Nonsmooth Speed-Gradient Algorithms*

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Abstract— **The first step toward development a "nonsmooth" version of Speed-Gradient (SG) method is made. As a basis for formal approach the Hadamard subdifferential and set-valued analysis are chosen. Nonsmooth versions of SG-algorithm in differential and finite form are formulated. Conditions for the control goal achievement are proposed. The results are illustrated by an example of asymptotic minimization of the qubic norm for a second order linear system.**

I. INTRODUCTION

The Speed-Gradient (SG) method was proposed in the end of the 1970s as a general framework for design of control, adaptation, identification algorithms for nonlinear systems [1]. Since then it was extended in different directions [2], [3], [4], [5] and applied to a variety of problems in physics and mechanics [6], [7], [8], [9], [10]. An intimate relation between applicability of SG method and passivity of controlled system was established [11]. In special case of affine controlled system the SG-algorithms encompass Jurdjevic-Quinn (LgV) algorithms [12].

Standard procedure of SG algorithms derivation requires differentiation of the goal function along trajectries of the controlled system. However in many cases the right hand sides of the system model are nonsmooth. Sometimes it may be profitable to introduce nonsmooth and even discontinuous terms into control algorithms in order to provide the desired system dynamics, e.g. finite time convergence. Therefore there is a need for a more general framework for design and analysis of SG-like algorithms in a general nonsmooth case. In this paper, an attempt is made to make the first step along this way.

As a basis for formal approach to nonsmooth problems we choose Hadamard subdifferential and set-valued analysis [13], [14]. It allows us to introduce and analyze the speed-subgradient algorithms in differential and finite forms. The main results of the paper are Theorem 1 and Theorem 2 providing conditions for achievement of the control goal. The results are illustrated by an example of asymptotic minimization of the qubic norm for a second order linear system.

II. PRELIMINARIES

Let us recall some notions from nonsmooth analysis [13] and set-valued analysis [14] that is used in the sequel. Denote $\mathbb{R}_+ = [0, +\infty)$, and denote by $|x|$ the Euclidean norm of a vector $x \in \mathbb{R}^n$. A function F that maps points from $\mathbb{R}^n$ to possibly empty subsets of $\mathbb{R}^m$ is called a *set-valued mapping* from $\mathbb{R}^n$ to $\mathbb{R}^m$. The set-valued mapping F is called *upper semicontinuous* at a point $x \in \mathbb{R}^n$ if for any open set $U \subset \mathbb{R}^m$ such that $F(x) \subset U$ there exists $\delta > 0$ such that for any $y \in \mathbb{R}^n$, $|x - y| < \delta$ one has $F(y) \subset U$. The set-valued mapping F is called *measurable* if for any open set $U \subset \mathbb{R}^n$ the set

$$\{x \in \mathbb{R}^n \mid F(x) \cap U \neq \emptyset\}$$

is measurable. Any upper semicontinuous set-valued mapping is measurable.

Let a real-valued function f be defined in a neighbourhood of a point $x \in \mathbb{R}^n$. The function f is called *Hadmard directionally differentiable* at the point x if for any $g \in \mathbb{R}^n$ there exists the finite limit

$$f'(x; g) = \lim_{[\alpha, g'] \to [+0, g]} \frac{f(x + \alpha g') - f(x)}{\alpha}.$$

The function $f'(x; \cdot)$ is positively homogeneous of degree one, i.e. for any $g \in \mathbb{R}^n$ and $\lambda \geq 0$ one has $f'(x; \lambda g) = \lambda f'(x; g)$. A convex positively homogeneous of degree one function $h \colon \mathbb{R}^n \to \mathbb{R}$ is referred to as an *upper convex approximation* of the function $f'(x; \cdot)$ if $f'(x; g) \leq h(g)$ for any $g \in \mathbb{R}^n$.

The function f is called *Hadamard subdifferentiable* at the point x if f is Hadamard directionally differentiable at this point, and the function $f'(x, \cdot)$ is convex or, equivalently, if there exists a compact convex set $\partial f(x) \subset \mathbb{R}^n$ such that

$$f'(x, g) = \max_{v \in \partial f(x)} v^T g \quad \forall g \in \mathbb{R}^n.$$

The set $\partial f(x)$ is called the *(Hadamard) subdifferential* of f at x. If the function f is convex then

$$\partial f(x) = \{v \in \mathbb{R}^n \mid f(y) - f(x) \geq v^T (y - x) \, \forall y \in \mathbb{R}^n\}.$$

Moreover, in this case the set-valued mapping $\partial f(\cdot)$ is upper semicontinuous.

III. NONSMOOTH SPEED-GRADIENT METHOD

A. Problem Formulation

Consider the controlled system

$$\dot{x} = F(x, u, t), \quad t \geq 0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of the system state, and $u \in \mathbb{R}^m$ is the control. We assume that the function $F \colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^n$ satisfies the Carathéodory condition, i.e. the mapping $(u, x) \to F(x, u, t)$ is continuous for almost all $t \geq 0$, and the mapping $t \to F(x, u, t)$ is measurable for all

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$x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. A solution of (1) is understood in the sense of Filippov [15].

We pose the general control problem as finding the control law $u(t) = U\{x(s), u(s) \colon 0 \le s \le t\}$ which ensures the control objective

$$Q_t \le \Delta \text{ when } t \ge t_*,$$

where $Q_t = Q(x(t), t)$, $Q(x, t)$ is a nonnegative function defined on $\mathbb{R}^n \times \mathbb{R}_+$, $\Delta \ge 0$ is some pre-specified threshold, and t^* is the time instant at which the control objective is achieved. The objective can be formulated also as

$$\limsup_{t\to\infty} Q_t \le \Delta,$$

which does not specify the value of t^*.

B. Nonsmooth Version of Speed-Gradient Algorithms

In order to design a control algorithm suppose that the function Q is Hadamard directionally differentiable, and choose a convex in u function $\omega(x, u, t)$ defined on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+$ and such that

$$\omega(x, u, t) \ge Q'(x, t; F(x, u, t), 1).$$

In particular, if F is linear in u, and $p(x, t; \cdot)$ is an upper convex approximation of the function $Q'(x, t; \cdot)$ then one can define

$$\omega(x, u, t) = p(x, t; F(x, u, t), 1).$$

Take the control algorithm in the form of differential inclusion [15], [16]

$$\dot{u} \in -\Gamma \partial_u \omega(x, u, t), \tag{2}$$

where Γ is a symmetric positive definite matrix, and $\partial_u \omega(x, u, t)$ is the subdifferential of the function $u \to \omega(x, u, t)$ at the point (x, u, t). The algorithm (2) is a natural generalization of the speed-gradient algorithm in differential form to the nonsmooth case. We shall call it the *Speed-Subgradient algorithm.*

Together with the Speed-Subgradient algorithm in differential from (2), let us consider an algorithm in the finite form

$$u \in u_0 - \Gamma \partial_u \omega(x, u, t),$$

where Γ is a positive definite gain matrix and u_0 is an initial value of the control variable. We will consider also a more general control algorithm

$$u = u_0 + \gamma \psi(x, u, t) \tag{3}$$

where $\gamma > 0$ is a scalar gain and the vector function ψ satisfies the "acute angle" condition: for any $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $t \ge 0$ there exists $v \in \partial_u \omega(x, u, t)$ such that

$$v^T \psi(x, u, t) \le 0$$

The algorithm of the form (3) is a generalization of the so-called speed-pseudogradient algorithms. Therefore it is natural to call it the *Speed-Pseudosubradient algorithm.*

C. Properties of the Nonsmooth Speed-Gradient Algorithms

Let us discuss the performance of the control systems with the proposed control algorithms. At first, we study the Speed-Subgradient algorithm in differential form. The theorem below is a generalization of the corresponding result in the smooth case.

Theorem 1. *Let the following assumptions hold true:*

1) *the function F satisfies the Carathéodory condition, and for any $r > 0$ there exists a locally integrable function $m_r \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that the inequality*
$$|F(x, u, t)| \le m_r(t) \quad \forall t \ge 0$$
holds true if $|x| \le r$ and $|u| \le r$;
2) *the set-valued mapping $(x, u) \to \partial_u \omega(x, u, t)$ is upper semicontinuous for a.e. $t \ge 0$, the set-valued mapping $t \to \partial_u \omega(x, u, t)$ is measurable for any x and u, and for any $r > 0$ there exists a locally integrable function $s_r \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that the inequality*
$$|v| \le s_r(t) \quad \forall v \in \partial_u \omega(x, u, t) \, \forall t \ge 0$$
holds true if $|x| \le r$ and $|u| \le r$;
3) *the function $Q(x, t)$ is nonnegative, uniformly continuous in any set of the form $\{(x, t) \colon |x| \le r, t \ge 0\}$ and radially unbounded, i.e.*
$$\inf_{t\ge 0} Q(x, t) \to +\infty \text{ as } |x| \to \infty;$$
4) *there exists $u^* \in \mathbb{R}^m$ and a positive definite continuous scalar function $\rho \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\rho(0) = 0$ and the inequality*
$$\omega(x, u^*, t) \le -\rho(Q(x, t))$$
holds for any $t \ge 0$ and $x \in \mathbb{R}^n$.

Then any solution $(x(t), u(t))$ of (1), (2) is defined on $\mathbb{R}_+$, bounded and

$$\lim_{t\to\infty} Q(x(t), t) = 0$$

for any $x(0) \in \mathbb{R}^n$ and $u(0) \in \mathbb{R}^m$.

Note that assumptions 1 and 2 ensure the existence of solutions of the system (1)-(2) at least on some finite time interval. The proof of the above theorem requires taking the Hadamard directional derivative of the following nonsmooth Lyapunov function

$$V(x, u, t) = Q(x, t) + (u - u^*)^T \Gamma^{-1} (u - u^*)/2.$$

Some details on the method of nonsmooth Lyapunov functions can be found, for instance, in [16].

Let us consider now the Speed-Pseudosubradient algorithm. Arguing similarly to the proof of Theorem 3.3 from [5] one can easily verify that the following result holds true.

Theorem 2. *Let the following assumptions be valid:*

1) *for any $\gamma > 0$, $u_0 \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $t \ge 0$ there exists a solution $u = \kappa(x, u_0, t, \gamma)$ of equation (3), and the function κ is locally bounded in x uniformly in t;*

2) *a solution of the system (1), (3) exists for all* $t \geq 0$, $x(0) \in \mathbb{R}^n$ *and* $u_0 \in \mathbb{R}^m$*;*
3) *the function* $Q(x,t)$ *is nonnegative, uniformly continuous in any set of the form* $\{(x,t)\colon |x| \leq r, t \geq 0\}$ *and radially unbounded;*
4) *there exist a locally bounded uniformly in* t *function* $u_*\colon \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^m$ *and a positive definite continuous scalar function* $\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\rho(0) = 0$ *and the inequality*

$$\omega(x, u_*(x,t), t) \leq -\rho(Q(x,t))$$

holds true for any $x \in \mathbb{R}^n$ *and* $t \geq 0$*;*
5) *there exists* $\beta > 0$ *such that for any* $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ *and* $t \geq 0$ *the inequality*

$$v^T \psi(x,u,t) \leq -\beta |v| \tag{4}$$

holds true for some $v \in \partial_u \omega(x,u,t)$*.*

Then there exists $\overline{\gamma} > 0$ *such that any solution* $(x(t), u(t))$ *of (1), (3) is bounded and*

$$\lim_{t\to\infty} Q(x(t),t) = 0 \tag{5}$$

for any $x(0) \in \mathbb{R}^n$, $u_0 \in \mathbb{R}^m$ *and* $\gamma > \overline{\gamma}$*. Moreover, if* $u_0 = 0$ *and the inequality*

$$u_*(x,t)^T v \geq 0 \quad \forall x \in \mathbb{R}^n \, \forall t \in \mathbb{R}_+ \tag{6}$$

holds for the same $v \in \partial_u \omega(x,u,t)$ *as in (4), then* $\overline{\gamma} = 0$*.*

Note that if in the theorem above the function ψ is bounded (in particular, $|\psi(x,u,t)| = 1$), does not depend on u, and the condition (6) holds true, then the control objective (5) can be achieved for arbitrarily small control input. Indeed, by choosing sufficiently small $\gamma > 0$ one can obtain that the inequality

$$|u(x,t)| = \gamma |\psi(x,u,t)| < \varepsilon \quad \forall x \in \mathbb{R}^n \, \forall t \geq 0$$

holds for an arbitrary small $\varepsilon > 0$.

D. Example

Let us illustrate the theory discussed above by a simple example. Consider the linear controlled system

$$\dot{x} = Ax + Bu,$$

and define

$$Q(x) = \|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}.$$

For the sake of simplicity, suppose that $n = 2$, $m = 1$ (i.e. $x \in \mathbb{R}^2$ and $u \in \mathbb{R}$) and $B = (1,1)^T$.

Let us design the control algorithm with the use of the Speed-Pseudosubradient algorithm (3). Define

$$\omega(x,u) = Q'(x; Ax + Bu).$$

Then

$$\omega(x,u) = \mathrm{sign}(x_1)(a_{11}x_1 + a_{12}x_2 + u)$$

in the case $|x_1| > |x_2|$,

$$\omega(x,u) = \mathrm{sign}(x_2)(a_{21}x_1 + a_{22}x_2 + u)$$

in the case $|x_2| > |x_1|$ and

$$\omega(x,u) = \max\{\mathrm{sign}(x_1)(a_{11}x_1 + a_{12}x_2 + u), \\ \mathrm{sign}(x_2)(a_{21}x_1 + a_{22}x_2 + u)\}$$

in the case $|x_1| = |x_2|$. Therefore according to the simplest version of the Speed-Pseudosubgradient algorithm we set

$$u(x) = -\gamma \nabla_u \omega(x,u) = -\gamma \, \mathrm{sign}(x_1) \tag{7}$$

if $|x_1| > |x_2|$, and

$$u(x) = -\gamma \nabla_u \omega(x,u) = -\gamma \, \mathrm{sign}(x_2) \tag{8}$$

if $|x_2| > |x_1|$. Here $\gamma > 0$ and $\nabla_u \omega(x,u)$ is the gradient of the function $u \to \omega(x,u)$.

Let now $|x_1| = |x_2|$. If $x_1 = x_2 > 0$, then

$$\omega(x,u) = \max\{a_{11}x_1 + a_{12}x_2 + u, a_{21}x_1 + a_{22}x_2 + u\}.$$

Hence $\partial_u \omega(x,u) = \{1\}$ for any $x \in \mathbb{R}^2$, $u \in \mathbb{R}$, and

$$u(x) = -\gamma. \tag{9}$$

Analogously, if $x_1 = x_2 < 0$, then

$$u(x) = \gamma. \tag{10}$$

Combining (7)–(10) one gets that

$$u(x) = -\gamma \, \mathrm{sign}(x_1 + x_2) \tag{11}$$

for any $x \in \mathbb{R}^2$ such that $x_1 \neq -x_2$.

Consider the case $x_1 = -x_2$, $x_1 > 0$. One has

$$\omega(x,u) = \max\{u + (a_{11} - a_{12})x_1, -u + (a_{22} - a_{21})x_1\}.$$

Solving the inclusion

$$u(x) \in -\gamma \partial_u \omega(x,u)$$

one gets that

$$u(x) = \begin{cases} -\gamma, \text{ if } \overline{a}x_1 < -2\gamma, \\ \overline{a}x_1/2, \text{ if } \overline{a}x_1 \in [-2\gamma, 2\gamma], \\ \gamma, \text{ if } \overline{a}x_1 > -2\gamma, \end{cases}$$

where

$$\overline{a} = a_{12} + a_{22} - a_{11} - a_{21}$$

Moreover, $u(x)$ has the same form in the case $x_1 = -x_2$, $x_1 < 0$. However, since in this paper we understand a solution of a differential equation with discontinuous right-hand side in the sense of Filippov, then one can define $u(x)$ as an arbitrary number in the segment $[-\gamma, \gamma]$ for any x such that $x_1 = -x_2$.

Let us check that the assumptions of Theorem 2 are valid. Then one can conclude that the control goal is achieved for the constructed control law (11).

Clearly, assumptions 1–3 are satisfied. Since by construction $u(x) \in -\gamma \partial_u \omega(x,u)$ then assumption 5 is also valid. It remains to check that assumption 4 holds true. For any $x \in \mathbb{R}^2$ such that $|x_1| > |x_2|$ define

$$u_*(x) = -x_1 - a_{11}x_1 - a_{12}x_2.$$

Then

$$\omega(x, u_*(x)) = -|x_1| = -Q(x).$$

Analogously, if

$$u_*(x) = -x_2 - a_{21}x_1 - a_{22}x_1$$

then for any x such that $|x_1| < |x_2|$ one has

$$\omega(x, u_*(x,t)) = -|x_2| = -Q(x).$$

If $x_1 = x_2$ then for

$$u_*(x) = -x_1 - \min\{\mathrm{sign}(x_1)(a_{11}x_1 + a_{12}x_2), \\ \mathrm{sign}(x_1)(a_{21}x_1 + a_{22}x_2)\}$$

one has

$$\omega(x, u_*(x)) \le -|x_1| = -Q(x).$$

Finally, if $x_1 = -x_2$ then defining

$$u_*(x) = a_0 x_1/2,$$

where

$$a_0 = a_{11} + a_{22} - a_{12} - a_{21},$$

one gets

$$\omega(x, u_*(x)) = \min_{u\in\mathbb{R}} \omega(x,u) = a_0|x_1|/2 = a_0 Q(x)/2.$$

Consequently, if $a_0 < 0$ then assumption 4 of Theorem 2 is satisfied with $\rho(s) \equiv |a_0|s/2$. Achievement of the control goal under condition $a_0 < 0$ now follows from Theorem 2. Extension to the case of arbitrary n and m can be made in a similar way.

IV. CONCLUSION

The Speed-Gradient algorithms are used in many nonlinear control and adaptation problems. However their extensions to the nonsmooth case were not available before. In this paper the first step toward development a nonsmooth version of Speed-Gradient method is made based on Hadamard subdifferentiation and set-valued analysis. Nonsmooth versions of SG-algorithm in differential and finite form are formulated and conditions for the control goal achievement are proposed. The results are illustrated by an example of asymptotic minimization of the qubic norm for a second order linear system.

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