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Conic *S*-procedure and constrained dissipativity for linear systems

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SUMMARY

A new proof and extension of the recent result of Iwasaki *et al*. (*Systems Control Lett*. 2005; **54**(7):681–691) concerning equivalence of frequency domain inequality on finite frequency range and constrained dissipativity property for linear systems is given. The results of this paper employ an extended version of *S*-procedure based on duality in the space of positive definite matrices and introduction of matrix Lagrange multipliers, proposed by Yakubovich in 1992. Copyright © 2006 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Quite a number of tools in the modern systems and control theory stem from celebrated Kalman–Yakubovich–Popov (KYP) lemma. Its first formulation was established by Yakubovich [1]. The lemma provided the basis for the discovery of equivalence between three fundamental properties for systems with associated quadratic constraints, expressed in frequency domain (as frequency domain inequalities—FDI), algebraic domain (as linear matrix inequalities—LMI) and in time domain (as existence of a special Lyapunov function, in the modern language called passivity or dissipativity). In the first proofs of Popov [2], and Yakubovich [3], a trick reducing a constrained problem to a non-constrained one was used called *S*-procedure.‡ Actually it was proved in References [2, 3] that such a reduction is lossless

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‡ This trick was first used by A. I. Lurie in 1944, while the term was coined by Aizerman and Gantmaher [4]. By the way, recently Yakubovich made a remark at a seminar that it was, perhaps, not the best choice: a better one would be to name it '*L*-procedure' after Lurie and Lagrange.

(non-conservative) for constraints specified by special quadratic forms. Later Yakubovich managed to prove losslessness of the S-procedure for general quadratic forms [5] and other S-procedure related results were established, see recent survey [6].

The author of this paper was lucky to attend the lectures of Yakubovich for undergraduates in 1970. In his usual lecturing style Professor was showing to students his fresh, unpublished results. One of them was S-procedure theorem from the paper [5] that was just in preparation. I literally fell in love with this beautiful, simple yet deep statement and chose S-procedure as the theme of my diploma project under supervision of Yakubovich. The project was defended in June, 1971 and published in 1973, see [7]. The key point of the paper [7] was interpretation of S-procedure theorems as duality relations in (possibly non-convex) extremal problem.

Recently a number of new tools for systems analysis and design related to frequency domain inequalities over a finite frequency range (so called *generalized KYP-lemma*) have been developed [8–10]. It follows from the results of Iwasaki *et al.* [8, 9] and Iwasaki and Hara [10] that fulfillness of a standard FDI in a finite frequency range is equivalent to validity of some non-classical linear matrix inequalities (LMI) for a pair of matrices P, Q replacing inequalities for a single matrix P appearing in the classical KYP-lemma. It was shown in Reference [11] that FDI, in turn, are equivalent to some time-domain inequality (TDI, dissipation inequality [12]), valid only over a part of the system trajectories, determined by an additional integral matrix inequality (restricted or constrained dissipativity [11]). Thus, a complete extension of the classical KYP-results on equivalence between FDI, TDI and LMI to the 'finite-frequency' case was obtained. Note that the proof of equivalence between FDI and TDI in Reference [11] goes along the lines of the necessity proof for the frequency-domain absolute stability criterion [13, 14]. However, a natural question arises: is it possible to exploit some, possibly new version of S-procedure to establish equivalence between TDI and LMI for the 'finite-frequency' case in a more simple way?

In this paper an affirmative answer to this question is provided. A new proof of the result of Iwasaki *et al.* [11] is given based on the losslessness result for a new version of the S-procedure, dealing with constraints in LMI form, or more generally, with conic inequalities in linear spaces.

In the next section new S-procedure results are presented. In Section 3 they are applied to the proof of equivalence between TDI and LMI, under more weak conditions than in Reference [11]. Section 4 is devoted to the 'generalized conic S-procedure', extending the framework of Iwasaki *et al.* [8].

In the paper the following notation is used. The set of square integrable functions on $[0, \infty)$ is denoted by $\mathscr{L}_2[0, \infty)$, M^*, where M is a matrix, stands for its transposition and complex conjugate of all elements. For a square matrix M, its Hermitian part is defined by $\mathrm{He}(M) := (M + M^*)/2$. The interior of a set $\mathbf{\Omega}$ is denoted by $\mathrm{Int}\,\mathbf{\Omega}$.

2. CONIC S-PROCEDURE

Let $X, Y_1, \ldots, Y_m$ be linear topological spaces, $G_j : X \to Y_j$, $j = 1, \ldots, m$, $F : X \to \mathbb{R}^1$ be continuous mappings.

Let for any $j = 1, \ldots, m$ a convex cone $K_j \subset Y_j$ be given defining inequality $G_j(x) \geqslant 0$ for $x \in X$ as inclusion $G_j(x) \in K_j$. Let Y_j^* denote the dual space to Y_j, i.e. the linear space of linear continuous functionals y_j^* on Y_j and $K_j^* \subset Y_j^*$ denote the dual cone to K_j, i.e. $K_j^* = \{y_j^* \in Y_j^* : \langle y_j^*, y_j \rangle \geqslant 0 \ \forall y_j \in K_j\}$, where $\langle y_j^*, y_j \rangle$ is the value of the functional y_j^* at the element y_j.

Obviously, if $Y = \mathbb{R}^1 \times Y_1 \times \cdots \times Y_m$, then $Y^* = \mathbb{R}^1 \times Y_1^* \times \cdots \times Y_m^*$ is the set of all corteges $(y_0^*, y_1^*, \ldots, y_m^*)$, where $y_0^* \in \mathbb{R}^1$, y_j^* is a linear functional from Y_j^*.

Consider the following two relations for the mappings F_0, G_1, ..., G_m:

(A) $F(x) \geqslant 0$ for $x \in X$, $G_j(x) \in K_j$, $j = 1, \ldots, m$;
(B) $\exists \tau_0 \geqslant 0$, $\tau_j \in K_j^*$: $\tau_0 F(x) - \sum_{j=1}^m \langle \tau_j, G_j(x) \rangle \geqslant 0 \ \forall_x \in X$.

Obviously, validity of (B) with $\tau_0 > 0$ implies (A). Indeed, if $x \in X$ satisfies inequalities $G_j(x) \in K_j$, $j = 1, \ldots, m$, then it follows from (B) that $\tau_0 F(x) \geqslant 0$, since $\langle \tau_i, G_i(x) \geqslant 0 \rangle$ for $j = 1, \ldots, m$. The opposite statement is not true even in the case of scalar constraints $Y_j = \mathbb{R}^1$, $j = 1, \ldots, m$, corresponding to the classical S-procedure as described by Yakubovich [5].

Similarly to the classical case we will say that S-procedure with conic constraints $G_j(x) \geqslant 0$ is *lossless*, if (A) implies (B). S-procedure with conic constraints was also introduced by Yakubovich [15, 16]. In this paper conditions of its losslessness slightly different from those of Yakubovich [15], Matveev and Yakubovich [16] and Matveev [17] will be presented.

It is well known [12] that losslessness of the classical S-procedure is equivalent to the duality theorem in the corresponding optimization problem. However, the problem is, in general, non-convex and only a few classes of functionals $F, G_1, \ldots, G_m$ are known to possess the losslessness property.

For example, classical S-procedure is lossless, if $m = 1$ and F, G_1 are quadratic forms on real or complex linear space X. It is also lossless, if $m = 2$ and F, G_1, G_2 are quadratic (Hermitian) forms on the complex linear space X. However, classical S-procedure for quadratic forms is, in general, lossy for $m \geqslant 2$ in real case and for $m \geqslant 3$ in complex case [14]. Megretski and Treil proved in 1990 [18, 19] that the classical S-procedure is lossless for all $m \geqslant 1$, if $F, G_1, \ldots, G_m$ are integral quadratic forms on $\mathscr{L}_2(0, \infty)$. Yakubovich extended this result to a more broad class of quadratic functionals, forming the so-called *S-system* [20].

Below a version of the results of [13] is formulated which is more suitable for the case of the S-procedure with conic constraints.

Theorem 1

Let $\bar{K}$ be the closure of the cone $K \subset Y$ generated by the set $\mathscr{F}(X) = \{(F(x), G_1(x), \ldots, G_m(x)) : x \in X\}$.

If the cone $\bar{K}$ is convex and has a non-empty relative interior Int K then the S-procedure with conic constraints is lossless.

If, in addition, constraints $G_j(x) \in K_j$ are regular, namely $\exists x_0 : G_j(x_0) \in \text{Int}\, K_j$, then one can choose $\tau_0 = 1$ in (B).

Proof

Condition (A) implies that $F(x) \geqslant 0$ for $G_j(x) \in \text{Int}\, K_j$, i.e. intersection of the set $\mathscr{F}(X)$ and the open cone $D = \{(-y_0, y_1, \ldots, y_m) : y_0 > 0, y_j \in \text{Int}\, K_j, j = 1, \ldots, m\}$ is empty: $D \cap \mathscr{F}(X) = \phi$. Therefore, $D \cap \bar{K} = \phi$. Applying separation theorem for cones, we obtain that there exists vector $\tau^* = (\tau_0^*, \tau_1^*, \ldots, \tau_m^*) \in Y^*$ such that $\langle \tau_0^*, F(x) \rangle + \sum_{j=1}^m \langle \tau_j^*, G(x) \rangle \geqslant 0$ for all $x \in X$ and $\langle \tau^*, y \rangle < 0$ for all $y \in D$, i.e. $\langle \tau_0^*, y_0 \rangle + \sum_{j=1}^m \langle \tau_j^*, y_j \rangle < 0$. For any $j = 1, \ldots, m$ pick up $y_j \in \text{Int}\, K_j$ and choose sequences $y_{0_k} \to 0, y_{s_k} \to 0$ as $k \to \infty$, such that $y_{0_k} > 0, y_{s_k} \in K_s, s \neq j$. If $k \to \infty$, then we obtain $\langle \tau_j^*, y_j \rangle \leqslant 0$, i.e. $-\tau_j^* \in K_j^*$. The first part of the theorem is proved.

Taking $x = x_0$ from regularity condition and $y_0 \neq 0$ yields $\tau_0^*, y_0 > 0$, i.e. $\tau_0^* > 0$. Dividing inequality (B) by τ_0, we arrive at the second statement of the theorem. End of the proof. □

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Let us recall the definition of S-system [20] extended to the case of conic constraints.

Definition 1
Let F_j, $j = 0, 1, \ldots, m$ be quadratic mappings from a Hilbert space $\mathbb{Z}$ to spaces of self-adjoint operators over corresponding Euclidean space $\mathbb{R}^{n_j}$, such that $F_j : \mathbb{Z} \to SR(n_j \times n_j)$.

We say that $F_0, F_1, \ldots, F_m$ form a S-system if there exists a subspace $\mathbb{Z}_0$ and a sequence of linear bounded operators $T_k : \mathbb{Z} \to \mathbb{Z}$, $k = 1, 2, \ldots$ such that

(i) $\langle T_k z_1, z_2 \rangle \to 0$ as $k \to \infty$ for all $z_1, z_2 \in \mathbb{Z}$;
(ii) $\mathbb{Z}_0$ is invariant for T_k for all $k = 1, 2, \ldots$;
(iii) $F_j(T_k z) \to F_j(z)$ as $k \to \infty$ for all $j = 0, 1, \ldots, m$, $z \in \mathbb{Z}_0$.

Lemma 1
Let F_j, $j = 0, 1, \ldots, m$ form S-system. Define the map $\mathscr{F} : \mathbb{Z} \to \prod_{j=0}^{n} \mathbb{R}^{n_j \times n_j}$ by means of the relation

$$\mathscr{F}(z) = F_0(z) \oplus F_1(z) \oplus \cdots \oplus F_m(z) \in \mathbb{R}^{n_0 \times n_0} \oplus \mathbb{R}^{n_1 \times n_1} \oplus \cdots \oplus \mathbb{R}^{n_m \times n_m}$$

Then the closure of the image $\mathscr{F}(\mathbb{Z})$ is a convex set in $\mathbb{R}^{n_0^2 + n_1^2 + \cdots + n_m^2}$.

In the special case $n_j = 1, j = 0, 1, \ldots, m$, Lemma 1 coincides with Lemma 1 of the paper [19] and its proof follows the same lines.

Example 1
An important series of examples for S-systems is provided by finite family of integral quadratic operators on the Hilbert space $\mathscr{L}_2[0, \infty)$ of square integrable functions with values $z(t) \in \mathbb{R}^{n_j}$. The mappings are defined for any $z \in \mathscr{L}_2[0, \infty)$ as follows:

$$F_j(z) = \mathrm{He} \int_0^{\infty} F_j' z(t) z^*(t) F_j''^* \, \mathrm{d}t \tag{1}$$

where F_j', F_j'' are $n_j' \times n_j$ matrices. In this case the family of the operators T_k can be chosen as time shifts: $T_k(z)(t) = z(t + k)$, while the subspace $\mathbb{Z}_0$ can be chosen as the set of solutions to the differential equation $\dot{x} = Ax + Bu$, with zero initial conditions.

The proof of the S-system property for Example 1 is again similar to Reference [20].

Losslessness of the S-procedure for S-systems is established by the following theorem.

Theorem 2
Let $n_0 = 1$. S-procedure with the objective function $F_0(z)$ and conic constraints $F_j(z) \in K_j$, where K_j is the convex cone of positive semidefinite $(n_j \times n_j)$-matrices, is lossless for any family of mappings $F_0, F_1, \ldots, F_m$ forming an S-system.

The proof follows immediately from Theorem 1 and Lemma 2. The result can be extended to the case of equality constraints and to the case of the so called *generalized S-procedure* introduced in Reference [8].

Theorem 2 applies to the Example 1, if $X = Z = \mathscr{L}_2(0, \infty)$, $n_0 = 1$ and F_j are defined as in (1). Note that the cone of positive semidefinite matrices is self-dual. Therefore, S-procedure with

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conic constraints determined by functions (1) deals with positive semidefinite matrix Lagrange multipliers.

3. CONSTRAINED DISSIPATIVITY

In this section, we first present a special case of the generalized KYP lemma [10], characterizing FDIs in the continuous-time setting. Let complex matrices A, B, Π, and real scalars ϖ_1, ϖ_2 ($\bar{\omega}_2 > \varpi_1$) be given. Define

$$\mathbf{\Omega} := \{\omega \in \mathbb{R} \mid (\omega - \varpi_1)(\omega - \varpi_2) \leqslant 0\} \tag{2}$$

Theorem 3 Iwasaki and Hara [10]
Suppose Π is Hermitian matrix, pair (A, B) is controllable, and $\mathbf{\Omega}$ has a non-empty interior. Then the following statements are equivalent.

(i) The frequency domain inequality

$$\begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Pi \begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix} \leqslant 0 \tag{3}$$

holds for all $\omega \in \mathbf{\Omega}$ such that $\det(i\omega I - A) \neq 0$, where $i^2 = -1$

(ii) There exist Hermitian matrices P and Q such that $Q \geqslant 0$ and the linear matrix inequality

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} -Q & P + i\varpi_o Q \\ P - i\varpi_o Q & -\varpi_1\varpi_2 Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Pi \leqslant 0 \tag{4}$$

holds, where $\varpi_o := (\varpi_1 + \varpi_2)/2$.

Choosing the parameters $\varpi_1 = -\varpi_2$ and tending ϖ_2 to $+\infty$, the set $\mathbf{\Omega}$ becomes the entire real numbers, and thus statement (i) becomes the FDI for all frequencies. In this case, the term associated with Q in the LMI (4) becomes positive semidefinite, and hence the best choice of Q for satisfaction of (4) is $Q = 0$. The resulting LMI with variable P is exactly the same as the one in the standard KYP lemma.

The following result extends the result of Iwasaki *et al.* [11]. It provides an equivalence between FDI and time domain dissipation inequality over a restricted class of input signals.

Theorem 4
Let complex matrices A, B, Π, and real scalars ϖ_1, ϖ_2 be given and $\mathbf{\Omega}$ be defined by (2). Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \in [0, \infty) \tag{5}$$

where $x(t) \in \mathbb{C}^n$ is the state and $u(t) \in \mathbb{C}^m$ is the input. Assume that (A, B) is controllable, Π is Hermitian, and $\mathbf{\Omega}$ has a non-empty interior. Then the following statements are equivalent:

(i) The frequency domain inequality (3) holds for $\omega \in \mathbf{\Omega}$.

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(ii) The time domain inequality

$$\int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^* \Pi \begin{bmatrix} x \\ u \end{bmatrix} \mathrm{d}t \leqslant 0 \tag{6}$$

holds for all solutions of (5) with $u \in \mathscr{L}_2[0,\infty)$ such that $x(0) = 0, x \in \mathscr{L}_2[0,\infty)$ and

$$\mathrm{He} \int_0^\infty (\varpi_1 x + i\dot{x})(\varpi_2 x + i\dot{x})^* \, \mathrm{d}t \leqslant 0 \tag{7}$$

Note that the corresponding result of Iwasaki *et al.* [11] was obtained under additional condition of asymptotic stability for the system (5) which is not required here.

In Theorem 4 a general frequency interval $\mathbf{\Omega}$ is considered for the FDI, and this has translated to the input constraint described by (7). Though the physical meaning of this constraint may be not clear in general, it becomes clear for the following special case.

Corollary 1

Let real matrices A, B, Π, and a positive scalar ϖ be given. Suppose Π is symmetric and consider the system (5) where $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the input. Assume that (A, B) is controllable. Then the following statements are equivalent.

(i) The frequency domain inequality (3) holds for all ω such that $|\omega| \leqslant \varpi$.

(ii) The time domain inequality (6) holds for all $u \in \mathscr{L}_2[0,\infty)$ such that $x \in \mathscr{L}_2[0,\infty)$ and

$$\int_0^\infty \dot{x}\dot{x}^\mathsf{T} \, \mathrm{d}t \leqslant \varpi^2 \int_0^\infty x x^\mathsf{T} \, \mathrm{d}t \tag{8}$$

The two statements obtained by replacement of signs '$\leqslant$' by '$\geqslant$' are equivalent too.

Loosely speaking, the first part of Corollary 1 states that the FDI in the low finite frequency range means that the system possesses property (6) for the input signals u that drive the states not too fast (slowly). The bound on the 'slowness' is given by ϖ in the sense of (8). The second part of Corollary 1 makes a similar statement for the FDI in the high frequency range. To derive Corollary 1 from Theorem 4 one needs to put $\omega_1 = -\omega_2 = -\varpi$.

Proof of Theorem 4

In view of Theorem 3 it is sufficient to prove equivalence of (ii) to the condition (ii) of Theorem 4 (solvability of matrix inequality (4)). The result follows from Theorem 2 of the previous section with $m = 2$ (one matrix constraint). Denote

$$z = \begin{bmatrix} x \\ u \end{bmatrix}, \quad F(z) = -\int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^* \Pi \begin{bmatrix} x \\ u \end{bmatrix} \mathrm{d}t$$

$$G_1(z) = -\mathrm{He} \int_0^\infty (\varpi_1 x + i\dot{x})(\varpi_2 x + i\dot{x})^* \, \mathrm{d}t \tag{9}$$

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Obviously, TDI (6), (7) correspond to the statement (A) of the conic S-procedure. Since the cone K of positive definite matrices is self-dual, $K^* = K$, the statement (B) for $\tau_0 = 1$ means existence of a positive definite $n \times n$-matrix τ^* satisfying inequality $F(z) - \langle \tau^*, G_1(z) \rangle \geqslant 0 \ \forall z$ or

$$\mathrm{He} \int_0^\infty \left(- \begin{bmatrix} x \\ u \end{bmatrix}^* \Pi \begin{bmatrix} x \\ u \end{bmatrix} - (\varpi_1 x + i\dot{x})^* \tau (\varpi_2 x + i\dot{x}) \right) \mathrm{d}t \geqslant 0 \tag{10}$$

Replacement of τ by Q and substitution of $\dot{x}$ from (5) transforms (10) into LMI (4). To verify regularity condition of Theorem 2 take $x_0 = (\bar{x}, \bar{u})$, where

$$\bar{x}(t) = -(A + \mu I)^{-1} B \exp(-\mu t), \quad \bar{u}(t) = \exp(-\mu t), \quad \mu > 0$$

and $\mu \in \mathrm{Int}\, \mathbf{\Omega}$. Application of Theorem 2 and the second part of Theorem 1 ends the proof. □

Remark 1
Similar results hold for discrete-time case.

Remark 2
Using the S-procedure (Theorem 2) with several matrix constraints in the backward direction provides characterization of constrained dissipativity with several constraints of form (7). Let, for example $0 \leqslant \omega' < \omega''$ and

$$\mathbf{\Omega} := \{\omega \in \mathbb{R} \,|\, (\omega' \leqslant |\omega| \leqslant \omega''\} \tag{11}$$

Then fulfillment of the FDI (3) for $\omega \in \mathbf{\Omega}$ is equivalent to validity of the TDI (6) for all solutions of (5) with $u \in \mathscr{L}_2[0, \infty)$ such that $x(0) = 0, x \in \mathscr{L}_2[0, \infty)$ and

$$\omega_1^2 \int_0^\infty x x^{\mathsf{T}} \, \mathrm{d}t \leqslant \int_0^\infty \dot{x} \dot{x}^{\mathsf{T}} \, \mathrm{d}t \leqslant \omega_2^2 \int_0^\infty x x^{\mathsf{T}} \, \mathrm{d}t \tag{12}$$

Each of properties (3) and (12) is equivalent to feasibility of the matrix inequality

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} -Q_1 + Q_2 & P \\ P & -\omega_1^2 Q_1 + \omega_2^2 Q_2 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Pi \leqslant 0 \tag{13}$$

4. GENERALIZED CONIC S-PROCEDURE

In the paper [8] a new formulation of S-procedure was proposed, extending the classical one to the case of infinite number of constraints. It was called *generalized S-procedure* and applied to proof of the generalized (finite-frequency) KYP-lemma. Although a connection between KYP-lemma and S-procedure (non-convex duality) was known earlier [6,21,23], a new version allowed straightforward derivation of the generalized (finite-frequency) version of KYP-lemma.

Let use introduce a generalized formulation of the conic S-procedure with infinite number of constraints. Consider the set Ξ of mappings $G : X \to Y$, where X, Y are linear topological spaces as before and the set of constraints $G(x) \in K, G \in \Xi$, where K is a given convex cone in Y. Combining conditions (A), (B) from the Section 2 and from Reference [8], formulate the following two conditions:

(A_0) $F(x) \geqslant 0$ for $G(x) \in K \ \forall G \in \Xi$,
(B_0) $\exists G \in \Xi, \tau^* \in K^*, \tau_0 \geqslant 0 : \tau_0 F(x) - \langle \tau^*, G(x) \rangle \geqslant 0 \ \forall x \in X$.

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Similarly the strict versions of $(A_0), (B_0)$ are introduced:

(A_1) $F(x) > 0$ for $x \neq 0, G(x) \in K \ \forall G \in \Xi$,
(B_1) $\exists G \in \Xi, \tau^* \in K^* : F(x) - \langle \tau^*, G(x) \rangle 0 \ \forall x \in X, x \neq 0$.

Similarly to the case with finite number of constraints analysed in Section 2, validity of (B_0) implies (A_0) and validity of (B_1) with $\tau_0 > 0$ implies (A_1) if the set Ξ is conic ($\lambda G \in \Xi$ if $G \in \Xi, \lambda \geqslant 0$). Again, we say that S-procedure with conic constraints $G(x) \geqslant 0, G \in \Xi$ is *lossless*, if (A_0) implies (B_0) or (A_1) implies (B_1).

Remark 3
The above properties are equivalent to those introduced in Section 2, i.e. $(A) \Leftrightarrow (A_0), (B) \Leftrightarrow (B_0)$, if Ξ is a finitely generated cone (a cone generated by a finite number of functions G_j: $\Xi = \mathscr{K}\{G_1, \ldots, G_m\}$). If all $G_J(x)$ are Hermitean forms, the generalized conic S-procedure coincides with generalized S-procedure of Iwasaki *et al.* [8].

5. CONCLUSIONS

The property of the system defined by the item (ii) of Theorem 4 and the corollary can be called *constrained dissipativity* or *restricted dissipativity*. It is weaker than standard passivity or dissipativity conditions and may better reflect specifications for real systems since it requires fulfillment of the dissipation inequality only along 'slow' trajectories. At the same time the 'slowness' described by inequality (8) leaves enough flexibility for analysis of systems.

The framework of this paper may shed new light on the intimate interrelations between S-procedure and KYP-lemma. It allows to extend classical S-procedure tools to make them suitable for analysis and design of robust systems with matrix inequality constraints. Application of conic S-procedure to robust control is an interesting direction of future research.

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