, \ldots, \bar{d}$ with the probabilities $p_k^{i,j}$.

Additionally, all these random variables and matrices are mutually independent.

For $t = 1, 2, \ldots,$ we define extending sequences of the σ-algebras of the probabilistic events $\mathcal{F}_t$ generated by the random elements $\bar{x}_0, A_0, \ldots A_{t-1}, d_0^{i,j}, \ldots, d_{t-1}^{i,j}, b_0^{i,j}, \ldots, b_{t-1}^{i,j}, w_0^{i,j}, \ldots, w_{t-1}^{i,j}$, $i, j \in N$, and $\tilde{\mathcal{F}}_t = \sigma\{\mathcal{F}_t, A_t\}$.

Let the random variable Q and σ-algebra of the probabilistic events $\mathcal{F}$ be given. The expectation Q relative to the σ-algebra $\mathcal{F}$ is denoted by $\mathrm{E}_{\mathcal{F}} Q$. We notice that the random variables $\bar{x}_t$ are measurable relative to the σ-algebra $\mathcal{F}_t$, that is, $\mathrm{E}_{\mathcal{F}_t}\bar{x}_t = \bar{x}_t$.

For $\bar{d} > 0$, we add $n\bar{d}$ new nodes to the network under consideration including new "dummy" agents with the states at the instant t equal to the corresponding states of the real agents at the preceding $\bar{d}$ time instants $t-1, t-2, \ldots, t-\bar{d}$. We denote $\bar{n} = n(\bar{d}+1)$ and define the $\bar{n} \times \bar{n}$ matrix $A_{\max}$ according to the rule

$$a_{\max}^{i,j} = p_{j \div n}^{i,((j-1) \bmod n)+1} p_a^{i,((j-1) \bmod n)+1} b^{i,((j-1) \bmod n)+1}, \quad i \in N, \quad j = 1, \ldots, \bar{n},$$
$$a_{\max}^{i,j} = 0, \quad i = n+1, \ldots, \bar{n}, \quad j = 1, \ldots, \bar{n}.$$

Here, the operation mod is the residue of division, and the operation $\div$ is the exact division.

If $\bar{A}_t$ is a sequence of random matrices with element defining link between all $\bar{n}$ agents at the time instant t, then, obviously, all of them are distributed identically, and the matrix $A_{\max}$ is their expectation. We assume that the following condition is satisfied:

(**A3**) The graph $(N, E_{\max})$ has a spanning tree and for any arc $(j, i) \in E_{\max}$ there exists at least one nonzero element among the elements $a_{\max}^{i,j}, a_{\max}^{i,j+n}, \ldots, a_{\max}^{i,j+\bar{d}n}$ of the matrix $A_{\max}$.

We assume that $\bar{x}_t \equiv 0$ for $-\bar{d} \leqslant t < 0$ and define $\bar{X}_t \in \bar{n}$, an extended state vector $\bar{X}_t = [\bar{x}_t, \tilde{x}_{t-1}, \ldots, \tilde{x}_{t-\bar{d}}]$, where $\tilde{x}_{t-k}$ is a vector consisting of x_{t-k}^i such that $\exists j \in N^i\ \exists k' \geqslant k : p_{k'}^{i,j} > 0$, that is, this value is involved in generation of at least one control action with a positive probability. In what follows, we assume for simplicity that the so-introduced extended vector of states is $\bar{X}_t = [\bar{x}_t, \bar{x}_{t-1}, \ldots, \bar{x}_{t-\bar{d}}]$, that is, includes all components with every possible delays not exceeding $\bar{d}$.

We rearrange the dynamics of the generalized network states in the vector-matrix form:

$$\bar{X}_{t+1} = U\bar{X}_t + F(\alpha_t, \bar{X}_t, \bar{w}_t), \tag{6}$$

where U is a $\bar{n} \times \bar{n}$ matrix given by

$$U = \begin{pmatrix} I & 0 & 0 & \ldots & 0 \\ I & 0 & 0 & \ldots & 0 \\ 0 & I & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & I & 0 \end{pmatrix}, \tag{7}$$

where I is the $n \times n$ identity matrix and $F(\alpha_t, \bar{X}_t, \bar{w}_t) : \mathbb{R} \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{n \times n} \to \mathbb{R}^{\bar{n}}$ is the vector function of the corresponding arguments

$$F(\alpha_t, \bar{X}_t, \bar{w}_t) = \begin{pmatrix} \ldots \\ f^i\left(x_t^i, \alpha_t \sum\limits_{j \in \bar{N}_t^i} b_t^{i,j}\left((x_{t-d_t^{i,j}}^j - x_t^i) + (w_t^{i,j} - w_t^{i,i})\right)\right) \\ \ldots \\ 0_{n\bar{d}} \end{pmatrix} \tag{8}$$

having nonzero components only at the n first places.

In this case, it is suggested to study the system dynamics with the use of the method of averaged models which lies in the approximate replacement of the original stochastic difference Eq. (6) describing the network dynamics by an averaged discrete equation.

Let us consider the averaged discrete model corresponding to (6):

$$\bar{Z}_{t+1} = U\bar{Z}_t + G(\alpha_t, \bar{Z}_t), \quad \bar{Z}_0 = \bar{X}_0, \tag{9}$$

where

$$G(\alpha,\bar Z)=G\begin{pmatrix}\alpha, & \begin{matrix} z^1\\ \vdots \\ z^{n(\bar d+1)}\end{matrix}\end{pmatrix}=\begin{pmatrix}\cdots\\ f^i(z^i,\alpha\bar s^i(\bar Z))\\ \cdots\\ 0_{n\bar d}\end{pmatrix}, \quad (10)$$

$$s^i(\bar Z)=\sum_{j\in N^i_{\max}} p_a^{i,j}b^{i,j}\left(\left(\sum_{k=0}^{\bar d} p_k^{i,j}z^{j+kn}\right)-z^i\right)=-d^i(A_{\max})z^i+\sum_{j=1}^{\bar n}a^{i,j}_{\max}z^j,\quad i\in N.$$

We note that under the above notation and satisfied last part of condition (**A1**),

$$\bar s(\bar Z)=[s^1(\bar Z),\dots,s^n(\bar Z)]=-\mathcal{L}(A_{\max})\bar Z.$$

If the last part of condition (**A1**) is not satisfied, then the definition

$$G(\alpha,\bar Z)=\mathrm{E}_{\bar Z}F(\alpha_t,\bar Z,\bar w_t) \quad (11)$$

can be used instead of (10); at that the formulations of the subsequent results undergo minor changes.

It turns put that at the time instant t the trajectories of the solution of the original system $\{\bar X_t\}$ from (6) are close in the root-mean-square sense to the trajectories of the averaged discrete system (9).

Theorem 1. *If conditions* (**A1**) *and* (**A2**) *are satisfied, then there exists* $\tilde\alpha$ *such that the following estimate is true for* $0<\alpha_t\leqslant\bar\alpha<\tilde\alpha$*:*

$$\mathrm{E}\max_{0\leqslant t\leqslant T}||\bar X_t-\bar Z_t||^2\leqslant c_1\tau_T e^{(c_2\tau_T)^2}\bar\alpha, \quad (12)$$

where $\tau_T=\alpha_0+\alpha_1+\ldots+\alpha_{T-1},\ c_1,\ c_2>0$ *are some constants*

$$c_1=2^{3+\bar d}n^3L_1^2\bar b^2\left(\sigma_w^2+4\left(\frac{nL_2L_c+(\bar\alpha nL_1\bar b)^2}{c_3-1}+||\bar X_0||^2\right)e^{T\ln c_3}\right),$$

$$c_2=2^{\tilde d/2}L_1\left(\frac{L_x}{\underline{\alpha}}+||\mathcal{L}(A_{\max})||\right),$$

$$\underline{\alpha}=\min_{0\leqslant t\leqslant T}\alpha_t,\quad \tilde d=0,\quad \textit{if}\quad \bar d=0,\quad \textit{or}\quad \tilde d=1,\quad \textit{if}\quad \bar d>0,$$

$$c_3=2^{\tilde d}+L_x\left(2^{1+\tilde d/2}L_1+L_2\right)+\bar\alpha 2^{1+\tilde d/2}L_1||\mathcal{L}(A_{\max})||$$
$$+\bar\alpha^2\left(L_2||\mathcal{L}(A_{\max})||^2+(2L_1(n-1)\bar b)^2\right).$$

See the proof in the Appendix.

We note that similar estimates can also be obtained for the general form of protocol (4).

Theorem 2. *If the conditions* (**A1**), (**A2**), *and* $0<\alpha_t\leqslant\bar\alpha$ *are satisfied, in the averaged discrete system* (9) *the* $\dfrac{\varepsilon}{4}$*-consensus is achieved at the time instant* T*, and the following estimate is valid for the constants* c_1 *and* c_2 *from the formulation of Theorem 1*

$$c_1\tau_Te^{c_2\tau_T^2}\bar\alpha\leqslant\frac{\varepsilon}{4}, \quad (13)$$

then the ε*-consensus is achieved in the stochastic discrete system* (6) *at the time instant* T.

See the proof in the Appendix.

Let us consider an important special case of $f^i(x,u) = u\ \forall i \in N$ and $\alpha_t = \alpha = \text{const}$ with the discrete averaged system (9) given by

$$\bar{Z}_{t+1} = (I - ((I-U) + \mathcal{L}(\alpha A_{\max})))\bar{Z}_t = (I - \bar{\mathcal{L}})\bar{Z}_t, \tag{14}$$

where in the matrix $\bar{\mathcal{L}}$ all element row sums are zero and all diagonal elements are positive and equal in magnitude to the sum of the rest of the negative row elements, that is, the matrix $\bar{\mathcal{L}}$ is the Laplacian of a certain graph.

Theorem 3. *Let $\alpha_t = \alpha > 0$, $f^i(x,u) = u$ for any $i \in N$. If the conditions* (**A2**), (**A3**), *and $\alpha < \dfrac{1}{d_{\max}(A_{\max})}$ are satisfied, then for any $\varepsilon > 0$ there will be time $\bar{T}$ such that after it the $\dfrac{\varepsilon}{4}$-consensus establishes in the averaged discrete system* (14). *Additionally, if there exists $T > \bar{T}$ for which the parameter α ensures satisfaction of the condition $\bar{C}_1 e^{\bar{C}_2^2}\alpha \leqslant \dfrac{\varepsilon}{4}$, where*

$$\bar{C}_1 = T\alpha 2^{3+\bar{d}} n^3 L_1^2 \bar{b}^2 \left(\sigma_w^2 + 4\left(\frac{(\bar{\alpha} n L_1 \bar{b})^2}{c_3 - 1} + ||\bar{X}_0||^2\right) e^{T \ln c_3}\right),$$

$$\bar{C}_2 = T\alpha 2^{\bar{d}/2}||\mathcal{L}(A_{\max})||,$$

$$c_3 = 2^{\bar{d}} + \bar{\alpha} 2^{1+\bar{d}/2}||\mathcal{L}(A_{\max})|| + \bar{\alpha}^2\left(||\mathcal{L}(A_{\max})||^2 + 4(n-1)^2\bar{b}^2\right),$$

then the ε-consensus is achieved in the stochastic discrete system (6) *at the time instants $t:\ \bar{T} \leqslant t \leqslant T$.*

See the proof in the Appendix.

We note that the necessary and sufficient condition for achieving the root-mean-square consensus in the case where the lengths of the steps α_t tend to zero was proved in [9] under some assumptions allied to the conditions of Theorem. 3. Consideration was given above to a more general case of the functions like $f^i(x_t^i, u_t^i)$ and α_t that do not tend to zero.

5. BALANCING THE LOAD OF THE DECENTRALIZED NETWORK NODES

The distributed systems of parallel computations for which the problem of sharing a task package between several computers is topical recently have found an increasing use. Such problems arise not only in the computer networks but also in the production networks [43], queuing networks [44], transport and logistic networks, and so on. They usually attempt to distribute the node tasks so as to make their loads uniform.

The majority of the dynamic balancing strategies may be classified with the centralized or completely distributed (decentralized) strategies. In the case of centralized strategy, there exists a special resource acquiring information about the entire network and making decision about allocation of the tasks to each node. For the completely decentralized strategies, a load balancing protocol exchanging the state information with other network nodes is executed at each node, and the tasks are moved only between the neighbor nodes.

We consider a model of the system of distribution of identical tasks between different nodes (agents) for parallel operation with feedback. Let $N = \{1,\ldots,n\}$ be a set of smart agents (nodes) executing the arriving tasks according to the queue principle. The tasks arrive to the system at different instants and to different nodes.

The state of agent i, $i = 1,\ldots,n$, at each time instant t is described by two characteristics:

(1) $q_t^i \geqslant 0$ is the length of queue of the atomic elementary tasks of the node i at the time instant t;

(2) $r_t^i > 0$ is the performance of the node i at the time instant t.

The agent's states obey the following equations:

$$q_{t+1}^i = q_t^i - r_t^i + z_t^i + u_t^i, \quad i = 1, \dots, n, \quad t = 0, 1, 2, \dots, \tag{15}$$

where z_t^i is a new task arriving to the node i at the time instant t, u_t^i is the result of relocating (adding or reducing) the tasks between the nodes by using the selected task redistribution protocol. We assume that in the equations of dynamics $\sum_i u_t^i = 0$, $t = 0, 1, 2, \dots$.

We assume that each node $i \in N$ at the time instant t has the following information to generate the control strategy:

- noisy data about its queue length

$$y_t^{i,i} = q_t^i + w_t^{i,i}, \tag{16}$$

- noisy observations of the neighbor queue lengths if $N_t^i \neq \emptyset$

$$y_t^{i,j} = q_{t-d_t^{i,j}}^j + w_t^{i,j}, \quad j \in N_t^i, \tag{17}$$

  where $w_t^{i,j}$ are interferences, $0 \leqslant d_t^{i,j} \leqslant \bar{d}$ is the integer delay, and $\bar{d}$ is the greatest possible delay.

- data about its performance r_t^i and that of the neighbor nodes r_t^j, $j \in N_t^i$.

We consider two formulations—stationary and nonstationary—of the problem.

<u>Stationary case.</u> All tasks arrive to different nodes of the system at the initial time instant, and the node performances do not vary in time.

If all tasks are executed only by the agent to which they arrived, then the time of executing all tasks obeys

$$T_{\max} = \max_{i \in N} \frac{q_0^i}{r_0^i}.$$

We notice that the form of system (15) in the stationary case corresponds to the difference Eq. (1).

<u>Nonstationary case.</u> New tasks may arrive to any of the n system nodes at different time instants t, and the node performances can vary in time.

For the time instant t, we define the time T_t until completion of all tasks at all nodes.

If in the stationary case beginning from the time instant t the tasks are not reallocated between the nodes, the time of executing all tasks is given by

$$T_t = \max_{i \in N} \frac{q_t^i}{r_t^i}. \tag{18}$$

The ratio $\frac{q_t^i}{r_t^i}$ will be called the load of the node i at the time instant t.

We pose the following aim of control:

$$T_t \to \min_{\bar{u}_t}. \tag{19}$$

It is only natural to use a protocol of task reallocation in time for achieving this aim. This allows one to increase the system throughput and reduce the time of order execution in the system.

Lemma (on the optimal control strategy). *In the stationary case, of all possible variants of distributing the total amount of tasks that are not processed by the time instant t the least time of system operation corresponds to that for which*

$$\frac{q_t^i}{r_t^i} = \frac{q_t^j}{r_t^j}, \quad \forall\, i, j \in N. \tag{20}$$

See the proof in the Appendix.

Corollary. *If* $x_t^i = \dfrac{q_t^i}{r_t^i} + t$ *is taken as the state of the node i of the dynamic network, then the aim of control—consensus in the network—corresponds to the optimal distribution of tasks between the nodes in the stationary case.*

Therefore, it suffices to consider the task of maintaining the uniform load of all network nodes. It is solved using the control protocol (5) where for all $i \in N$ we define $\bar{N}_t^i = N_t^i$ and $b_t^{i,j} = \dfrac{r_t^j}{r_t^i}$, $j \in N_t^i$ for all t.

As a matter of fact, the use of protocol (5) is justified in practice if additional assumptions are satisfied. First of all, it is assumed that the corresponding input data are exchanged instantaneously. Additionally, one has to verify coordination of package forwarding because various collisions can be created by the delays and arrival of information with interferences. In particular, in practice in the problem of balancing the load of nodes of the decentralized computer network protocol (5) requires additional coordinations of the sizes of the packages transmitted between the agents. Neither "overconsumption" nor "underconsumption" is permitted at relocation of the resources (or tasks between the nodes). Additional checks and coordinations between the neighbors are executed to satisfy this condition. With the use of the local voting protocol, each node determines how many tasks it can "give away" or "receive." Then the nodes ready to accept tasks send requests to the neighbors about the amounts they are ready to give at the given instant. Each "accepting" node sends in response to these requests a confirmation of how much tasks it can accept from one or another node and coordinates this amount with it (at that, each node orients to its current values $\tilde{u}_t^i$ recommended by the local voting protocol). It is assumed that the procedure of task coordination and transmission need much less time than one cycle of the dynamic system. For example, in the problem of order allocation at organization of cargo transportation the tomorrow's tasks are coordinated at night upon completion of the current workday (see [45]).

If beginning from the time instant $\bar{t}$ no new tasks arrive to the system ($z_t^i = 0, t \geqslant \bar{t}$) and the performances do not vary ($r_t^i = r^i, t \geqslant \bar{t}$), then Theorem 1 from the last section enables one to reduce the study of the dynamics of node load balancing to that of the corresponding discrete averaged model which can be executed either analytically or numerically.

With regard for possible additional coordinations according to the packet transmission protocol on the basis of protocol (5), in the case under consideration the dynamics of system (1) is approximately given by

$$x_{t+1}^i = x_t^i + \alpha_t \sum_{j \in N_t^i} b_t^{i,j} \left(\frac{y_t^{i,j}}{r_t^j} - \frac{y_t^{i,i}}{r_t^i} \right), \tag{21}$$

where α_t is a sequence of positive certain sizes of steps, y_t^{ij} are the noisy observations of the queue length of the jth node, and z_t^i is the new task arriving to the node i at the time instant t.

It is assumed that at each step the size of the transmitted tasks is small as compared with the current task queue length. For this reason and also because consideration is given to the approximate consensus, we regard the node state and queue length as real variables.

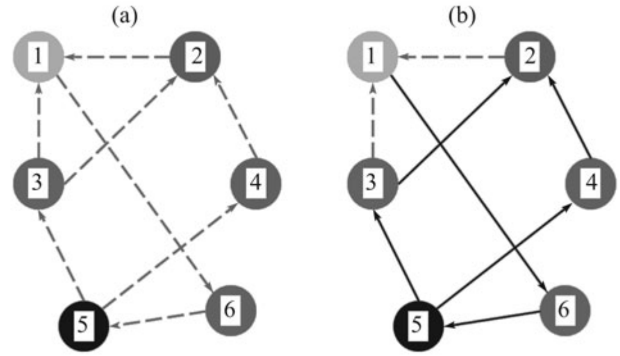

Fig. 1. (a) Maximal set of the communication channels $E_{\max}$; (b) network topology.

It is clear that roundoff is done at realizing protocol (5). The algorithm is efficient if the task queues are sufficiently long. In this case, the roundoff does not exert appreciable influence on the result because consideration is given to the approximate consensus.

We consider by way of example a system consisting of six computing units (Fig. 1). At each time instant the network topology is random. Link 2—1 exists with the probability 1/2, as well as link 3—1 (Fig. 1b). The arcs 2—1 and 3—1 have the same probabilities $p_a^{13} = p_a^{12} = 1/2$ of their occurrence.

In the case of uniformly distributed measurement delays where the integer delay d_t^{ij} is 0 or 1 with the probability 1/2, $\bar{d} = 1$, $p_0^{ij} = p_1^{ij} = 1/2$, we expand the state space:

$$\bar{X}_t = [x_t^1, \dots, x_t^n, x_{t-1}^1, \dots x_{t-1}^n] \in \mathbb{R}^{2n}. \tag{22}$$

The matrix G of the corresponding averaged discrete model (9) is as follows:

$$G = \begin{pmatrix} \frac{1}{2}H\alpha & \frac{1}{2}H\alpha \\ 0 & 0 \end{pmatrix}, \tag{23}$$

where

$$H = \begin{pmatrix} 0 & \frac{1}{2}b^{1,2} & \frac{1}{2}b^{1,3} & 0 & 0 & 0 \\ 0 & 0 & 0 & b^{2,4} & 0 & 0 \\ 0 & 0 & 0 & 0 & b^{3,5} & 0 \\ 0 & 0 & 0 & 0 & b^{4,5} & 0 \\ 0 & 0 & 0 & 0 & 0 & b^{5,6} \\ b^{6,1} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{24}$$

6. SIMULATION

To illustrate the theoretical findings, we give an example of simulating the problem of load balancing of a decentralized computer network depicted in Fig. 1. Predefined were the initial states $q_0^1 = 5000$, $q_0^2 = 3500$, $q_0^3 = 2300$, $q_0^4 = 3150$, $q_0^5 = 7400$, $q_0^6 = 1100$ and node performances $r^1 = 2$, $r^2 = 0.75$, $r^3 = 1.2$, $r^4 = 1.7$, $r^5 = 3.5$, $r^6 = 2.1$. The node performances were time invariable.

The observations of the current queue lengths were carried out against the background of the centered independent interferences from the interval $[-500; 500]$ that were distributed uniformly.

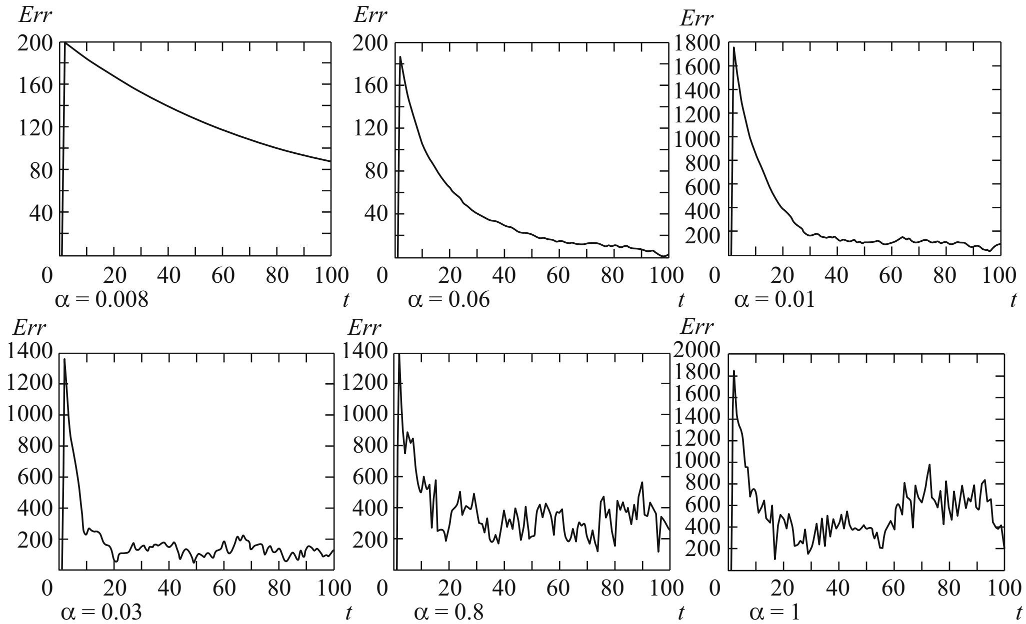

Fig. 2. Comparison of the mean mismatches for different parameters of the step α.

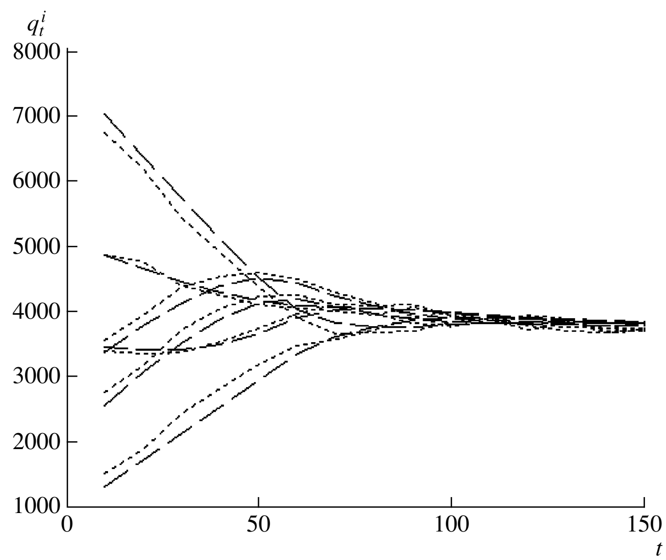

Fig. 3. Behavior of the trajectories of system (21) and the corresponding averaged model.

The performance of protocol (5) (convergence of the trajectories to the consensus $x^\star$) is characterized by the mean mismatch $Err = \sqrt{\sum_i \frac{(x_t^i - x^\star)^2}{n}}$. For different constant parameters of the step α, the mean mismatches are depicted in Fig. 2. As can be seen, the greater α, the faster convergence to the consensus. However, from a certain time instant the behavior of the mean mismatches becomes largely oscillatory.

The results of comparing the trajectories of the discrete stochastic system (dotted lines) and the corresponding aforementioned averaged discrete model (dashed lines) are shown in Fig. 3 for the constant step $\alpha_t = 0.1$.

7. CONCLUSIONS

The present paper considered the problem of achieving consensus in the multiagent stochastic system with nonlinear dynamics, interferences and delay in the measurement, and switching topology. An algorithm of the type of stochastic approximation with a nonvanishing step was used to solve the problem.

A mathematical model of the problem of decentralized balancing of the network loading under incomplete information about the states of nodes and variable topology was discussed as a practical example.

The system behavior was studied with the use of the method of averaged models. It is planned to consider in future the impact of various interferences on the algorithm.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research, project no. 11-08-01218-a, and the Federal Goal-oriented Program "Scientific and Educational Personnel of Innovative Russia" (state contracts nos. 16.740.11.0042 and 14.740.11.0942).

The authors should like to thank their reviewers for constructive remarks which enabled them to substantially improve the quality of this paper.

APPENDIX

Proof of Theorem 1. We denote

$$v_t = F(\alpha_t, \bar{X}_t, \bar{w}_t) - G(\alpha_t, \bar{X}_t). \tag{A.1}$$

In virtue of condition (**A2**), we establish by averaging successively relative to the σ-algebras $\tilde{\mathcal{F}}_t$ and $\mathcal{F}_t$ that $\mathrm{E}_{\mathcal{F}_t} v_t = 0$.

The following auxiliary propositions will be used to prove Theorem 1.

Proposition 1. *If conditions* (**A2**) *are satisfied, then* $\bar{s}(\bar{X}_t) = \dfrac{1}{\alpha_t}\mathrm{E}_{\mathcal{F}_t}\bar{u}_t$ *and the estimate*

$$\mathrm{E}_{\mathcal{F}_t}\left\|\frac{1}{\alpha_t}\bar{u}_t - \bar{s}(\bar{X}_t)\right\|^2 \leqslant n^2\bar{b}^2\sigma_w^2 + 4(n-1)^2\bar{b}^2||\bar{X}_t||^2, \quad i \in N \tag{A.2}$$

is valid.

Proof. We obtain in virtue of the definition of protocol (5) that

$$\frac{1}{\alpha_t}u_t^i = \sum_{j \in \bar{N}_t^i} b_t^{i,j}\left(\left(x_{t-d_t^{i,j}}^j - x_t^i\right) + \left(w_t^{i,j} - w_t^{i,i}\right)\right).$$

By taking the conditional expectation under the condition of the σ-algebras $\tilde{\mathcal{F}}_t$ of both sides of the last formula, we establish in virtue of assumptions (**A2**), (**a**), (**c**), and (**d**) about centering of the observation interferences and independence of the corresponding random variables that

$$\frac{1}{\alpha_t}\mathrm{E}_{\tilde{\mathcal{F}}_t}u_t^i = \sum_{j \in \bar{N}_t^i}\sum_{k=0}^{\bar{d}} b^{i,j}p_k^{i,j}(x_{t-k}^j - x_t^i) = \sum_{j \in \bar{N}_t^i}\sum_{k=0}^{\bar{d}} b^{i,j}p_k^{i,j}(\bar{x}_t^{j+kn} - \bar{x}_t^i)$$

from which the first proposition $s^i(\bar{x}) = \frac{1}{\alpha_t}\mathrm{E}_{\mathcal{F}_t} u_t^i$ follows after averaging relative to the σ-algebra $\mathcal{F}_t$. Using this fact, we obtain for the left side of (A.2) that

$$\mathrm{E}_{\mathcal{F}_t}\left\|\frac{1}{\alpha_t}\bar{u}_t - \bar{s}(\bar{X}_t)\right\|^2 = \frac{1}{\alpha_t^2}\mathrm{E}_{\mathcal{F}_t}||\bar{u}_t||^2 - 2\mathrm{E}_{\mathcal{F}_t}\frac{1}{\alpha_t}\bar{u}_t^{\mathrm{T}}\bar{s}(\bar{X}_t) + ||\bar{s}(\bar{X}_t)||^2$$
$$= \frac{1}{\alpha_t^2}\mathrm{E}_{\mathcal{F}_t}||\bar{u}_t||^2 - ||\bar{s}(\bar{X}_t)||^2.$$

We estimate $\frac{1}{\alpha_t^2}\mathrm{E}_{\mathcal{F}_t}||\bar{u}_t||^2$. Having considered first the conditional expectation relative to the σ-algebra $\tilde{\mathcal{F}}_t$, we obtain successively in virtue of centering of the observation interferences and boundedness of the weight coefficients in the control protocol (according to the conditions (**A2**), (**a**), (**c**), (**d**) that

$$\frac{1}{\alpha_t^2}\mathrm{E}_{\tilde{\mathcal{F}}_t}||\bar{u}_t||^2 = \sum_{i\in N}\mathrm{E}_{\tilde{\mathcal{F}}_t}\left(\sum_{j\in\bar{N}_t^i} b_t^{i,j}\left(\left(x^j_{t-d_t^{i,j}} - x_t^i\right) + \left(w_t^{i,j} - w_t^{i,i}\right)\right)\right)^2$$
$$= \sum_{i\in N}\mathrm{E}_{\tilde{\mathcal{F}}_t}\left(\sum_{j\in\bar{N}_t^i} b_t^{i,j}\left(x^j_{t-d_t^{i,j}} - x_t^i\right)\right)^2 + \mathrm{E}_{\tilde{\mathcal{F}}_t}\left(\sum_{j\in\bar{N}_t^i} b_t^{i,j}\left(w_t^{i,j} - w_t^{i,i}\right)\right)^2$$
$$\leqslant \sum_{i\in N}\mathrm{E}_{\tilde{\mathcal{F}}_t}\sum_{j\in\bar{N}_t^i}\left(x^j_{t-d_t^{i,j}} - x_t^i\right)^2\mathrm{E}_{\tilde{\mathcal{F}}_t}\sum_{j\in\bar{N}_t^i}\left(b_t^{i,j}\right)^2 + \mathrm{E}_{\tilde{\mathcal{F}}_t}\sum_{j\in\bar{N}_t^i}\left(b_t^{i,j}\right)^2\left(\left(w_t^{i,j}\right)^2 + \left(w_t^{i,i}\right)^2\right)$$
$$\leqslant n^2\bar{b}^2\sigma_w^2 + (n-1)\bar{b}^2\sum_{i\in N}\mathrm{E}_{\tilde{\mathcal{F}}_t}\sum_{j\in\bar{N}_t^i}\sum_{k=0}^{\bar{d}} p_k^{i,j}\left(x^j_{t-k} - x_t^i\right)^2.$$

By averaging relative to the σ-algebra $\mathcal{F}_t$, one can easily establish

$$\frac{1}{\alpha_t^2}\mathrm{E}_{\mathcal{F}_t}||\bar{u}_t - \bar{s}(\bar{X}_t)||^2$$
$$\leqslant n^2\bar{b}^2\sigma_w^2 + 2(n-1)\bar{b}^2\sum_{i\in N}\sum_{j\in N^i_{\max}}(\bar{x}_t^i)^2 + \sum_{k=0}^{\bar{d}} p_k^{i,j}(\bar{x}_t^{j+kn})^2$$
$$\leqslant n^2\bar{b}^2\sigma_w^2 + 4(n-1)^2\bar{b}^2||\bar{X}_t||^2.$$

Proposition 2.

$$||U\bar{X}||^2 \leqslant 2^{\bar{d}}||\bar{X}||^2,\ \ldots,\ ||U^{\bar{d}}\bar{X}||^2 \leqslant 2^{\bar{d}}||\bar{X}||^2,\ \ldots,\ ||U^k\bar{X}||^2 \leqslant 2^{\bar{d}}||\bar{X}||^2.$$

Proof. The first inequality can be readily determined in virtue of the definition of the matrix U, and the following inequalities are determined by induction on k with provision for

$$\forall k > \bar{d}\quad U^k = U^{\bar{d}} = \begin{pmatrix} I & 0 & 0 & \ldots & 0 \\ I & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ I & 0 & 0 & \ldots & 0 \end{pmatrix}. \tag{A.3}$$

Proposition 3. *The estimate*

$$\mathrm{E}\max_{0\leqslant t\leqslant T}\left\|\sum_{i=0}^{t} v_t\right\|^2 \leqslant 4n\sum_{t=0}^{T}\mathrm{E}||v_t||^2$$

is valid if conditions (**A2**) *are satisfied.*

Proof. If conditions (**A2**) are satisfied, the random elements v_t are the martingale differences, that is, they are centered under the conditional averaging in the prehistory: $\mathrm{E}_{\mathcal{F}_{t-1}} v_t = 0$. The proof is similar to the proof of Corollary 1 to Lemma 1 from Section 3 [46] if one takes $v_t = \zeta_t - \zeta_{t-1}$. The size of the vectors v_t is $\bar{n}$, yet since only the n first components of the vectors v_t are other than zero, in the formula n may be used for estimation instead of $\bar{n}$.

Proposition 4. *Let the sequence of numbers* $\mu_t \geqslant 0$, $\alpha_t \geqslant 0$, $\tau_t \geqslant 0$, $t = 0, 1, \ldots, T$ *satisfy the inequalities*

$$\mu_{t+1} \leqslant \bar{\alpha} c_1 \tau_t + c_2^2 \tau_t \sum_{k=1}^{t} \alpha_k \mu_k,$$

where $c_1, c_2, \bar{\alpha} \geqslant 0$. *Then,*

$$\mu_t \leqslant c_1 \tau_t e^{(c_2 \tau_t)^2} \bar{\alpha}, \quad t = 0, 1, \ldots, T.$$

Proof. The proposition follows immediately from the corresponding result in [47] and represents a discrete counterpart of the Gronwall–Bellman lemma.

Proposition 5. *If conditions* (**A1**), (**A2**) *are satisfied, then*

$$\mathrm{E}||\bar{X}_t||^2 \leqslant \left(\frac{2nL_2 + (\bar{\alpha} n L_1 \bar{b})^2}{c_3 - 1} + ||\bar{X}_0||^2 \right) e^{t \ln c_3}.$$

Proof. We set down Eq. (6) as

$$\bar{X}_{t+1} = U \bar{X}_t + G(\alpha_t, \bar{X}_t) + v_t. \tag{A.4}$$

We obtain for the square of the norm $\bar{X}_{t+1}$ that

$$||\bar{X}_{t+1}||^2 = ||U\bar{X}_t + G(\alpha_t, \bar{X}_t)||^2 + 2(U\bar{X}_t + G(\alpha_t, \bar{X}_t))^{\mathrm{T}} v_t + ||v_t||^2. \tag{A.5}$$

By taking the conditional expectation of both sides of (A.5) in the σ-algebra $\mathcal{F}_t$, that is, under fixed $\bar{X}_t$, in virtue of the centrality of v_t we obtain that

$$\begin{aligned} \mathrm{E}_{\mathcal{F}_{t-1}} ||\bar{X}_{t+1}||^2 &= ||U\bar{X}_t + G(\alpha_t, \bar{X}_t)||^2 + \mathrm{E}_{\mathcal{F}_{t-1}} ||v_t||^2 \\ &= ||U\bar{X}_t||^2 + 2(U\bar{X}_t)^{\mathrm{T}} G(\alpha_t, \bar{X}_t) + ||G(\alpha_t, \bar{X}_t)||^2 + E_{\mathcal{F}_{t-1}} ||v_t||^2. \end{aligned} \tag{A.6}$$

In virtue of the form of v_t and Lipschitzianity in u of the functions $f^i(\cdot,\cdot)$ (by assumption (**A1**)) for $||v_t||^2$, we obtain

$$\begin{aligned} ||v_t||^2 = \sum_{i \in N} \left| f^i \left(x_t^i, \alpha_t \sum_{j \in \bar{N}_t^i} b_t^{i,j} \left(x^j_{t - d_t^{i,j}} - x_t^i + w_t^{i,j} - w_t^{i,i} \right) \right) - f^i \left(x_t^i, \alpha_t s_t^i(\bar{X}_t) \right) \right|^2 \\ \leqslant L_1^2 ||\bar{u}_t - \alpha_t \bar{s}_t||^2. \end{aligned}$$

With regard for Proposition 1, we obtain

$$\mathrm{E}_{\mathcal{F}_{t-1}} ||v_t||^2 \leqslant \alpha_t^2 L_1^2 \left(n^2 \bar{b}^2 \sigma_w^2 + 4(n-1)^2 \bar{b}^2 ||\bar{X}_t||^2 \right). \tag{A.7}$$

By estimating successively all terms in the right side of (A.6) and allowing for the properties of the functions $f^i(\cdot,\cdot)$ (by assumption (**A1**)) and the results of Propositions 1 and 2, we deduce

$$\begin{aligned} \mathrm{E}_{\mathcal{F}_t} ||\bar{X}_{t+1}||^2 \leqslant 2^{\tilde{d}} ||\bar{X}_t||^2 + 2^{1+\tilde{d}/2} ||\bar{X}_t|| L_1 (L_x ||\bar{X}_t|| + \alpha_t ||\bar{s}||) \\ + L_2 \left(nL_c + L_x ||\bar{X}_t||^2 + \alpha_t^2 ||\bar{s}||^2 \right) + \alpha_t^2 L_1^2 \left(4(n-1)^2 \bar{b}^2 ||\bar{X}_t||^2 + n^2 \bar{b}^2 \sigma_w^2 \right) \\ \leqslant \Big(2^{\tilde{d}} + 2^{1+\tilde{d}/2} L_1 L_x + L_2 L_x + \alpha_t 2^{1+\tilde{d}/2} L_1 ||\mathcal{L}(A_{\max})|| \\ + \alpha_t^2 \left(L_2 ||\mathcal{L}(A_{\max})||^2 + 4(n-1)^2 L_1^2 \bar{b}^2 \right) \Big) ||\bar{X}_t||^2 + nL_2 L_c + \alpha_t^2 n^2 L_1^2 \bar{b}^2 \sigma_w^2 \leqslant \bar{c} + c_3 ||\bar{X}_t||^2, \end{aligned}$$

where $\bar{c} = nL_2 L_c + \alpha_t^2 \tilde{c}$.

By taking the unconditional expectation of both sides of the last inequality and iterating successively in t, we establish the conclusion of Proposition 5:

$$\begin{gathered}\mathrm{E}||\bar{X}_t||^2 \leqslant \bar{c} + c_3\mathrm{E}||\bar{X}_{t-1}||^2 \leqslant \bar{c} + \bar{c}c_3 + c_3^2\mathrm{E}||\bar{X}_{t-2}||^2\\ \leqslant \bar{c}(1 + c_3 + c_3^2 + \ldots + c_3^{t-1}) + c_3^t||\bar{X}_0||^2 \leqslant \bar{c}\frac{c_3^t - 1}{c_3 - 1} + c_3^t||\bar{X}_0||^2\\ \leqslant \left(\frac{\bar{c}}{c_3 - 1} + ||\bar{X}_0||^2\right) c_3^t \leqslant (\bar{c}_4 + ||\bar{X}_0||^2)e^{t\ln c_3},\end{gathered}$$

where $\bar{c}_4 = \bar{c}/c_3 - 1$.

Now we proceed to the direct proof of Theorem 1. By iterating (6) for $t, t-1, \ldots, 0$, we obtain

$$\begin{gathered}\bar{X}_{t+1} = U\bar{X}_t + G(\alpha_t, \bar{X}_t) + v_t\\ = U^2\bar{X}_{t-1} + UG(\alpha_{t-1}, \bar{X}_{t-1}) + G(\alpha_t, \bar{X}_t) + Uv_{t-1} + v_t\\ = \cdots = U^{t+1}\bar{X}_0 + \sum_{k=0}^{t} U^{t-k}G(\alpha_k, \bar{X}_k) + \sum_{k=0}^{t} U^{t-k}v_k\end{gathered} \tag{A.8}$$

and similarly

$$\bar{Z}_{t+1} = U^{t+1}\bar{X}_0 + \sum_{k=0}^{t} U^{t-k}G(\alpha_k, \bar{Z}_k). \tag{A.9}$$

We estimate $||\bar{X}_t - \bar{Z}_t||^2,\ t = 1, \ldots, T$. By subtracting (A.9) from (A.8) and quadrating the result, we obtain

$$\begin{gathered}||\bar{X}_t - \bar{Z}_t||^2 = \left\|\sum_{k=0}^{t} U^{t-k}v_k + \sum_{k=0}^{t} U^{t-k}\left(G(\alpha_k, \bar{X}_k) - G(\alpha_k, \bar{Z}_k)\right)\right\|^2\\ \leqslant 2\left\|\sum_{k=0}^{t} U^{t-k}v_k\right\|^2 + 2\left\|\sum_{k=0}^{t} U^{t-k}\left(G(\alpha_k, \bar{X}_k) - G(\alpha_k, \bar{Z}_k)\right)\right\|^2\\ \leqslant 2\left\|\sum_{k=0}^{t} U^{t-k}v_k\right\|^2 + 2\tau_t\sum_{k=0}^{t}\frac{1}{\alpha_t}\left\|U^{t-k}(G(\alpha_k, \bar{X}_k) - G(\alpha_k, \bar{Z}_k))\right\|^2.\end{gathered} \tag{A.10}$$

Using Proposition 2 and Lipschitzianity of $f^i(\cdot,\cdot)$ (assumption (**A1**)), we deduce for the addends in the second sum of (A.10) that

$$\begin{gathered}||U^{t-k}(G(\alpha_k, \bar{X}_k) - G(\alpha_k, \bar{Z}_k))||^2\\ \leqslant 2^{\bar{d}}L_1^2\sum_{i=1}^{n}\left(L_x|x_k^i - z_k^i| + \alpha_k|s(x_k^i) - s(z_k^i)|\right)^2\\ \leqslant 2^{1+\bar{d}}L_1^2\sum_{i=1}^{n} L_x|x_k^i - z_k^i|^2 + \alpha_k^2 s(x_k^i - z_k^i)^2\\ \leqslant 2^{1+\bar{d}}L_1^2(L_x + \alpha_k||\mathcal{L}(A_{\max})||)^2||\bar{X}_k - \bar{Z}_k||^2.\end{gathered}$$

By taking the expectation of both sides of (A.10), denoting $\mu_T = \max\limits_{0\leqslant t\leqslant T}\mathrm{E}||\bar{X}_t - \bar{Z}_t||^2$ and applying to the first addend Proposition 3 and the above estimate of the second addend, we obtain

$$\mu_T \leqslant 2^{3+\bar{d}}n\sum_{k=0}^{T}\mathrm{E}||v_k||^2 + 2\tau_T L_1^2\sum_{k=0}^{t}\alpha_k\left(\frac{L_x}{\underline{\alpha}} + ||\mathcal{L}(A_{\max})||\right)^2\mu_k. \tag{A.11}$$

Making use of the above relation (A.7) and the result of Proposition 5, we deduce for the estimate $\mathrm{E}||v_k||^2$ that

$$\mathrm{E}||v_k||^2 \leqslant \alpha_k^2 \left(n^2\sigma_w^2 + 4(n-1)^2(\bar{c}_4 + ||\bar{X}_0||^2)e^{k\ln c_3}\right)$$

and, consequently,

$$2^{3+\bar{d}}n\sum_{k=0}^{T}\mathrm{E}||v_k||^2 \leqslant \bar{\alpha}2^{3+\bar{d}}n^3L_1^2\bar{b}^2\tau_T\left(\sigma_w^2 + 4(\bar{c}_4 + ||\bar{X}_0||^2)e^{T\ln c_3}\right). \tag{A.12}$$

With regard for the estimates (A.12) from (A.11), we obtain in virtue of the simple relation $\sum_{k=0}^{t}\alpha_k^2 \leqslant \bar{\alpha}\sum_{k=1}^{t}\alpha_k = \bar{\alpha}\tau_t$ that

$$\mathrm{E}\mu_T \leqslant \bar{\alpha}c_1\tau_T + c_2^2\tau_T\sum_{k=1}^{T}\alpha_k\mathrm{E}\mu_k, \tag{A.13}$$

from which Theorem 1 is proved by applying Proposition 4.

Proof of Theorem 2. The ε-consensus is achieved in the averaged discrete system in virtue of the first condition. We denote by $x^\star$ the value of the ε-consensus of the averaged discrete system (9). It follows from the first conditions of Theorem 2 that the condition of Theorem 1 is satisfied, that is, that its result is true. From the rest of the conditions of Theorem 2 and the result of Theorem 1 we obtain

$$\mathrm{E}||\bar{X}_t - x^\star\underline{1}||^2 \leqslant 2\mathrm{E}||\bar{X}_t - \bar{Z}_t||^2 + 2||\bar{Z}_t - x^\star\underline{1}||^2 \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leqslant \varepsilon.$$

Proof of Theorem 3. The result of Theorem 3 is an immediate conclusion of Theorem 2. All sums along the rows of the elements of the matrix $\bar{\mathcal{L}} = (I-U) + \mathcal{L}(\alpha A_{\max})$ are zero and, moreover, all diagonal elements are positive and equal in magnitude to the sum of the rest elements of the row which are negative. Consequently, the matrix $\bar{\mathcal{L}}$ is a Laplacian of some graph.

If condition (**A3**) is satisfied, then the graph corresponding to the Laplacian $\bar{\mathcal{L}}$ has a spanning tree because the graph of the n first vertices has a spanning tree by condition (**A3**), and the unities on the $(n+1)$st diagonal connect successively the first vertex with the $(n+1)$st one, the second vertex with the $(n+2)$th one, and so on. In such discrete system an asymptotic consensus is achieved [25]. Since condition $\alpha < \dfrac{1}{d_{\max}}$ is satisfied in virtue of the assumptions of Theorem 3, asymptotic consensus is achieved in the averaged discrete system in virtue of the assertion on pp. 171–172 of [24], that is, any arbitrarily small level of proximity to the state of consensus is achievable in a finite time.

To satisfy the conditions of Theorem 2, it remains to prove that the constants $\bar{C}_1$ and $\bar{C}_2$ coincide with the corresponding constants of Theorem 1 which follows from the fact that in the case at hand $L_1 = L_2 = 1$, $L_x = L_c = 0$.

Proof of lemma. We take $x_t^i = \dfrac{q_t^i}{r_t^i}$ as the state of node i and carry out the proof from the contrary. We assume that for some optimal strategy not all x_t^i are equal to each other, that is, there exists a node numbered $k \in N$ and a subset of nodes $\tilde{N}_t$ such that $x_t^k > x_t^j$, $\forall j \in \tilde{N}_t$. We denote by $l = |\tilde{N}_t|$ the number of nodes in $\tilde{N}_t$. The states of the rest of $n-l$ nodes are x_t^k.

Let the difference between the state of the kth node and the greatest one in the set $\tilde{N}_t$ be ϵ_t, that is,

$$\epsilon_t = x_t^k - \max_{j\,\in\tilde{N}_t} x_t^j. \tag{A.14}$$

Let us consider a new strategy of task allocation and reduce the loads of all $n-l$ nodes having the greatest load by $\frac{\epsilon}{2(n-l)}$, that is, only by $\frac{\epsilon}{2}$, and add these $\frac{\epsilon}{2}$ tasks to any of the l nodes of $\tilde{N}_t$. For the new strategy, we see that the time of task processing in the system is less by $\frac{\epsilon}{2(n-l)}$ than the original time, that is, less than the minimal time by assumption, which gives rise to contradiction.

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This paper was recommended for publication by B.T. Polyak, a member of the Editorial Board