Passification Based Synchronization of Nonlinear Systems Under Communication Constraints and Bounded Disturbances

Alexander L. Fradkov [a,b,c] , Boris Andrievsky [a,b] and Mikhail S. Ananyevskiy [a,b]

[a] *Institute for Problems of Mechanical Engineering of the RAS, Saint Petersburg, Russia*

[b] *St.Petersburg State University, Saint Petersburg, Russia*

[c] *ITMO University, Saint Petersburg, Russia*

Abstract

In the brief the synchronization problem for nonlinear systems under communication constraints and bounded exogenous disturbances is analyzed. The main contribution is in the evaluation of the synchronization error as a function of transmission rate and the upper bounds of the disturbances. Relevance of passifiability condition for controlled synchronization of master-slave nonlinear systems for first order coder/decoder pair is demonstrated. Experimental results obtained at three-computer setup, illustrating the theory are presented.

Key words: passification method, nonlinear systems, synchronization, state estimation, communication constraints

1 Introduction

The limitations of control under constraints imposed by a finite capacity of information channel have been investigated in detail in the control literature, see the surveys [2, 3, 15], the monograph [13] and the references therein. It has been shown that stabilization of linear systems under information constraints is possible if and only if the capacity of the information channel exceeds the entropy production of the system at the equilibrium (*data rate theorem*) [14]. Results of the previous works on control systems analysis under information constraints do not apply to synchronization systems since in a synchronization problem trajectories in the phase space converge to a set (a manifold) rather than to a point, i.e. the problem cannot be reduced to simple stabilization. Moreover, the data rate theorem is difficult to extend to nonlinear systems [11].

One of the first approaches to synchronization of nonlinear systems under information constraints [10] is based on passification. In [10] the *output feedback* controlled synchronization of two nonlinear systems assuming that the coupling is implemented via the control signal, transmitted over a limited-band communication channel was considered. Key tools used to solve the problem are quadratic Lyapunov functions and the *passification method* [1, 7, 8]. Passification (rendering the closed loop system passive by output feedback) provides a simple design tool for a simple adaptive controller.

The present work is focused on the problem of synchronization and state estimation of nonlinear systems over the communication network under information constraints in presence of the exogenous model disturbances and measurement errors. Although the transmission delay and transmission channel distortions usually appear in practice, it is assumed here that the coded symbols are available at the receiver side at the same sampling instant as they are generated by the coder and transmission channel distortions are absent. We assume that only plant output (instead of all the components of the state vector) can be measured, and base our remote synchronization and state estimation schemes on transmission

⋆ Corresponding author B. Andrievsky. IPME RAS, 61, Bolshoy Ave. V.O., Saint Petersburg, 199178, Russia. Tel. +7 812 321 4766. Fax +7 812 321 4771.

Email addresses: alexander.fradkov@gmail.com (Alexander L. Fradkov), boris.andrievsky@gmail.com (Boris Andrievsky), msaipme@gmail.com (Mikhail S. Ananyevskiy).

URL: http://www.ipme.ru/ipme/labs/ccs/alf.htm (Alexander L. Fradkov).

of the scalar signal rather than a state vector.

Recently the authors of [5] investigated application of the passivity approach to collective coordination and synchronization problems in the presence of quantized measurements. It is assumed that the system interaction is performed in a completely asynchronous fashion and no common sampling time is required. Since the quantization interval is not explicitly bounded, then the channel bitrate of [5] is potentially unlimited. The other distinctive feature of our approach is that we assume that the system is rather *passifiable*, than *passive*.

This brief is organized as follows. The binary coding procedure of [10] is briefly recalled in Section 2. Section 3 contains main results concerning evaluation of the synchronization error under communication constraints, bounded exogenous disturbances and measurement errors. Experimental results are presented in Section 4.

2 Preliminaries

2.1 Coding procedure

Starting from the results of [4, 10, 14], in the present brief the following time-varying coder with memory and a binary quantizer are used.

Let signal $z(t)$ be transmitted over the digital communication channel at sampling instants $t_k = kT$, where T is a constant sampling period, $k = 0, 1, \dots$ are integer numbers. At each instant k, *deviation signal* $\partial z[k]$ between transmitted signal $z[k]$ and a certain *central number* $c[k]$ (defined below) is calculated as $\partial z[k] = z[k] - c[k]$. Then signal $\partial z[k]$ is subjected to the following binary quantization scheme:

$$\bar{\partial} z[k] = M[k]\,\mathrm{sign}(\partial z[k]). \tag{1}$$

where $\mathrm{sign}(\cdot)$ is the signum function, $M[k] > 0$ may be reffered to as a time-varying *quantization range*.

Output signal $\bar{\partial} z[k]$ of the quantizer is transmitted over the communication channel to the decoder. Central numbers $c[k]$ are defined by the following recursive algorithm:

$$c[k+1] = c[k] + \bar{\partial} z[k], \quad c[0] = 0. \tag{2}$$

Quantization range $M[k]$ is represented by the following time sequence:

$$M[k] = (M_0 - M_\infty)\rho^k + M_\infty, \quad k = 0, 1, \dots, \tag{3}$$

where $0 < \rho \leqslant 1$ is the *decay parameter*, M_∞ stands for the limit value of M. Initial value M_0 should be large enough to capture all the region of possible initial values of ∂z. For practice, to avoid computations of powers of ρ, it is advisable to calculate $M[k]$ in the following recursive form: $M[k+1] = \rho M[k] + m$, where $m = (1 - \rho)M_\infty$, $M[0] = M_0$.

Equations (1)–(3) describe the coder algorithm. A similar algorithm is represented by the decoder: the sequence of $M[k]$ is reproduced at the receiver node utilizing (3); values of $\bar{\partial} z[k]$ are restored with given $M[k]$ from the received binary codeword; central numbers $c[k]$ are found in the decoder in accordance with (2). Then decoder output $\bar{z}[k]$ is found as a sum of $c[k]$ and $\bar{\partial} z[k]$, $\bar{z}[k] = c[k] + \bar{\partial} z[k]$.

It is worth mentioning that the considered coding scheme corresponds to the channel data rate of one bit per step or $R = T^{-1}$ bits per second.

2.2 Hyper-minimum-phaseness and passification

Below the following definition is used.

Definition 1 *Consider a linear time invariant system*

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \tag{4}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$, A, B, C are constant real matrices of appropriate dimensions. System (4) is called minimum phase *if the polynomial*

$$\varphi_0(s) = \det \begin{bmatrix} sI_n - A & -B \\ C & 0 \end{bmatrix}$$

where I_n stands for an $(n \times n)$ identity matrix, is Hurwitz; hyper minimum phase *(HMP), if it is minimum phase and $CB = (CB)^{\mathrm{T}} > 0$.*

For the special case $m = 1$ (SISO systems) considered in this paper the HMP property coincides with the standard minimum phase relative degree one property.

We need the following corollary from Passification theorem [1].

Corollary 1 *There exist $n \times n$ matrix $P = P^{\mathrm{T}} > 0$, $m \times m$ matrix K and a scalar $\mu > 0$ such that*

$$PA_K + A_K^{\mathrm{T}} P < -\mu I, \;\; PB = C^{\mathrm{T}}, \;\; A_K = A + BKC. \tag{5}$$

if and only if the system (4) *is HMP.*

If HMP condition holds then existence of matrix P and a scalar $\mu > 0$ satisfying (5) is guaranteed if $K = -\kappa I_m$, where $\kappa > 0$ is sufficiently large.

3 Controlled synchronization of passifiable nonautonomous Lurie systems

Fradkov *et. al* [9,10] studied a unidirectional controlled synchronization of nonlinear systems over the limited-band communication channel. In this Section these results are extended to the case of exogenously disturbed systems.

3.1 Master-slave unidirectional synchronization

Consider two identical dynamical systems modeled in the Lurie form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\varphi(y_1) + f_1(t), \\ y_1(t) &= Cx(t) + v_1(t), \end{aligned} \tag{6}$$

$$\begin{aligned} \dot{z}(t) &= Az(t) + B\varphi(y_2) + Bu(t) + f_2(t), \\ y_2(t) &= Cz(t) + v_2(t), \end{aligned} \tag{7}$$

where $x(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^n$ are n-dimensional vectors of state variables; $y_1(t) \in \mathbb{R}^1$, $y_2(t) \in \mathbb{R}^1$ are the scalar output variables; A is $(n\times n)$-matrix; B is $(n\times 1)$-matrix; C is $(1 \times n)$-matrix, $\varphi(y)$ is a continuous nonlinearity; $f_1(t)$, $f_2(t) \in \mathbb{R}^n$, $v_1(t)$, $v_2(t) \in \mathbb{R}^1$, denote input signals. Signals $f_i(t)$, $v_i(t)$, $i = 1, 2$ are piecewise continuous, not necessarily bounded. Boundedness will be required from their differences, see below. System (6) is called *a master system*. System (7), called *a slave system*, is controlled by a scalar control signal $u(t) \in \mathbb{R}^1$. The signal $u(t)$ is produced based on the dataflow available over the limited capacity communication channel. The goal is to find $u(t)$ such that the *state synchronization error* $e(t)$, defined as $e(t) = x(t) - z(t)$, becomes small as t becomes large.

3.2 Case of transmitting the master system output through a communication channel

The case when the observation signal $y_1(t)$ is mapped to a finite alphabet of symbols at discrete time instants $t_k = kT$ ($k = 0, 1, 2, \ldots$; T is the sampling period) before transmission to the estimator over a digital communication channel is studied by [9]. The *zero-order extrapolation* is used for converting the digital sequence $\bar{y}_1[k]$ to the continuous-time input of the response system $\bar{y}_1(t)$. This leads to the following notion of the *transmission error* $\delta_y(t)$:

$$\delta_y(t) = y_1(t) - \bar{y}_1(t). \tag{8}$$

It is assumed that the controller on the receiver side can use only the signal $\bar{y}_1(t)$ instead of $y_1(t)$. The control law is taken in the form of a static linear feedback

$$u(t) = -K\sigma(t), \tag{9}$$

where $\sigma(t) = y_2(t) - \bar{y}_1(t)$, K is the scalar controller gain.

The result of this Section may be summarized in the following theorem.

Theorem 1 *Let the following assumptions hold:*

- *A1.1.* *Transfer function $W(\lambda) = C(\lambda I - A)^{-1}B$ with A, B, C from* (6) *is HMP;*
- *A1.2.* *Function $\varphi(y)$ in* (6), (7) *is Lipschitz continuous with the Lipschitz constant L_φ;*
- *A1.3.* *The growth rate of $y_1(t)$ is uniformly bounded with the exact bound L_y and transmission error is bounded: $|\delta_y(t)| \leqslant \Delta$ for $\Delta > 0$.*
- *A1.4.* *Differences between disturbances $f_1(t) - f_2(t)$ and $v_1(t) - v_2(t)$ are bounded, $|f_1(t) - f_2(t)| \leqslant \Delta_1$, $|v_1(t) - v_2(t)| \leqslant \Delta_2$, for all $t \geqslant 0$.*

Then the system (6), (7), (9) *is input to output stable (IOS) for input $w = (\delta_y, f_1 - f_2, v_1 - v_2)$ and output $e = x - z$. Namely, there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that*

$$\begin{aligned} \|e(t)\| \leqslant \beta\Big(\big\|\mathrm{col}\big((x(0), z(0)\big)\big\|, t\Big) \\ + \gamma\Big(\sup_{0\leqslant s\leqslant t} \big\|w(s)\big\|\Big). \end{aligned} \tag{10}$$

Moreover the following inequality is valid

$$\begin{aligned} \overline{\lim_{t\to\infty}}\|e(t)\| \leqslant \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\frac{\|B\|}{\mu} \times \\ \Big(\big(|K| + L_\varphi\big)(\Delta + \Delta_2) + L_\varphi\Delta_2 + \lambda_{\max}(P)\Delta_1\Big), \end{aligned} \tag{11}$$

where $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ are, respectively, the maximal and minimal eigenvalues of matrix P in (5).

Remark 1 *In the case of equal input signals $\Delta_1 = \Delta_2 = 0$, and* (11) *is reduced to the result of [9]:*

$$\overline{\lim_{t\to\infty}}\|e(t)\| \leqslant C_e\Delta, \tag{12}$$

where $C_e = \sqrt{\dfrac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\dfrac{L_\varphi + |K|}{\mu}\|B\|.$

Inequality (12) *means that the total synchronization error $\overline{\lim}_{t\to\infty}\|e(t)\|$ is proportional to the upper bound on the transmission error Δ. The latter for a given quantizer and the master system dynamics, depends on sampling time T. As is stated in [10] assuming that the growth rate* *of $y_1(t)$ is uniformly bounded by L_y, then the transmission error satisfies the inequality:*

$$|\delta_y(t)| \leqslant \frac{M}{2} + L_y T, \tag{13}$$

where M denotes the quantization range (1). [1]

Remark 2 *The exact bound L_y for the rate of $y_1(t)$ may be found as $L_y = \sup_{x\in\Omega} |C\dot{x}|$, where $\dot{x}$ is from* (6), *$x(0) \in \Omega_0$.*

Remark 3 *Theorem 1 can be easily extended to the case of m-dimensional y_1, y_2 with $m < n$.*

The proof of Theorem 1 is given in Appendix A.

3.3 Case of transmitting the signal of the error between the master and the slave systems

Now consider the case when the error signal between the master and the slave systems (6), (7) is transmitted over the channel. Such a case may appear in various applications. Particularly both master and slave systems may have the ability to estimate the dynamics of another system so that the output error signal can be directly computed. The following example is originated from command guidance, where the tracking center is used for remote control, cf. [12].

Let us take the control signal in the following form:

$$u(t) = -K\bar{\varepsilon}(t), \tag{14}$$

where $\bar{\varepsilon}(t) = \bar{\varepsilon}[k]$ as $t_k < t < t_{k+1}$; $\bar{\varepsilon}[k]$ is the result of transmission of the output error signal $\varepsilon(t) = y_2(t) - y_1(t)$ over the channel, $t_k = kT$, $k = 0, 1, \ldots$.

According to the quantization algorithm (1), the quantized error signal $\bar{\varepsilon}[k]$ becomes

$$\bar{\varepsilon}[k] = M[k]\,\mathrm{sign}(\varepsilon(t_k)), \tag{15}$$

where the range $M[k]$ is defined by (3).

The key point of justification and error analysis of the presented synchronization scheme is application of the so-called *method of continuous models*: analysis of the hybrid nonlinear system via analysis of its continuous-time approximate model [6].

Theorem 2 *Let A1.1, A1.2, A1.4 hold, the controller gain K satisfies relations* (5) *for some positive definite matrix P and the coder parameters ρ, T be chosen to meet the inequalities*

$$e^{\mu T}(e^{L_F T} - 1) \leqslant \frac{L_F}{\|C\|(K\|B\| + L_F)}, \quad e^{-\mu T} < \rho < 1, \tag{16}$$

where $L_F = \|A\| + L_\varphi\|B\| \cdot \|C\|$, μ is from (5). *Let the coder range $M[k]$ be specified as*

$$M[k] = M_0\rho^k. \tag{17}$$

Then the system (6), (7), (14) *is IOS for input $w = \mathrm{col}\, f_1 - f_2, v_1 - v_2$ and output $e = x - z$. Moreover, if $f_1 = f_2$ and $v_1 = v_2$, then for all initial conditions $x(0)$, $z(0)$ in* (6), (7) *such that $e(0)^{\mathrm{T}} P e(0) \leqslant M_0^2$ the synchronization error decays exponentially: $|\varepsilon[k]| \leqslant \|e[k]\| \leqslant M_0\rho^k$. In addition, $|\varepsilon(t)| \leqslant |\varepsilon[k]|$ for $t_k \leqslant t < t_{k+1}$.*

Proof of the theorem follows the lines of the proof of Theorem 1 and [10, Theorem 1].

Remark 4 *For practice, it is reasonable to choose the coder range $M[k]$ separated from zero and use the coder range goverened by* (3) *as*

$$M[k] = (M_0 - M_\infty)\rho^k + M_\infty, \quad k = 0, 1, \ldots,$$

instead of (17) *for zooming. Evidently, this leads to the residual synchronization error.*

4 Illustrative example

4.1 Master-slave system description

Consider the following example, inspired by a long standing problem of the proportional-integral (PI) control under the condition of the input signal saturation. The so-called *wind-up* phenomenon may arise in the such a kind of systems [18]. The integrator wind-up may cause coexistence of different steady-state solutions for the same reference input [19] or, in the other words, loss of the *convergence property* [16]. The illustrative example below demonstrates application of the design method of Section 3 (Theorem 2) for ensuring synchronization (convergence of trajectories) of two systems, coupled over the digital communication channel.

Let master system (6) be modeled as

$$\begin{cases} \dot{x}_1(t) = \varphi\big(y_1(t)\big), \\ \dot{x}_2(t) = r(t) - x_1(t), \end{cases} \tag{18}$$

$$y_1(t) = k_P\big(x_1(t) - r(t)\big) - k_I x_2(t), \tag{19}$$

$$\varphi(y_1) = -\,\mathrm{sat}(y_1), \tag{20}$$

[1] The present brief deals with a binary quantizer (1). The case of a uniform quantizer is considered in [10].

where $r(t)$ is a differentible in t exogenous signal, $\mathrm{sat}(\cdot)$ denotes the *saturation* nonlinearity, k_I, k_P are positive gains (system parameters). Note, that (18)–(20) may be treated as a model of the PI-controlled servosystem with saturated input and reference signal $r(t)$ [19].

Matrices A, B, C, functions $f(t)$, $v(t)$ of the master system model (6) for system (18), (19) have the following form:

$$A = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} k_P, & -k_I \end{bmatrix}, \qquad (21)$$

$$f(t) = \begin{bmatrix} 0 \\ r(t) \end{bmatrix}, \quad v(t) = -k_P r(t). \qquad (22)$$

Let slave system (7) be described by the similar equations with an additional control input $u(t)$, acting in the span of matrix B:

$$\begin{cases} \dot z_1(t) = \varphi\big(y_2(t)\big) + u(t), \\ \dot z_2(t) = r(t) - z_1(t), \end{cases} \qquad (23)$$

$$y_2(t) = k_P\big(z_1(t) - r(t)\big) - k_I z_2(t), \qquad (24)$$

$$\varphi(y_2) = -\,\mathrm{sat}(y_2), \qquad (25)$$

Consider the master-slave synchronization problem under communication constraints. Let the error $\varepsilon(t) = y_1(t) - y_2(t)$ be measured and transmitted over the communication channel with a limited capacity. In the case of *binary* quantization, the quantized error signal $\bar\varepsilon[k]$ has the form

$$\bar\varepsilon[k] = M[k]\,\mathrm{sign}(\varepsilon(t_k)), \qquad (26)$$

where range $M[k]$ is defined by (17), $t_k = kT$, $k = 0, 1, \ldots$, and T denotes the sampling period.

Let us check the conditions of Theorem 2 for considered system (18)–(20).

Condition A1.1. Transfer function $W(\lambda) = C(\lambda I - A)^{-1}B = \dfrac{\beta(\lambda)}{\alpha(\lambda)}$ of system (18), (19) with matrices (22) has a form $W(\lambda) = \dfrac{k_P\lambda + k_I}{\lambda^2}$. Its denominator $\alpha(\lambda) = \lambda^2$ is a polynomial of degree $n = 2$. For all positive k_P, k_I, numerator $\beta(\lambda) = k_P\lambda + k_I$ is a Hurwitz polynomial of degree 1. Therefore $W(\lambda)$ is HMP, and condition A2 of Theorem 2 is satisfied as gains k_I, k_P are positive. In what follows, $k_P = 10$, $k_I = 20$ are chosen, ensuring the 5%-settling time less than 1 s and the overshoot 12% for the step response of the closed-loop linear (non-saturated) system. For the chosen controller gains, stability degree μ_0 of numerator $\beta(\lambda)$ is equal to 2.

Let $K = 1$ be chosen in (14). Then A_K in (5) is as follows:

$$A_K = \begin{bmatrix} -10 & 20 \\ -1 & 0 \end{bmatrix}.$$

It may be easily checked that if matrix P is chosen as

$$P = \begin{bmatrix} 10 & -20 \\ -20 & 80 \end{bmatrix}.$$

then (5) is valid for any $0 < \mu < \mu_0$, with $\mu_0 = 2$.

Let us pick up decay parameter ρ in (3) in the form $\rho = e^{-\mu T}$, where $\mu = 0.6$ is chosen to satisfy the inequality $0 < \mu < 2$. Obviously, such a choice implies the second inequality of (16) if $e^{-\mu T} < \rho < 1$. To find a sufficient upper bound for sampling time T consider the first inequality of (16). Taking into account that $L_\varphi = 1$ and calculating H_2-norms of the matrices, one may found that $L_F = \|A\| + L_\varphi\|B\| \cdot \|C\| = 23.4$. Therefore the right-hand side of (16) may be estimated as $\dfrac{L_F}{\|C\|(K\|B\| + L_F)} = 0.043$. By means of a searching procedure with respect to parameter T one can find that the upper bound for T, ensuring fulfillment of the inequality $e^{\mu T}(e^{L_F T} - 1) \leqslant 0.043$, is approximately $1.75 \cdot 10^{-3}$ s.

Condition A1.2. Nonlinearity $\varphi(y_1) = -\,\mathrm{sat}(y_1)$ is Lipschitz continuous with $L_\varphi = 1$.

Condition A1.4. We assume that all the disturbances are bounded, according to the definition of IOS.

Therefore, conditions of Theorem 2 for given system equations and the chosen paramenter values are fulfilled and control (14)

$$u(t) = -K\bar\varepsilon(t), \qquad (27)$$

where $\bar\varepsilon(t) = \bar\varepsilon[k]$ as $t_k < t < t_{k+1}$, may be applied to achieve synchronization of systems (18)–(20), (23)–(25) for $K = 1$, $T = 1.75$ ms.

4.2 Experimental results

The master-slave synchronization was emulated in real-time over LAN. Differential equations of master and slave systems (20), (25) were numerically solved by the Euler method with time step $\Delta t = 1.75 \cdot 10^{-3}$ s.

Data transfer was realized over localnet by TCP/IP protocol. All three subsystems (Fig. 1) were implemented on different computers (AMD FX-8350 3.5Ghz, 8GB RAM, OS Gentoo Linux). The master and slave systems data were transmitted to the controller without limitations on the throughput channel capacity. Controller data were transmitted to the slave system according to the described algorithm. For doing this only 570 bit/s from 100 Kbit/s data rate flow were actually used for control.

Parameters of system (17), (18)–(27) were taken as follows: sampling time $T = 1.75 \cdot 10^{-3}$ s ($R = 570$ bit/s); harmonic excitation signal $r(t) = A\sin(\omega t)$, where $\omega = 0.15$ s^{-1}, $A = 2.5$; initial conditions: $x_1(0) = 0.1$, $x_2(0) = 0$, $z_1(0) = 3$, $z_2(0) = 1$. The coder parameters: initial quantizer range $M[0] = 2$, decay parameter ρ was found as $\rho = \exp(-\mu T)$ for $\mu = 0.6$, which led to $\rho = 0.99895$.

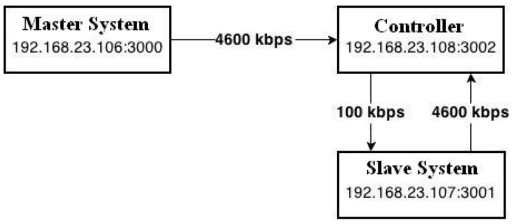

Fig. 1. Structure of the data flow between the servers for emulation.

The parameters or real communication experiment between three different PCs were used in the numerical example for the simulations. To represent the data transfer in one bit per sample, the communications have been emulated via the standard TCP/IP protocol.

The emulation results are depicted in Figs. 2, 3. Time histories of state space variables $x_1(t)$, $z_1(t)$ of systems (18)–(20), (23)–(25) for $u(t) \equiv 0$ are shown in Fig. 2. The phenomenon of non-synchronized motion caused by an absence of the mentioned convergence property is visible, demonstrating that the synchronization may not be acheived despite that the same reference signal is applied to identical systems.

Successfull synchronization over the digital communication channel by means of control law (27) and binary coder (17), (26) is demonstrated in Fig. 3, where the time histories $x_1(t)$, $z_1(t)$, $u(t)$ for the given above controller/coder parameters are plotted. It is seen from the plots that the synchronization transient time is about 6 s, which well corresponds to chosen decay parameter $\rho = \exp(-\mu T)$ for $\mu = 0.6$.

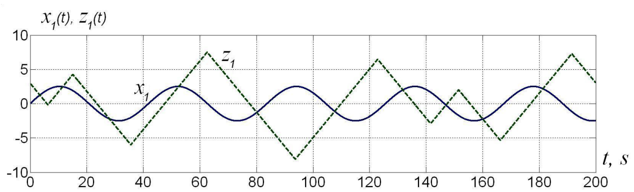

Fig. 2. Time histories of $x_1(t)$, $z_1(t)$ for $u(t) \equiv 0$. No synchronization occurs.

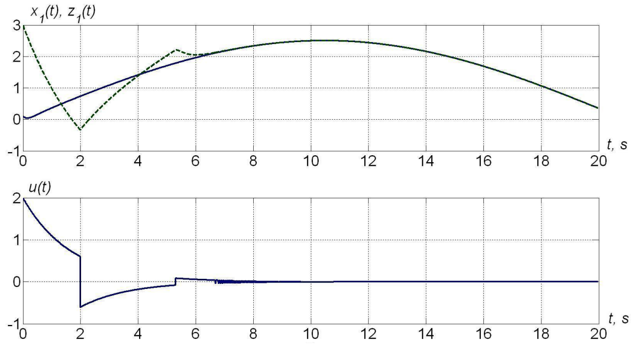

Fig. 3. Time histories of $x_1(t)$, $z_1(t)$, $u(t)$ for the case of control (27). Sampling period $T = 1.75 \cdot 10^{-3}$ s ($R = 570$ bit/s).

The system parameters have been chosen to satisfy Theorem 2 conditions. Since the value of K in control law (14) should be taken sufficiently large (cf. the remark below Eq. (5)), we have made 50 experiments for K varying from 0.1 to 50 demonstrating dependence of synchronization dynamics on K. Some representative examples are plotted in Fig. 4. It is seen from the plots, that for small K ($K = 0.2$) the synchronization fails; for sufficiently large K ($K \geqslant 0.5$), the synchronization occurs. As also seen from the plots, for large values of K dependence of the synchronization rate on K is weaker.

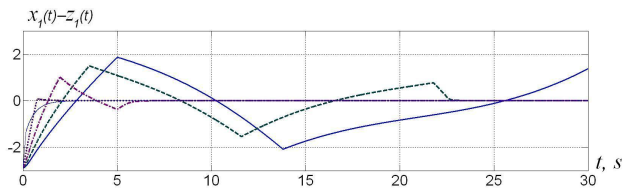

Fig. 4. Synchronization error $x_1(t) - z_1(t)$ time histories for various $K \in \{0.2, 0.5, 1.0, 2.0, 5.0\}$.

The case of different disturbances $f_1(t)$, $f_2(t)$ is illustrated in Fig. 5. In this experiment, disturbance $f_1(t)$ was taken as above, $f_1(t) = \left[0,\ r(t)\right]^{\mathrm{T}}$, while $f_2(t)$ has an additional harmonic term $\psi(t)$: $f_2(t) = \left[0,\ r(t) + \psi(t)\right]^{\mathrm{T}}$, where for the experiment $\psi(t) = 0.25\sin(\omega_\psi t)$, $\omega_\psi = 1$ rad/s, was taken. The coder range was governed by (3), where $M_0 = 2$, $M_\infty = 0.5$.

5 Conclusions

In the brief the synchronization and state estimation problems for nonlinear systems in the Lurie form under communication constraints, bounded exogenous disturbances and bounded measurement errors are analyzed. The main contribution (Theorems 1, 2) is in the evaluation of the synchronization error as a function of transmission error and the upper bounds of the disturbances. It allows one to evaluate transmission rate required for achievement of prespecified synchronization error. Ex-

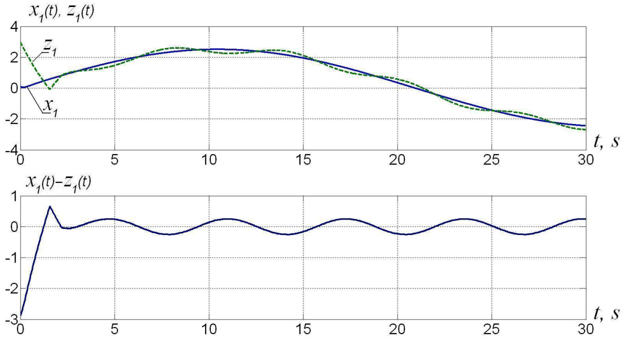

Fig. 5. Time histories of $x_1(t)$, $z_1(t)$ and synchronization error $x_1(t) - z_1(t)$, case of different $f_1(t)$, $f_2(t)$. $M_0 = 2$, $M_\infty = 0.5$.

perimental results demonstrate that a nonvanishing difference between disturbances acting on master and slave systems implies nonvanishing synchronization error. Experimental results qualitatively illustrating the theory are presented.

Relevance of passifiability condition for controlled synchronization of master-slave nonlinear systems is demonstrated.

For nonlinear systems in the Lurie form with strictly passifiable linear part it is possible to ensure synchronization over the limited-capacity communication channel by means of measuring and transmitting only the scalar output of the master system as opposed to the full state vector measurement and transmission, cf. [17].

Further research could be devoted to extension to the systems with multiple nonlinearities and complex networks. Applications may include various physical and mechanical systems including oscillatory systems, communicating via digital communication channels.

A Proof of Theorem 1

Represent the exogenous signals $f_2(t)$ and $v_2(t)$ in (7) as $f_2(t) = f_1(t) + \xi_1(t)$, $v_2(t) = v_1(t) + \xi_2(t)$, where $\|\xi_1(t)\|_\infty \leqslant \Delta_1$, $\|\xi_2(t)\|_\infty \leqslant \Delta_2$, for certain $\Delta_1 \geqslant 0$, $\Delta_2 \geqslant 0$.

Introduce the differentiable vector-functions $w(t) \in \mathbb{R}^n$ and $\tilde{x}(t) = x(t) - w(t)$. Then (6), (7) may be represented in the following form:

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\varphi(\tilde{y}_1) + \tilde{f}(t),\ y_1(t) = C\tilde{x}(t), \quad \text{(A.1)}$$

$$\begin{aligned}\dot{\tilde{z}}(t) = A\tilde{z}(t) + B\varphi(\tilde{y}_2) + Bu(t) + \tilde{f}(t) + \xi_1(t),\\ \tilde{y}_2(t) = C\tilde{z}(t) + \xi_2(t),\end{aligned} \quad \text{(A.2)}$$

where $\tilde{f}(t) = f_1(t) + Aw(t) - \dot{w}(t)$.

Since $x(t) = \tilde{x}(t)+w(t)$ we have $y_1(t) = C\tilde{x}(t)+Cw(t)+v_1(t)$ and (6) reads

$$\begin{aligned}&\dot{\tilde{x}}(t) + \dot{w}(t) = A\tilde{x}(t) + Aw(t)\\ &+B\varphi(C\tilde{x} + Cw + v_1) + \tilde{f}(t).\end{aligned} \quad \text{(A.3)}$$

Let us represent row-vector C as $C = [c_1, c_2, \dots, c_n]$ and assume that $\|C\| \neq 0$. Then there exists $i \in \{1, 2, \dots, n\}$ such that $c_i \neq 0$. Picking up $w(t)$ such that $w(t) = -c_i^{-1}v(t)$, and $w_j(t) \equiv 0$ for all $j \in \{1, 2, \dots, n\}$ as $j \neq i$, one obtains that $Cw(t) + v_1(t) \equiv 0$ and, therefore, $y_1(t) = C\tilde{x}(t)$, and (A.3) may be rewritten as follows:

$$\dot{\tilde{x}}(t)=A\tilde{x}(t)+B\varphi(\tilde{y})+\tilde{f}(t),\ y_1(t)=C\tilde{x}(t). \quad \text{(A.4)}$$

The similar transformation of (7) leads to (A.2). Since $\tilde{x}(t) = x(t) - w(t)$ and $\tilde{z}(t) = z(t) - w(t)$ then the synchronization error $e(t) = x(t) - z(t)$ may be represented as $e(t) = \tilde{x}(t) - \tilde{z}(t)$. Substracting (A.2) from (A.1) leads to the following error equation

$$\dot{e}(t)=Ae(t)-Bu(t)+B\big(\varphi(\tilde{y}_1)-\varphi(\tilde{y}_2+\xi_2)\big)-\xi_1. \quad \text{(A.5)}$$

Taking into account control law (9) $u = -K\sigma = -K\big(-Ce(t) + \delta_y(t) + \xi_2(t)\big)$, (A.5) reads as

$$\dot{e}(t) = A_K e(t) + B\big(\varphi(\tilde{y}_1) - \varphi(\tilde{y}_2 + \xi_2)\big) - \xi(t), \quad \text{(A.6)}$$

where $\xi(t) = \xi_1(t)+BK\big(\delta_y(t)+\xi_2(t)\big)$, $A_K = A-BKC$.

Introduce the extended nonlinearity

$$\zeta(t) = \varphi\big(\tilde{y}_1(t)) - \varphi\big(\tilde{y}_2(t)\big) + L_\varphi\sigma(t). \quad \text{(A.7)}$$

It follows from Lipschitz condition A1.2 that

$$\zeta(t)\sigma(t) \geqslant 0 \ \text{ for all } \ t \geqslant 0, \quad \text{(A.8)}$$

i.e. $\zeta(t)$ satisfies sector condition. Make the change $K := K + L_\varphi$. Then error equation (A.6) reads

$$\dot{e}(t) = A_K e(t) + B\zeta(t) - \tilde{\xi}(t), \quad \text{(A.9)}$$

where $\zeta(t)$ satisfies sector condition (A.8), and

$$\begin{aligned}\tilde{\xi} &= \xi_1 - B(K + L_\varphi)\delta_y(t) + \xi_2(t))\\ &+BK\big(\varphi(\tilde{y}_2) - \varphi(\tilde{y}_2 + \xi_2)\big).\end{aligned} \quad \text{(A.10)}$$

Apparently, $\big|K\big(\varphi(\tilde{y}_2) - \varphi(\tilde{y}_2 + \xi_2)\big)\big| \leqslant |K|L_\varphi\Delta_2$.

Consider the Lyapunov function candidate $V(e) = e^{\mathrm{T}}Pe$, where $P = P^{\mathrm{T}} > 0$. From Passification Theorem [1, 8] and simple algebra establish existence of function $V(e)$, and controller gain K such that

$$\dot{V}(e) \leqslant -2\mu V(e) + \mu_0\sqrt{V}, \tag{A.11}$$

where $\mu_0 = \sqrt{V(B)}\Big(\big(|K| + L_\varphi\big)(\Delta + \Delta_2) + L_\varphi\Delta_2 + \sqrt{\lambda_m}\Delta_1\Big)$, $\lambda_m = \lambda_{\max}(P)$. Denoting $W(e) = \sqrt{(V(e))}$ we transform (A.11) to the linear differential inequality $\dot{W} \leqslant -\mu/2W + \mu_0/2$. It implies IOS property with $\beta(\cdot, t)$ exponentially tending to zero as $t \to \infty$. Since $\dot{V} < 0$ in the set $\sqrt{V(e)} > \dfrac{\mu_0}{\mu}$, the value of $\overline{\lim_{t\to\infty}} V\big(e(t)\big)$ cannot exceed

$$\frac{\mu_0^2}{\mu^2} = \frac{V(B)}{\mu^2}\Big(\big(|K| + L_\varphi\big)(\Delta + \Delta_2) + L_\varphi\Delta_2 + \lambda_{\max}(P)\Delta_1\Big)^2. \tag{A.12}$$

Therefore

$$\overline{\lim_{t\to\infty}} \|e(t)\| \leqslant \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\frac{\|B\|}{\mu} \times \Big(\big(|K| + L_\varphi\big)(\Delta + \Delta_2) + L_\varphi\Delta_2 + \lambda_{\max}(P)\Delta_1\Big), \tag{A.13}$$

i.e. (11) is proven.

Acknowledgements

Partially supported by the Russian Foundation for Basic Research (Proj. # 13-08-01014, 14-08-01015), by the Program of basic research of RAS # 40 "Actual problems of robotics", by the Saint Petersburg State University (grant 6.38.230.2015) and by the Government of Russian Federation, Grant 074-U01. The experiments (Section 4.2) are performed in IPME RAS and supported by the Russian Scientific Foundation (grant 14-29-00142).

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