Passification-based decentralized adaptive synchronization of dynamical networks with time-varying delays[☆]

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Abstract

This paper is aimed at application of the passification based adaptive control to decentralized synchronization of dynamical networks. We consider Lurie type systems with hyper-minimum-phase linear parts and two types of nonlinearities: Lipschitz and matched. The network is assumed to have both instant and delayed time-varying interconnections. Agent model may also include delays. Based on the speed-gradient method decentralized adaptive controllers are derived, i. e. each controller measures only the output of the node it controls. Synchronization conditions for disturbance free networks and ultimate boundedness conditions for networks with disturbances are formulated. The proofs are based on Passification lemma in combination with Lyapunov-Krasovskii functional's technique. Numerical examples of 4 and 100 interconnected Chua systems are presented to demonstrate the efficiency of the proposed approach.

Keywords: networks, adaptive control, uncertain systems, delays

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1. Introduction

Adaptive synchronization of dynamical networks has attracted a growing interest during recent years [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. It is motivated by a broad area of potential applications: networks of robots, formations of flying and underwater vehicles, control of industrial, electrical, communication, and production networks, etc. Although problems of decentralized control for networks of coupled systems were studied before, most of the existing works, e. g. [3, 4, 5, 6, 7, 8], deal with full state feedback and linear interconnections. Moreover, control variables usually appear in all equations of the network model. Such system models are quite restrictive for applications, where uncertainties of the system, nonlinear interconnections, switching structure of the network topology, nonlinear dynamics of the local subsystems and incomplete measurement of their states should be taken into account.

The key to solve the above problem is application of the passification approach. It was initially proposed in 1974 for a SIMO plant [12] and later was extended to a broad class of MIMO linear and nonlinear systems. Related versions are also known under names "adaptive systems with implicit reference models" [13], "adaptive control based on feedback Kalman-Yakubovich lemma" [14] and "simple adaptive control" [15, 16]. Adaptive system design proposed in the 1970s was sensitive to disturbances: an arbitrary small disturbance was able to destroy boundedness of the trajectories. Later regularization tricks to overcome difficulties were proposed, e. g. negative parametric feedback used in this paper. In the early papers on the passification based approach the restrictive hyper-minimum-phase condition was imposed. However later the so called "parallel feedforward compensator" (shunt) was proposed by Barkana in [17, 18] and extended in [19] that allowed one to relax hyper-minimum-phase condition requiring only minimum phaseness, without "relative degree one" property. Thus, relative degree one restriction has been removed. To simplify exposition and make more clear basic ideas we do not use shunts in this paper. The idea of shunt trick can be found in [19, 20] while detailed exposition is to appear elsewhere.

A passification based approach to decentralized adaptive synchronization of the Lurie type networks with incomplete measurements and incomplete control was proposed in [21]. Here we extend these results to the case of time-varying unknown interconnection delays and bounded disturbances.

For the synchronization of networks with delayed couplings and disturbances quite a number of papers have already been published [22, 23, 24, 25, 26, 27, 28, 29]. However, again, adaptive control laws were derived only for a narrow class of networks, such as fully-controlled and fully-measured agents. Some of these works deal with non-switching topology or provide non-adaptive control.

In the current work we propose an adaptive decentralized algorithm for synchronization of networks with nonlinear delayed couplings that depend on time. We consider partly unknown Lurie type nonlinear systems with delayed interconnections and bounded disturbances. The controller does not use any information about system parameters, but to ensure synchronization it is required that all subsystems belong to a special class described below (see conditions of Theorems 1, 2, 3, 4). Our approach is based on Passification lemma [30] and Lyapunov-Krasovskii method.

Notations used throughout the paper is fairly standard. The fields of real and complex numbers are denoted by $\mathbb{R}$, $\mathbb{C}$. $\mathbb{R}^n$ is n-dimensional Euclidean space with Euclidean norm $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$. $\mathcal{C}[a,b]$ is a space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with a norm $\|\phi\|_{\mathcal{C}} = \max_{s\in[a,b]} \|\phi(s)\|$. As usual I is an identity matrix, A^T is transposed matrix A, $\lambda_{\max}(A)$ is the maximum eigen value of a square matrix A, $\operatorname{sign} p = -1$ for $p < 0$, 0 for $p = 0$ and 1 for $p > 0$.

Some preliminary results were presented in [1].

1.1. Passification method

Definition 1. *For given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $C \in \mathbb{R}^{l\times n}$, $g \in \mathbb{R}^l$ a transfer function $g^T W(s) = g^T C(sI - A)^{-1} B$ is called hyper-minimum-phase if the polynomial $g^T W(s) \det(sI - A)$ is Hurwitz and $g^T CB$ is a positive number.*

To formulate main results we will need Passification lemma in the following form [31].

Lemma 1 (Passification lemma). *Let the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $C \in \mathbb{R}^{l\times n}$, $g \in \mathbb{R}^l$ be given. Then for existence of a positive-definite $n \times n$-matrix $P = P^T > 0$ and a vector $\theta_* \in \mathbb{R}^l$ such that*

$$PA_* + A_*^T P < 0, \quad PB = C^T g, \tag{1}$$

where $A_* = A - B\theta_*^T C$, it is necessary and sufficient that the function $g^T W(s) = g^T C(sI - A)^{-1}B$ is hyper-minimum-phase.

Remark 1. *Consider a linear system*

$$\dot{x} = Ax + Bu, \quad y = Cx. \tag{2}$$

It follows from Passification lemma (see [20] for details) that if $g^T C(sI-A)^{-1}B$ is hyper-minimum-phase then there exists θ_* such that the substitution $u = -\theta_*^T y + v$ makes the system (2) strictly passive with respect to a new input v, i. e. there exist a nonnegative scalar function $V(x)$ and a scalar function $\rho(x)$, where $\rho(x) > 0$ for $x \neq 0$, such that

$$V(x) \le V(x_0) + \int_0^t \left[v(t)^T g^T y(t) - \rho(x(t))\right] dt$$

for any solution of the system (2) satisfying $x(0) = x_0$.

The last inequality has a simple physical interpretation. Function $V(x)$ is an analog of system total energy. The term $v(t)^T g^T y(t)$ can be treated as the power transmitted to the system. Therefore, $\int_0^t v(t)^T g^T y(t)\, dt$ is the energy transmitted to the system. The term $\rho(x(t))$ reflects dissipation rate that arises due to energy loss (friction, for instance). In brief, the last inequality is an energy balance for a system without internal energy sources.

It follows from Passification lemma that if $g^T W(s)$ is hyper-minimum-phase then there exist $P > 0$, θ_*, $\varepsilon > 0$ such that

$$PA_* + A_*^T P < -\varepsilon I, \quad PB = C^T g, \tag{3}$$

where $A_* = A - B\theta_*^T C$.

The first inequality means that matrix A_* degree of stability is $\varepsilon\lambda_{\max}^{-1}(P)$. The value $\varepsilon\lambda_{\max}^{-1}(P)$ has a crucial meaning for synchronization and we would like it to be as big as possible. The second relation $PB = C^T g$ will be used to construct a realizable controller.

2. Problem statement

We will study networks dynamics of which are given by the following equations:

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + \varphi_0\big(t, x_i(t)\big) + \sum_{j=1}^{N} \varphi_{ij}\big(t, x_j(t)\big) \\ &\quad + \sum_{j=1}^{N} \psi_{ij}\big(t, x_j(t-\tau(t))\big) + Bu_i(t), \\ y_i(t) &= Cx_i(t), \quad t \geq t_0, \quad i = 1, \ldots, N, \end{aligned} \tag{4}$$

with states $x_i \in \mathbb{R}^n$, inputs $u_i \in \mathbb{R}$, measurable outputs $y_i \in \mathbb{R}^l$, and constant matrices A, B, C having appropriate dimensions. Time-varying delay $\tau(t)$ is assumed to be a differentiable function such that $-h < t - \tau(t) < t$ ($h > 0$) and $\dot{\tau}(t) \leq d < 1$ for all $t \geq t_0$. Functions φ_0, φ_{ij} and ψ_{ij} describe local dynamics of the nodes and their interactions. Note that the network model (4) admits delay in local agent dynamics described by the term $\psi_{ii}(t, x_i(t - \tau(t)))$. Throughout the paper we assume that φ_{ij} and ψ_{ij} satisfy Lipschitz condition with respect to the second argument with nonnegative constants L_{ij} and M_{ij}, i. e. for all $t \geq t_0$ and any $x, y \in \mathbb{R}^n$

$$\begin{aligned} \|\varphi_{ij}(t, x) - \varphi_{ij}(t, y)\| &\leq L_{ij}\|x - y\|, \\ \|\psi_{ij}(t, x) - \psi_{ij}(t, y)\| &\leq M_{ij}\|x - y\|. \end{aligned} \tag{5}$$

Functions φ_0, φ_{ij} and ψ_{ij} are assumed satisfying standard conditions for existence and uniqueness of solutions of (4) for any piecewise continuous $u_i(t)$ (see, e. g. [32] for details). Discontinuity of φ_{ij}, ψ_{ij} in t reflects the switching character of the network.

Initial conditions for the system (4) are given by continuous functions $x_i^0 \in \mathcal{C}[-h, 0]$, $i = 1, \ldots, N$ as follows:

$$x_i(t) = x_i^0(t), \quad \forall t \in [-h, 0]. \tag{6}$$

Here we deal with the problem of synchronization, therefore it is necessary to assume that the network (4) admits a synchronous solution $\bar{x}(t)$. Suppose that the system is synchronized and we do not need to control it, i. e. $x_1(t) = \ldots = x_N(t) = \bar{x}(t)$ and $u_1(t) = \ldots = u_N(t) = 0$ for all $t \geq t_0$. By substituting this values in equations (4) we derive that there should exist functions $\Phi(t, x)$ and $\Psi(t, x)$ such that for all $i = 1, \ldots, N$ and all $t \geq t_0$

$$\begin{aligned} \sum_{j=1}^{N} \varphi_{ij}(t, \bar{x}(t)) &= \Phi(t, \bar{x}(t)), \\ \sum_{j=1}^{N} \psi_{ij}(t, \bar{x}(t)) &= \Psi(t, \bar{x}(t)). \end{aligned} \tag{7}$$

Here we assume that the controller of the i-th subsystem does not possess any information about other nodes. Then, to synchronize the network, a *leader* system is required:

$$\begin{aligned} &\dot{x}_L(t) = Ax_L(t) + \varphi_0(t, x_L(t)) + \Phi(t, x_L(t)) + \Psi\left(t, x_L\left(t - \tau(t)\right)\right) + Bu_L(t), \\ &y_L(t) = Cx_L(t), \end{aligned} \tag{8}$$

where u_L is a known input signal. Initial condition for this system is given by $x_L^0 \in \mathcal{C}[-h, 0]$.

We will also assume that the controller does not know all system parameters. Therefore, in the control law the entries of A, B, C will not be used, although to prove the convergence we need to know that the system belongs to a special class of systems given below.

The problem is formulated as follows: find functions $u_i = U_i(t, y_i, y_L, u_L)$ such that for all solutions of the system (4), (6), (8) for all $i = 1, \ldots, N$

$$\lim_{t \to \infty} \|x_i(t) - x_L(t)\| = 0. \tag{9}$$

The problem is complicated by the fact that the system (4) is not fully controlled: subsystems are n-dimensional while control signals u_i are scalars. Therefore, the goal (9) cannot be always achieved (e. g. when $B = 0$). Nevertheless, (9) can be satisfied in a special case, namely, we assume the following.

Assumption 1. *There exists $g \in \mathbb{R}^l$ such that $g^T C(sI - A)^{-1}B$ is hyper-minimum-phase.*

3. Controller design

First, by taking the difference between (4) and (8) we derive equations for the errors $e_i(t) = x_i(t) - x_L(t)$

$$\begin{aligned} \dot{e}_i(t) &= Ae_i(t) + \big[\varphi_0\big(t, x_i(t)\big) - \varphi_0\big(t, x_L(t)\big)\big] \\ &\quad + \sum_{j=1}^{N} \big[\varphi_{ij}\big(t, x_j(t)\big) - \varphi_{ij}\big(t, x_L(t)\big)\big] \\ &\quad + \sum_{j=1}^{N} \big[\psi_{ij}\big(t, x_j(t-\tau(t))\big) - \psi_{ij}\big(t, x_L(t-\tau(t))\big)\big] \\ &\quad + B\big[u_i(t) - u_L(t)\big], \\ y_i(t) - y_L(t) &= C\big[x_i(t) - x_L(t)\big], \quad i = 1, \ldots, N. \end{aligned} \tag{10}$$

The idea of the control algorithm is the following. If the system is synchronized than in view of (7) it is sufficient to apply zero forces to the subsystems (10), i. e. $u_i = u_L$. If the system is not synchronized than it is reasonable that the bigger difference $y_i - y_L$ is the bigger force we should apply. Thereby, we arrive to the controllers:

$$u_i(t) - u_L(t) = -\theta_i^T \big[y_i(t) - y_L(t)\big]. \tag{11}$$

Since the system is uncertain the values of θ_i are adjusted adaptively using *the speed-gradient method* [33].

Let us fix $i = 1, \ldots, N$. Consider a goal function $V_0(e_i) = \frac{1}{2} e_i^T P e_i$. Denote $\omega_i(e_i, \theta_i) = \big[\nabla_{e_i} V_0(e_i)\big]^T \dot{e}_i$, where $\dot{e}_i$ is given by (10), (11). Decentralized speed-gradient algorithm is introduced as follows:

$$\dot{\theta}_i = -\Gamma_i \nabla_{\theta_i} \omega_i(e_i, \theta_i, t) = \Gamma_i (e_i^T P B)[y_i - y_L],$$

$i = 1, \dots, N$, where $\Gamma_i = \Gamma_i^T > 0$ is $l \times l$-matrix. As soon as the conditions (3) are satisfied, $PB = C^T g$, therefore $\dot{\theta}_i = \Gamma_i (e_i^T C^T g)[y_i - y_L] = \Gamma_i ([y_i - y_L]^T g)[y_i - y_L]$. The term $[y_i - y_L]^T g$ is a scalar, thus we can rewrite this equation in the form $\dot{\theta}_i = \Gamma_i [y_i - y_L][y_i - y_L]^T g$. Finally, we derived the following adaptive controllers:

$$\begin{aligned} u_i(t) &= -\theta_i(t)^T \left[y_i(t) - y_L(t)\right] + u_L(t), \\ \dot{\theta}_i(t) &= \Gamma_i \left[y_i(t) - y_L(t)\right] \left[y_i(t) - y_L(t)\right]^T g. \end{aligned} \tag{12}$$

Initial values for $\theta_i(t)$ can be chosen arbitrarily.

Remark 2. *The control law* (12) *includes undefined terms* Γ_i. *Synchronization conditions will be proved for all* $\Gamma_i > 0$. *The concrete values of* Γ_i *determine the speed of convergence. If* Γ_i *is too small, then the convergence will be slow. At the same time, large* Γ_i *may cause undesirable oscillations of* θ_i. *Therefore, the question of optimal definition of* Γ_i *is still to be investigated. In the simulations presented here we took* $\Gamma_i = I$.

Adaptive decentralized controller (12) of the i-th node does not require the knowledge of y_j with $j \neq i$. At the same time the terms φ_{ij}, ψ_{ij} depend on y_j and may prevent the system from synchronization. Therefore, to synchronize the system with (12) one need to ensure that the influence of φ_{ij}, ψ_{ij} is small enough. Note that unlike the so-called pinning control [34, 35, 36, 37] we do not impose any conditions that guarantee that pinning terms ϕ_{ij}, ψ_{ij} have a positive effect on synchronization. In what follows we derive conditions on Lipschitz constants L_{ij}, M_{ij} such that (12) ensures (9) for the network under consideration.

4. Synchronization conditions

In order to formulate synchronization conditions for the system (4), (6), (8), (12) we introduce notations:

$$\overline{L} = \max_{i=1,\dots,N} \sum_{j=1}^{N} [L_{ij} + L_{ji}],$$
$$\overline{M} = \max_{i=1,\dots,N} \sum_{j=1}^{N} \left[M_{ij} + \frac{M_{ji}}{1-d} \right], \tag{13}$$

where L_{ij}, M_{ij} are from (5), d is the upper bound for derivative of a time-varying delay: $\dot{\tau}(t) \leq d$. Values $\overline{L}$ and $\overline{M}$ have the meaning of couplings' strengths. As soon as the controllers (12) are decentralized this values are required to be small.

4.1. Lipschitz type nonlinearity

Synchronization conditions will be formulated for two types of nonlinearity φ_0. We begin with Lipschitz type nonlinearity.

Assumption 2. *Function $\varphi_0(t,x)$ satisfies Lipschitz condition with respect to x uniformly on $t \geq t_0$ with a positive constant L_0, that is for all $t \geq t_0$ and any $x, y \in \mathbb{R}^n$*

$$\|\varphi_0(t,x) - \varphi_0(t,y)\| \leq L_0 \|x - y\|.$$

Theorem 1 (Lipschitz nonlinearity). *Consider the network* (4) *subject to* (5) *and the leader system* (8)*. Let Assumption 1 hold with $g \in \mathbb{R}^l$ and, thus,* (3) *is feasible for some $P > 0$, $\varepsilon > 0$, and θ_*. Let Assumption 2 be valid with some $L_0 > 0$. If the following inequality holds*

$$2L_0 + \overline{L} + \overline{M} < \frac{\varepsilon}{\lambda_{\max}(P)} \tag{14}$$

where $\overline{L}$ and $\overline{M}$ are given by (13)*, then the adaptive control algorithm (12) ensures synchronization (9). Moreover, all tunable parameters $\theta_i(t)$ will tend to constant values.*

Proof. Denote $e_i^t = e_i(t+\theta), \theta \in [-\tau(t), 0]$ and consider the following Lyapunov-Krasovskii functional

$$V(t, e_1^t, \dots, e_N^t) = V_1 + V_2 + V_3, \tag{15}$$

where

$$V_1 = \sum_{i=1}^{N} e_i^T(t) P e_i(t), \quad V_2 = \sum_{i=1}^{N} (\theta_i - \theta_*)^T \Gamma_i^{-1} (\theta_i - \theta_*),$$

$$V_3 = \sum_{i=1}^{N} \int_{t-\tau(t)}^{t} e_i^T(s) Q_i e_i(s)\, ds,$$

with $Q_i = \frac{\lambda_{\max}(P)}{1-d} \sum_{j=1}^{N} M_{ji} I \geq 0$.

Now calculate a derivative of V along the trajectories of the system (10), (12).

$$\begin{aligned}
\dot{V}_1 &= \sum_{i=1}^{N} [e_i^T(t) P \dot{e}_i(t) + \dot{e}_i^T(t) P e_i(t)] = \sum_{i=1}^{N} e_i^T(t) \big[PA + A^T P\big] e_i(t) \\
&+ 2 \sum_{i=1}^{N} e_i^T(t) P \big[\varphi_0\big(t, x_i(t)\big) - \varphi_0\big(t, x_L(t)\big)\big] \\
&+ 2 \sum_{i=1}^{N} e_i^T(t) P \sum_{j=1}^{N} \big[\varphi_{ij}\big(t, x_j(t)\big) - \varphi_{ij}\big(t, x_L(t)\big)\big] \\
&+ 2 \sum_{i=1}^{N} e_i^T(t) P \sum_{j=1}^{N} \big[\psi_{ij}\big(t, x_j(t-\tau)\big) - \psi_{ij}\big(t, x_L(t-\tau)\big)\big] \\
&- 2 \sum_{i=1}^{N} e_i^T(t) P B \theta_i^T(t) \big[y_i(t) - y_L(t)\big].
\end{aligned}$$

In view of Assumption 2

$$2 \sum_{i=1}^{N} e_i^T(t) P \big[\varphi_0\big(t, x_i(t)\big) - \varphi_0\big(t, x_L(t)\big)\big] \leq 2\lambda_{\max}(P) L_0 \sum_{i=1}^{N} \|e_i(t)\|^2. \tag{16}$$

Further,

$$\begin{aligned}
&\left|2\sum_{i=1}^{N} e_i^T(t)P\sum_{j=1}^{N}\big[\varphi_{ij}\big(t,x_j(t)\big)-\varphi_{ij}\big(t,x_L(t)\big)\big]\right| \\
&\le \left|\sum_{i=1}^{N}\sum_{j=1}^{N} 2\lambda_{\max}(P)L_{ij}e_i^T(t)e_j(t)\right| \\
&\le \sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{\max}(P)L_{ij}\big[\|e_i(t)\|^2+\|e_j(t)\|^2\big] \\
&= \lambda_{\max}(P)\sum_{i=1}^{N}\|e_i(t)\|^2\sum_{j=1}^{N}\big[L_{ij}+L_{ji}\big] \\
&\le \lambda_{\max}(P)\overline{L}\sum_{i=1}^{N}\|e_i(t)\|^2
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
&\left|2\sum_{i=1}^{N} e_i^T(t)P\sum_{j=1}^{N}\big[\psi_{ij}\big(t,x_j(t-\tau)\big)-\psi_{ij}\big(t,x_L(t-\tau)\big)\big]\right| \\
&\le \left|\sum_{i=1}^{N}\sum_{j=1}^{N} 2\lambda_{\max}(P)M_{ij}e_i^T(t)e_j(t-\tau)\right| \\
&\le \sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{\max}(P)M_{ij}\big[\|e_i(t)\|^2+\|e_j(t-\tau)\|^2\big].
\end{aligned}$$

Thus,

$$\begin{aligned}
\dot{V}_1 &\le \sum_{i=1}^{N} e_i^T(t)\big[PA+A^TP\big]e_i(t) \\
&+ 2\lambda_{\max}(P)L_0\sum_{i=1}^{N}\|e_i(t)\|^2+\lambda_{\max}(P)\overline{L}\sum_{i=1}^{N}\|e_i(t)\|^2 \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{\max}(P)M_{ij}\big[\|e_i(t)\|^2+\|e_j(t-\tau(t))\|^2\big] \\
&- 2\sum_{i=1}^{N} e_i^T(t)PB\theta_i^T(t)\big[y_i(t)-y_L(t)\big].
\end{aligned}$$

Now keeping in mind that $C^T g = PB$ we calculate a derivative of V_2:

$$\begin{aligned}
\dot{V}_2 &= 2\sum_{i=1}^{N}(\theta_i(t)-\theta_*)^T\Gamma_i^{-1}\dot{\theta}_i(t)\\
&= 2\sum_{i=1}^{N}(\theta_i(t)-\theta_*)^T\big[y_i(t)-y_L(t)\big]\big[y_i(t)-y_L(t)\big]^T g\\
&= 2\sum_{i=1}^{N}(\theta_i(t)-\theta_*)^T\big[y_i(t)-y_L(t)\big]e_i^T(t)C^T g\\
&= 2\sum_{i=1}^{N}e_i^T(t)PB\theta_i^T(t)\big[y_i(t)-y_L(t)\big]\\
&- 2\sum_{i=1}^{N}e_i^T(t)PB\theta_*^T Ce_i(t).
\end{aligned}$$

Finally, a derivative of V_3 is:

$$\begin{aligned}
\dot{V}_3 &= \sum_{i=1}^{N}\big[e_i^T(t)Q_ie_i(t)-(1-\dot{\tau}(t))e_i^T(t-\tau)Q_ie_i(t-\tau)\big]\\
&\le \sum_{i=1}^{N}\left[\|e_i(t)\|^2\frac{\lambda_{\max}(P)}{1-d}\sum_{j=1}^{N}M_{ji}-(1-d)\|e_i(t-\tau(t))\|^2\frac{\lambda_{\max}(P)}{1-d}\sum_{j=1}^{N}M_{ji}\right].
\end{aligned}$$

Summing up all derivatives and using notation $A_* = A - B\theta_*^T C$ we obtain:

$$\begin{aligned}
\dot{V} &\le \sum_{i=1}^{N}e_i^T(t)\big[PA_*+A_*^TP\big]e_i(t)+(2L_0\lambda_{\max}(P)\\
&+\overline{L}\lambda_{\max}(P)+\overline{M}\lambda_{\max}(P))\sum_{i=1}^{N}\|e_i(t)\|^2\\
&\le(-\varepsilon+2L_0\lambda_{\max}(P)+\overline{L}\lambda_{\max}(P)+\overline{M}\lambda_{\max}(P))\sum_{i=1}^{N}\|e_i(t)\|^2.
\end{aligned}$$

Thus,

$$\dot{V} \le -\mu\sum_{i=1}^{N}\|e_i(t)\|^2 \le 0,$$

where $\mu = \varepsilon - 2L_0\lambda_{\max}(P) - \overline{L}\lambda_{\max}(P) - \overline{M}\lambda_{\max}(P) > 0$. Function $V_t(t) = V(t, e_1^t, \dots, e_N^t)$ can be presented as

$$V_t(t) = V_t(0) + \int_0^t \dot{V}_t(s)\,ds \le V_t(0) - \mu\int_0^t\sum_{i=1}^{N}\|e_i(s)\|^2\,ds.$$

As far as $x_i^0, x_L^0 \in \mathcal{C}([-h,0])$, i. e. bounded functions, $V_t(0) < \infty$ and thus $V_t(t)$ is bounded. But if $\exists i = 1,\dots,N : \theta_i(t) \xrightarrow[t\to\infty]{} \infty$ then $V_t(t) \xrightarrow[t\to\infty]{} \infty$ which is not possible. Thus all $\theta_i(t)$ are bounded.

As soon as V_t is bounded and $V_t(0)$ is finite, $\int_0^t \sum_{i=1}^N \|e_i(s)\|^2\, ds < \infty$. By applying Barbalat's lemma [38] we conclude that $e_i(t) \to 0$ while $t \to \infty$ for all $i = 1,\dots,N$. In other words, zero solution of the system (10), (12) is asymptotically stable. Since $\varphi_0(t,x)$ satisfies Lipschitz condition all solutions of (4) and (8) exist for all $t \geq t_0$. Therefore $\lim_{t\to\infty} \|x_i(t) - x_L(t)\| = 0$ for $i = 1,\dots,N$.

Finally, to prove that all $\theta_i(t)$ tend to some constant values let us integrate the second equation of (12):

$$\begin{aligned}\theta_i(t) =& \theta_i(0) + \Gamma_i \int_0^t \left[y_i(s) - \bar{y}(s)\right]\left[y_i(s) - \bar{y}(s)\right]^T g\, ds = \\ & \theta_i(0) + \Gamma_i \int_0^t e_i^T(s) C^T g C e_i(s)\, ds.\end{aligned}$$

The term $\int\limits_0^\infty e_i^T(s) C^T g C e_i(s)\, ds$ is finite as far as $\int\limits_0^\infty e_i^T(s) P e_i(s)\, ds < \infty$ and therefore there exist finite $\lim_{t\to\infty} \theta_i(t) = \theta_i(0) + \Gamma_i \int_0^\infty e_i^T(s) C^T g C e_i(s)\, ds$. ■

Remark 3. *Note that the boundedness of $x_i(t)$ is not proved in the theorem. In fact the trajectories x_i may be unbounded. However, if $x_L(t)$ is bounded then $x_i(t)$ are bounded too.*

4.2. Matched nonlinearity

Now we consider the second class of nonlinearities.

Assumption 3. *There exists a function $h_0(t, Cx)\colon [t_0,\infty) \times \mathbb{R}^l$ such that*

$$\varphi_0(t,x) = B h_0(t, Cx)$$

and for all initial conditions from $\mathcal{C}[-h,0]$ and piecewise continuous u_i equations (4), (8) have solutions for all $t \geq t_0$.

Function φ_0 that satisfies Assumption 3 is called *matched nonlinearity* since it can be canceled by a control signal $u = -h_0(t,y)$. Further we consider the case where h_0 is unknown.

Theorem 2 (Matched nonlinearity). *Consider the network* (4) *subject to* (5) *and the leader system* (8). *Let Assumption 1 hold with* $g \in \mathbb{R}^l$ *and, thus,* (3) *is feasible for some* $P > 0$, $\varepsilon > 0$, *and* θ_*. *Let Assumption 3 be valid and assume that* h_0 *satisfies*

$$(\zeta_1 - \zeta_2)^T g(h_0(t, \zeta_1) - h_0(t, \zeta_2)) \leq 0, \quad \forall \zeta_1, \zeta_2 \in \mathbb{R}^l. \tag{18}$$

If the following inequality holds

$$\overline{L} + \overline{M} < \frac{\varepsilon}{\lambda_{\max}(P)}, \tag{19}$$

where $\overline{L}$ *and* $\overline{M}$ *are given by* (13), *then the adaptive control algorithm (12) ensures synchronization (9). Moreover, all tunable parameters* $\theta_i(t)$ *tend to constant values.*

Proof is similar to the proof of Theorem 1. Consider the functional (15) with the same V_1, V_2, V_3. Calculating the bound for $\dot{V}$ yields:

$$\begin{aligned} \dot{V} \leq{} & -\mu' \sum_{i=1}^{N} \|e_i(t)\|^2 \\ & + 2 \sum_{i=1}^{N} e_i^T(t) P \big[\varphi_0\big(t, x_i(t)\big) - \varphi_0\big(t, x_L(t)\big)\big], \end{aligned}$$

where $\mu' = \varepsilon - \overline{L}\lambda_{\max}(P) - \overline{M}\lambda_{\max}(P) > 0$. As far as $\varphi_0(t, x) = Bh_0(t, Cx)$, $PB = C^T g$ and h_0 satisfies (18), we obtain:

$$\begin{aligned} & 2 \sum_{i=1}^{N} e_i^T(t) P \big[\varphi_0\big(t, x_i(t)\big) - \varphi_0\big(t, x_L(t)\big)\big] \\ & = 2 \sum_{i=1}^{N} e_i^T(t) PB \big[h_0\big(t, Cx_i(t)\big) - h_0\big(t, Cx_L(t)\big)\big] \\ & = 2 \sum_{i=1}^{N} \big[y_i(t) - y_L(t)\big]^T g \big[h_0\big(t, y_i(t)\big) - h_0\big(t, y_L(t)\big)\big] \leq 0. \end{aligned}$$

Therefore, $\dot{V} \leq -\mu' \sum_{i=1}^{N} \|e_i(t)\|^2$. The end of the proof is similar to the end of the proof for Theorem 1.■

Remark 4. *Note that* (14) *turns into* (19) *when* $L_0 = 0$. *That is, condition* (19) *are less restrictive. This relaxation is received by imposing structural conditions on* φ_0. *Hence we can conclude that if* φ_0 *is matched nonlinearity with* h_0 *satisfying* (18) *then it is reasonable to use Theorem 2. If it is not then Theorem 1 should be applied.*

Remark 5. *Results of Theorems 1, 2 are delay-independent, i. e. it is not important how big the value of* $\tau(t)$ *is.*

Remark 6. *Sometimes it is necessary to consider a case of nonequal delays. In this case the delayed term in* (4) *is replaced by* $\sum_{j=1}^{N} \psi_{ij}\big(t, x_j(t - \tau_{ij}(t))\big)$, *where* $\tau_{ij}(t)$ *are such that* $\dot{\tau}_{ij} \le d$. *For this instance the convergence conditions are same as in Theorems 1, 2. To prove that one should take* $V_3 = \frac{\lambda_{\max}(P)}{1-d} \sum_{i=1}^{N} \sum_{j=1}^{N} M_{ji} \int_{t-\tau_{ij}(t)}^{t} e_i^T(s) e_i(s)\, ds$. *Unfortunately, to ensure the existence of the synchronous solution for the system* (4) *with* $u_i \equiv 0$ *we should assume that* $\forall i, k$

$$\sum_{j=1}^{N} \psi_{ij}(t, x(t - \tau_{ij}(t))) = \sum_{j=1}^{N} \psi_{kj}(t, x(t - \tau_{kj}(t))).$$

This assumption is too formal because its fulfillment in general depends mainly on the values of the particular process $x(t)$ *in different moments of time. That seems to have no practical implementation.*

5. Ultimate boundedness of disturbed system

An important issue for control system design is providing its robustness with respect to disturbances unmodelled dynamics. It is well known however that many adaptive systems do not possess such a property that makes their behavior very sensitive to inevitable impreciseness of the plant model. Even boundedness of the closed loop system trajectories cannot be guaranteed in many cases. Among various robustification methods one of the most popular ones is introduction of negative feedback into the adaptation algorithm (σ-modification).

However, this method was not examined before for the plants affected by delay. Below it is demonstrated that σ-modification ensures robust behavior and ultimate boundedness for the controlled network affected by both delays and bounded disturbances.

Consider the system (4) with disturbances:

$$\begin{aligned}\dot{x}_i(t) &= Ax_i(t) + \varphi_0\big(t, x_i(t)\big) + \sum_{j=1}^{N} \varphi_{ij}\big(t, x_j(t)\big)\\ &\quad + \sum_{j=1}^{N} \psi_{ij}\big(t, x_j(t-\tau(t))\big) + Bu_i(t) + w_i(t),\\ y_i(t) &= Cx_i(t), \quad t \geq t_0, \quad i = 1, \ldots, N,\end{aligned} \tag{20}$$

where x_i, u_i, y_i, A, B, C, φ_0, φ_{ij}, ψ_{ij} are the same as in (4) and $w_i \in \mathbb{R}^n$ are unknown bounded disturbances: $\|w_i\| \leq \Delta_i$. In contrast to (4) here we assume that time-varying delay $\tau(t)$ is a *bounded* differentiable function such that $0 \leq \tau(t) \leq h$ and $\dot{\tau}(t) \leq d < 1$ for all $t \geq t_0$.

Since the system contains disturbances instead of (9) we consider the following control goal:

$$\lim_{t\to\infty} \sum_{i=1}^{N} \|x_i(t) - x_L(t)\|^2 < b. \tag{21}$$

It turns out that in this case under the control law (12) tuning parameters θ_i tend to infinity, that is $\|\theta_i\| \to \infty$ while $t \to \infty$. To ensure boundedness of θ_i a regularized controller will be used:

$$\begin{aligned}u_i(t) &= -\theta_i(t)^T \big[y_i(t) - y_L(t)\big] + u_L(t),\\ \dot{\theta}_i(t) &= \Gamma_i \big[y_i(t) - y_L(t)\big]\big[y_i(t) - y_L(t)\big]^T g - \alpha\theta_i(t),\end{aligned} \tag{22}$$

where $\Gamma_i = \Gamma_i^T > 0$ is $l \times l$-matrix and $\alpha > 0$.

To formulate the following result we introduce notation:

$$\overline{M}_h = \max_{i=1,\ldots,N} \sum_{j=1}^{N} \left[e^{\alpha h} M_{ij} + \frac{M_{ji}}{1-d}\right], \tag{23}$$

where L_{ij}, M_{ij} are from (5), h and d are upper bounds for the time-varying delay and its derivative: $0 \leq \tau(t) \leq h$, $\dot{\tau}(t) \leq d < 1$, and α is a controller parameter.

As previously two types of nonlinearities φ_0 will be considered: Lipschitz continuous and matched nonlinearities.

5.1. Lipschitz type nonlinearity

Theorem 3 (Boundedness with Lipschitz nonlinearity). *Consider the network* (20) *subject to* (5) *and the leader system* (8)*. Let Assumption 1 hold with* $g \in \mathbb{R}^l$ *and, thus,* (3) *is feasible for some* $P > 0$*,* $\varepsilon > 0$*, and* θ_**. Let Assumption 2 be valid with some* $L_0 > 0$*. If*

$$\mu = \frac{\varepsilon}{\lambda_{\max}(P)} - 2L_0 - \overline{L} - \overline{M}_h - \alpha \geq 0 \tag{24}$$

where $\overline{L}$ *and* $\overline{M}_h$ *are given by* (13) *and* (23)*, then the adaptive control algorithm* (22) *ensures* (21) *with*

$$b = \frac{\lambda_{\max}(P)}{\alpha\mu\lambda_{\min}(P)} \sum_{i=1}^{N} \Delta_i^2 + \frac{1}{\lambda_{\min}(P)} \sum_{i=1}^{N} \theta_*^T \Gamma_i^{-1} \theta_*. \tag{25}$$

Moreover, all tunable parameters $\theta_i(t)$ *stay bounded on the time interval* $[0, \infty)$ *for all* $i = 1, \dots, N$*.*

Proof. Denote $e_i^t = e_i(t+\theta), \theta \in [-\tau(t), 0]$ and consider the following functional

$$V(t, e_1^t, \dots, e_N^t) = V_1 + V_2 + V_4, \tag{26}$$

where V_1 and V_2 are the same as in (15) and

$$V_4 = \sum_{i=1}^{N} \int_{t-\tau(t)}^{t} e^{-\alpha(t-s)} e_i^T(s) Q_i e_i(s)\, ds,$$

with $Q_i = \frac{\lambda_{\max}(P)}{1-d} \sum_{j=1}^{N} M_{ji} I \geq 0$.

By subtracting (8) from (20) we derive equations for the errors $e_i(t)$. Derivative of V is given by:

$$
\begin{aligned}
\dot{V}_1 &= \sum_{i=1}^{N}[e_i^T(t)P\dot{e}_i(t) + \dot{e}_i^T(t)Pe_i(t)] = \sum_{i=1}^{N} e_i^T(t)\big[PA + A^TP\big]e_i(t)\\
&+ 2\sum_{i=1}^{N} e_i^T(t)P\big[\varphi_0\big(t, x_i(t)\big) - \varphi_0\big(t, x_L(t)\big)\big]\\
&+ 2\sum_{i=1}^{N} e_i^T(t)P\sum_{j=1}^{N}\big[\varphi_{ij}\big(t, x_j(t)\big) - \varphi_{ij}\big(t, x_L(t)\big)\big]\\
&+ 2\sum_{i=1}^{N} e_i^T(t)P\sum_{j=1}^{N}\big[\psi_{ij}\big(t, x_j(t-\tau)\big) - \psi_{ij}\big(t, x_L(t-\tau)\big)\big]\\
&- 2\sum_{i=1}^{N} e_i^T(t)PB\theta_i^T(t)\big[y_i(t) - y_L(t)\big] + 2\sum_{i=1}^{N} e_i^T(t)Pw_i(t).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\left|2\sum_{i=1}^{N} e_i^T(t)P\sum_{j=1}^{N}\big[\psi_{ij}\big(t, x_j(t-\tau)\big) - \psi_{ij}\big(t, x_L(t-\tau)\big)\big]\right|\\
&\leq \left|\sum_{i=1}^{N}\sum_{j=1}^{N} 2\lambda_{\max}(P)M_{ij}e_i^T(t)e_j(t-\tau(t))\right|\\
&\leq \sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{\max}(P)M_{ij}\big[e^{\alpha h}\|e_i(t)\|^2 + e^{-\alpha h}\|e_j(t-\tau)\|^2\big]
\end{aligned}
$$

and

$$
2\sum_{i=1}^{N} e_i^T(t)Pw_i(t) \leq \mu\sum_{i=1}^{N} e_i^T(t)Pe_i(t) + \frac{1}{\mu}\sum_{i=1}^{N} w_i^T(t)Pw_i(t).
$$

Using the last two inequalities and (16), (17) we find that

$$
\begin{aligned}
\dot{V}_1 &\le \sum_{i=1}^{N} e_i^T(t)\big[PA + A^T P\big]e_i(t) \\
&+ 2\lambda_{\max}(P)L_0 \sum_{i=1}^{N} \|e_i(t)\|^2 + \lambda_{\max}(P)\overline{L} \sum_{i=1}^{N} \|e_i(t)\|^2 \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_{\max}(P)M_{ij}\big[e^{\alpha h}\|e_i(t)\|^2 + e^{-\alpha h}\|e_j(t-\tau(t))\|^2\big] \\
&- 2\sum_{i=1}^{N} e_i^T(t)PB\theta_i^T(t)\big[y_i(t) - y_L(t)\big] + \mu \sum_{i=1}^{N} e_i^T(t)Pe_i(t) \\
&+ \frac{1}{\mu}\sum_{i=1}^{N} w_i^T(t)Pw_i(t).
\end{aligned}
$$

Now keeping in mind that $C^T g = PB$ we calculate a derivative of V_2:

$$
\begin{aligned}
\dot{V}_2 &= 2\sum_{i=1}^{N}(\theta_i(t) - \theta_*)^T \Gamma_i^{-1}\dot{\theta}_i(t) \\
&= 2\sum_{i=1}^{N}(\theta_i(t) - \theta_*)^T \big[y_i(t) - y_L(t)\big]\big[y_i(t) - y_L(t)\big]^T g \\
&- 2\alpha\sum_{i=1}^{N}(\theta_i(t) - \theta_*)^T \Gamma_i^{-1}\theta_i(t) \\
&= 2\sum_{i=1}^{N} e_i^T(t)PB\theta_i^T(t)\big[y_i(t) - y_L(t)\big] - 2\sum_{i=1}^{N} e_i^T(t)PB\theta_*^T Ce_i(t) \\
&- \alpha\sum_{i=1}^{N}(\theta_i(t) - \theta_*)^T \Gamma_i^{-1}(\theta_i(t) - \theta_*) + \alpha\sum_{i=1}^{N}\theta_*^T\Gamma_i^{-1}\theta_*.
\end{aligned}
$$

Derivative of V_4 is:

$$
\begin{aligned}
\dot{V}_4 &= \sum_{i=1}^{N}\big[e_i^T(t)Q_i e_i(t) \\
&- (1 - \dot{\tau}(t))e^{-\alpha\tau(t)}e_i^T(t-\tau(t))Q_i e_i(t-\tau(t))\big] - \alpha V_4 \\
&\le \sum_{i=1}^{N}\Bigg[\|e_i(t)\|^2\frac{\lambda_{\max}(P)}{1-d}\sum_{j=1}^{N} M_{ji} \\
&- (1-d)e^{-\alpha h}\|e_i(t-\tau(t))\|^2\frac{\lambda_{\max}(P)}{1-d}\sum_{j=1}^{N} M_{ji}\Bigg] - \alpha V_4.
\end{aligned}
$$

Summing up all derivatives we obtain:

$$\dot{V} + \alpha V - \beta \leq \eta \sum_{i=1}^{N} \|e_i(t)\|^2 + \frac{\lambda_{\max}(P)}{\mu} \sum_{i=1}^{N} \Delta_i^2 + \alpha \sum_{i=1}^{N} \theta_*^T \Gamma_i^{-1} \theta_* - \beta,$$

where $\eta = -\varepsilon + 2L_0\lambda_{\max}(P) + \overline{L}\lambda_{\max}(P) + \overline{M}_h\lambda_{\max}(P) + \mu\lambda_{\max}(P) + \alpha\lambda_{\max}(P)$. From the conditions of the theorem it follows that there exists $\mu > 0$ such that $\eta < 0$. Let $\beta = \frac{\lambda_{\max}(P)}{\mu} \sum_{i=1}^{N} \Delta_i^2 + \alpha \sum_{i=1}^{N} \theta_*^T \Gamma_i^{-1} \theta_*$. Then

$$\dot{V} \leq -\alpha V + \beta.$$

From the comparison principle [38] it follows that:

$$V(t, e_1^t, \ldots, e_N^t) \leq \left(V(t_0, e_1^{t_0}, \ldots, e_N^{t_0}) - \frac{\beta}{\alpha} \right) e^{-\alpha(t-t_0)} + \frac{\beta}{\alpha}. \tag{27}$$

Therefore,

$$\lim_{t \to \infty} \sum_{i=1}^{N} \|e_i(t)\|^2 \leqslant b$$

with

$$b = \frac{\beta}{\lambda_{\min}(P)\alpha} = \frac{\lambda_{\max}(P)}{\alpha\mu\lambda_{\min}(P)} \sum_{i=1}^{N} \Delta_i^2 + \frac{1}{\lambda_{\min}(P)} \sum_{i=1}^{N} \theta_*^T \Gamma_i^{-1} \theta_*.$$

From (27) it follows that V is bounded, therefore all θ_i are bounded. ■

5.2. Matched nonlinearity

Theorem 4 (Boundedness with matched nonlinearity). *Consider the network* (20) *subject to* (5) *and the leader system* (8). *Let Assumption 1 hold with* $g \in \mathbb{R}^l$ *and, thus,* (3) *is feasible for some* $P > 0$, $\varepsilon > 0$, *and* θ_*. *Let Assumption 3 be valid and assume that* h_0 *satisfies*

$$(\zeta_1 - \zeta_2)^T g(h_0(t, \zeta_1) - h_0(t, \zeta_2)) \leq 0, \quad \forall \zeta_1, \zeta_2 \in \mathbb{R}^l. \tag{28}$$

If the following inequality holds

$$\mu = \frac{\varepsilon}{\lambda_{\max}(P)} - \overline{L} - \overline{M}_h - \alpha \geq 0, \tag{29}$$

where $\overline{L}$ *and* $\overline{M}_h$ *are given by* (13) *and* (23), *then the adaptive control algorithm* (22) *ensures* (21) *with*

$$b = \frac{\lambda_{\max}(P)}{\alpha\mu\lambda_{\min}(P)} \sum_{i=1}^{N} \Delta_i^2 + \frac{1}{\lambda_{\min}(P)} \sum_{i=1}^{N} \theta_*^T \Gamma_i^{-1} \theta_*. \tag{30}$$

Moreover, all tunable parameters $\theta_i(t)$ stay bounded on the time interval $[0, \infty)$ for all $i = 1, \ldots, N$.

Proof of Theorem 4 is similar to the proof of Theorems 2 and 3 and, therefore, is omitted here.

6. Numerical example

To demonstrate the efficiency of the proposed algorithm we make use of a celebrated Chua circuit [39]. A network of four connected Chua circuits with disturbances where the first component of each system is measured and controlled can be presented in the form:

$$
\begin{aligned}
\dot{s}_i(t) &= As_i(t) + Bh_0(\xi_i(t)) + Bu(t) + \sum_{j=1}^{4} \varphi_{ij}(t, s_j(t)) \\
&\quad + \sum_{j=1}^{4} \psi_{ij}(t, s_j(t - \tau(t))) + w_i(t), \\
\xi_i(t) &= Cs_i(t), \quad i = 1, \ldots, 4,
\end{aligned}
\tag{31}
$$

where $s_i = (x_i, y_i, z_i)^T$, $h_0(\xi) = -\frac{p}{2}(m_0 - m_1)(|\xi + 1| - |\xi - 1| - 2\xi)$, $p > 0$, $q > 0$, $m_0 < m_1 < 0$,

$$
A = \begin{pmatrix} -(1 + m_0)p & p & 0 \\ 1 & -1 & 1 \\ 0 & -q & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T.
$$

Suppose that the values of m_0 and m_1 are known while p, q are unknown. The only information that we possess about p, q is that they belong to some intervals of possible values, i. e. $p \in [p_1, p_2]$, $q \in [q_1, q_2]$ $(p_1 > 0,\ q_1 > 0)$.

Consider the following interconnections:

$$
\begin{aligned}
&\varphi_{12}(t,s_2) = \sigma(0.5\sin x_2, 0, 0)^T, && \varphi_{13}(t,s_3) = \sigma(0, 0.5y_3\sin t, 0)^T,\\
&\varphi_{21}(t,s_1) = \sigma(0.5\cos x_1 \operatorname{sign}(\sin t), 0, 0)^T, && \varphi_{24}(t,s_4) = \sigma(0, 0, 0.5z_4\operatorname{sign}(\cos t))^T,\\
&\varphi_{32}(t,s_2) = \sigma(0, 0.5y_2\sin t, 0)^T, && \varphi_{34}(t,s_4) = \sigma(0, 0, 0.5\sin z_4)^T,\\
&\varphi_{41}(t,s_1) = \sigma(0.5\cos x_1, 0.5\cos y_1, 0)^T, && \varphi_{43}(t,s_3) = \sigma(0, 0, 0.5z_3\operatorname{sign}(\cos t))^T,\\
&\psi_{12}(t,s_2) = \sigma(0, 0, 0.45\cos z_2)^T, && \psi_{13}(t,s_3) = \sigma(0.45\sin x_3\cos t, 0, 0)^T,\\
&\psi_{21}(t,s_1) = \sigma(0, 0.45\sin y_1\operatorname{sign}(\sin t), 0)^T, && \psi_{24}(t,s_4) = \sigma(0.45x_4, 0, 0)^T,\\
&\psi_{31}(t,s_1) = \sigma(0, 0.45y_1\operatorname{sign}(\cos t), 0)^T, && \psi_{34}(t,s_4) = \sigma(0, 0, 0.45\cos z_4)^T,\\
&\psi_{42}(t,s_2) = \sigma(0.45\sin x_2, 0, 0)^T, && \psi_{43}(t,s_3) = \sigma(0, 0.45y_3\sin t, 0)^T,\\
&\varphi_{ii}(t,s_i) = -\sum_{\substack{j=1\\ j\neq i}}^{4}\varphi_{ij}(t,s_i), && \psi_{ii}(t,s_i) = -\sum_{\substack{j=1\\ j\neq i}}^{4}\psi_{ij}(t,s_i),
\end{aligned}
$$

with $\sigma = 0.01$. Other φ_{ij}, ψ_{ij} are assumed to be zeroes. Note that φ_{ij} and ψ_{ij} depend on the states $s_i(t)$ and $s_i(t-\tau(t))$ correspondingly. Calculating (13) and (23) for $h = 9$ yields $\overline{L} = 0.04$ and $\overline{M}_h \approx 0.04$.

Along with the system (31) consider the leader system of the form (8) with $u_L(t) \equiv 0$, $\Phi(t,x) \equiv 0$, $\Psi(t,x) \equiv 0$.

For the system (31) Assumption 3 is fulfilled with matched nonlinearity $\varphi_0(s_i) = Bh_0(\xi_i)$ where h_0 satisfies (18) for any $g > 0$. Therefore, Theorem 4 can be applied.

Assumption 1 is fulfilled since for all $p > 0$, $q > 0$ and $g > 0$, $\varphi(\lambda) = g^T W(\lambda)\det(\lambda I - A) = gp(\lambda^2 + \lambda + q)$ is Hurwitz and $g^T CB = g > 0$.

To check the condition (29) we try to enlarge $\varepsilon\lambda_{\max}^{-1}(P)$ such that (3) are satisfied. Introducing $P_1 = \frac{1}{\varepsilon}P$, $\eta = \frac{\lambda}{\varepsilon}$ we reformulate the task in terms of matrix inequalities, where θ_* will be treated as a tuning parameter:

$$
\begin{aligned}
&\eta \to \min\\
&(A - B\theta_*^T C)^T P_1 + P_1(A - B\theta_*^T C) < -I, \qquad (32)\\
&P_1 < \eta I, \quad \varepsilon P_1 B = C^T g, \quad P_1 > 0.
\end{aligned}
$$

Obviously, if η_* is a solution of this task then $\varepsilon\lambda_{\max}^{-1}(P) = 1/\eta_*$. Since (32) is

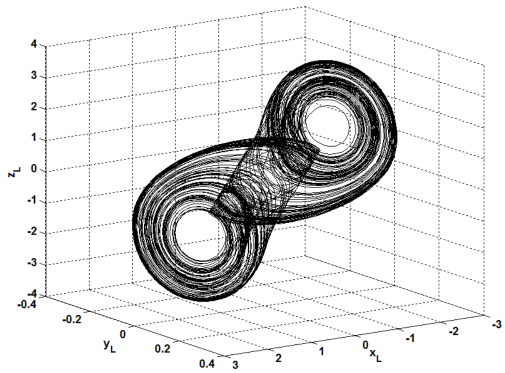

Figure 1: Phase portrait of the leader system.

affine in A, one have to solve linear matrix inequalities (32) simultaneously for the four vertices given by $A^1 = A|_{\substack{p=p_1\\q=q_1}}$, $A^2 = A|_{\substack{p=p_1\\q=q_2}}$, $A^3 = A|_{\substack{p=p_2\\q=q_1}}$, $A^4 = A|_{\substack{p=p_2\\q=q_2}}$, with the same tunable parameter θ_* and the same decision variables $P_1 > 0$ and η.

For simulations we take $m_0 = -8/7$, $m_1 = -5/7$ and suppose that $p \in [5; 15]$, $q \in [14; 15]$. In this case the numerical solution of (32) for $\theta_* = 150$ yields:

$$P_1 = \begin{pmatrix} 0.8191 & 0 & 0 \\ 0 & 9.9520 & -0.5448 \\ 0 & -0.5448 & 0.7117 \end{pmatrix}, \quad \eta_* = 9.9863.$$

Thus, $g = 0.8191$ and $\varepsilon\lambda_{\max}^{-1}(P) = 1/\eta_* = 0.1$. We take $\alpha = 0.01$. In this case $\mu = 0.1$ and, therefore, (29) is true. Thereby an adaptive control algorithm

$$u_i(t) = -\theta_i(t)^T\big[\xi_i(t) - \xi_L(t)\big],$$
$$\dot{\theta}_i(t) = 0.8191 \cdot \Gamma_i\big[\xi_i(t) - \xi_L(t)\big]^2 - 0.01 \cdot \theta_i(t),$$

with any $\Gamma_i > 0$ ensures the achievement of the goal

$$\lim_{t\to\infty} \sum_{i=1}^{4} \|\xi_i(t) - \xi_L(t)\|^2 < b,$$

where the value of b depends on Γ_i and the noise bounds Δ_i.

For simulations we take $p = 9$, $q = 14.286$. For simplicity $\Gamma_i = 1$ for all $i = 1, \ldots, 4$. Initial functions $s_i^0 = \begin{pmatrix} x_i^0 & y_i^0 & z_i^0 \end{pmatrix}^T$ are random linear functions

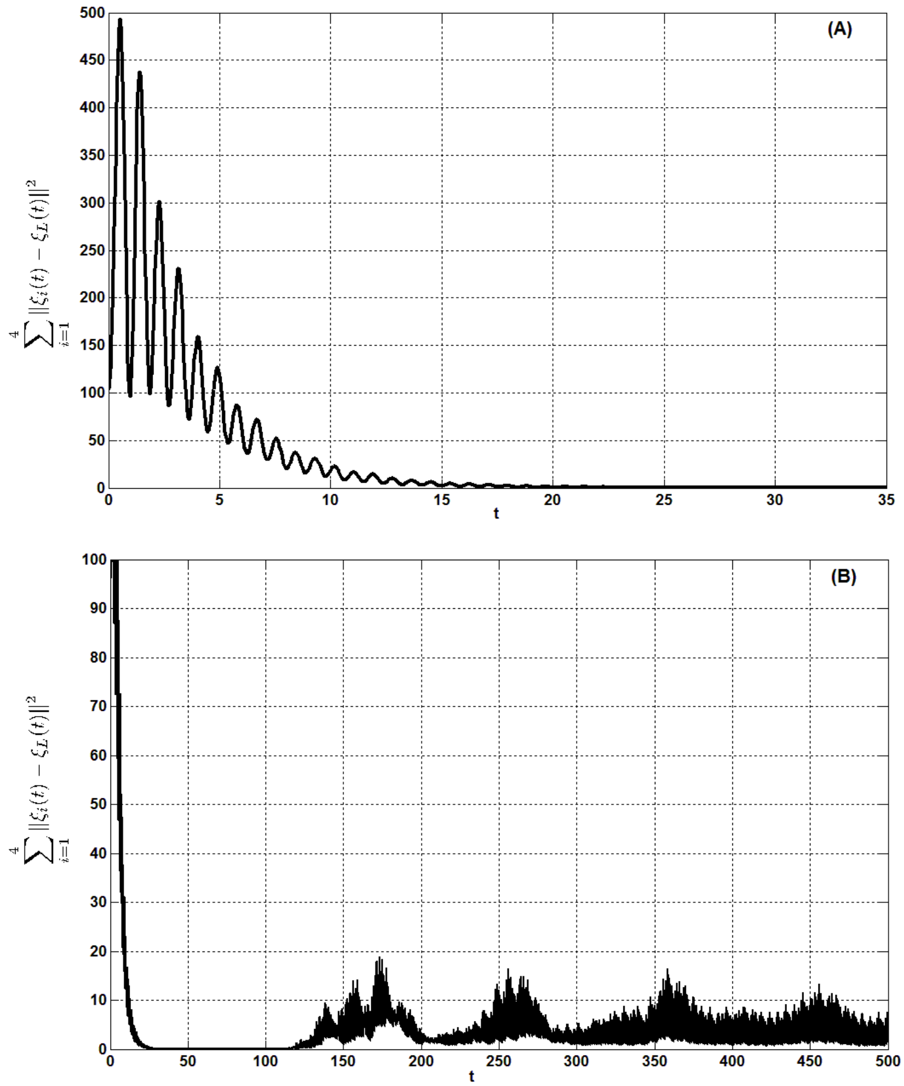

Figure 2: The value of $\sum_{i=1}^{4} \|\xi_i(t) - \xi_L(t)\|^2$: A — during 35 seconds of simulation; B — during 500 seconds of simulation

such that $\|x_i^0\|_{\mathcal{C}} < 5$, $\|y_i^0\|_{\mathcal{C}} < 5$, $\|z_i^0\|_{\mathcal{C}} < 5$. Initial function for the leader system is chosen as $s_L^0(t) = \begin{pmatrix} 0.1 & 0.1 & 0.1 \end{pmatrix}^T$ for $t \in [-9, 0]$. Initial values for all θ_i are zeroes.

In Figure 1 a phase portrait of the leader system is presented. It is a well

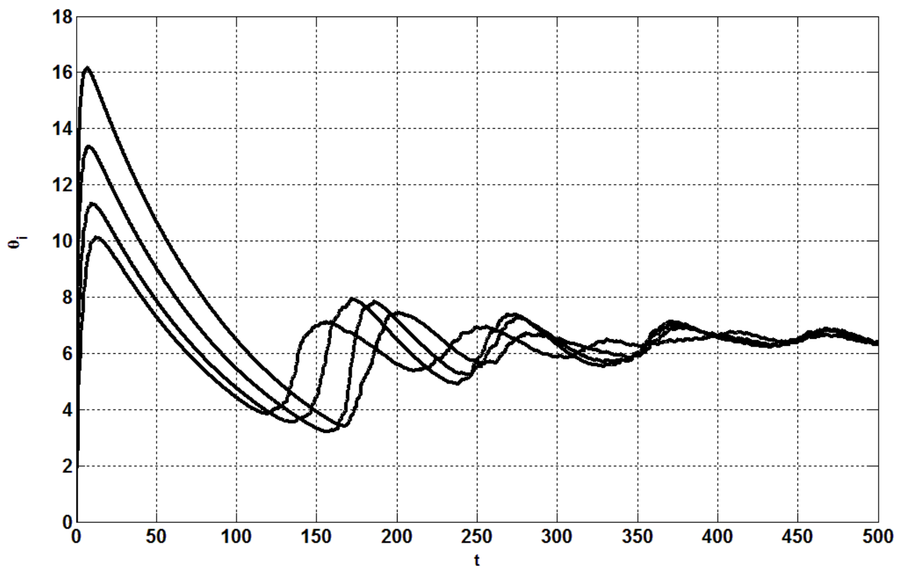

Figure 3: Evolution of θ_i $(i = 1, \ldots, 4)$

known fact that for chosen values of system parameters Chua circuit exhibits a chaotic behavior. In Figure 2 one can see the value of $\sum_{i=1}^{4} \|\xi_i(t) - \xi_L(t)\|^2$ stays bounded during the time of simulation. In Figure 3 the evolution of θ_i is depicted.

Note that for big enough θ_i (e.g. for $\theta_i = \theta_*$ which solves (3) subject to (19)) static output feedback (11) ensures synchronization of the system (4), (8). In this case θ_i may have big magnitudes leading to high-gain control which can cause undesirable behavior of the closed loop system. On the other hand, the adaptive controller (12) perform adaptive tuning of the unknown parameters θ_i with a smaller gain. In the presented example the task (32) is not feasible for $\theta_* < 10$. For $\theta_* < 150$ smaller values of $\lambda_{\max}^{-1}(P)\varepsilon$ are obtained. At the same time, as it can be seen in Figure 3, all θ_i after the transient period are smaller than 8. That is, the adaptive controller (12) allows one to ensure ultimate boundedness of a network (4), (8) with a small enough control gain.

In Figure 4 one can see the results of numerical simulations for 100 interconnected Chua circuits. All system parameters are same as previously and

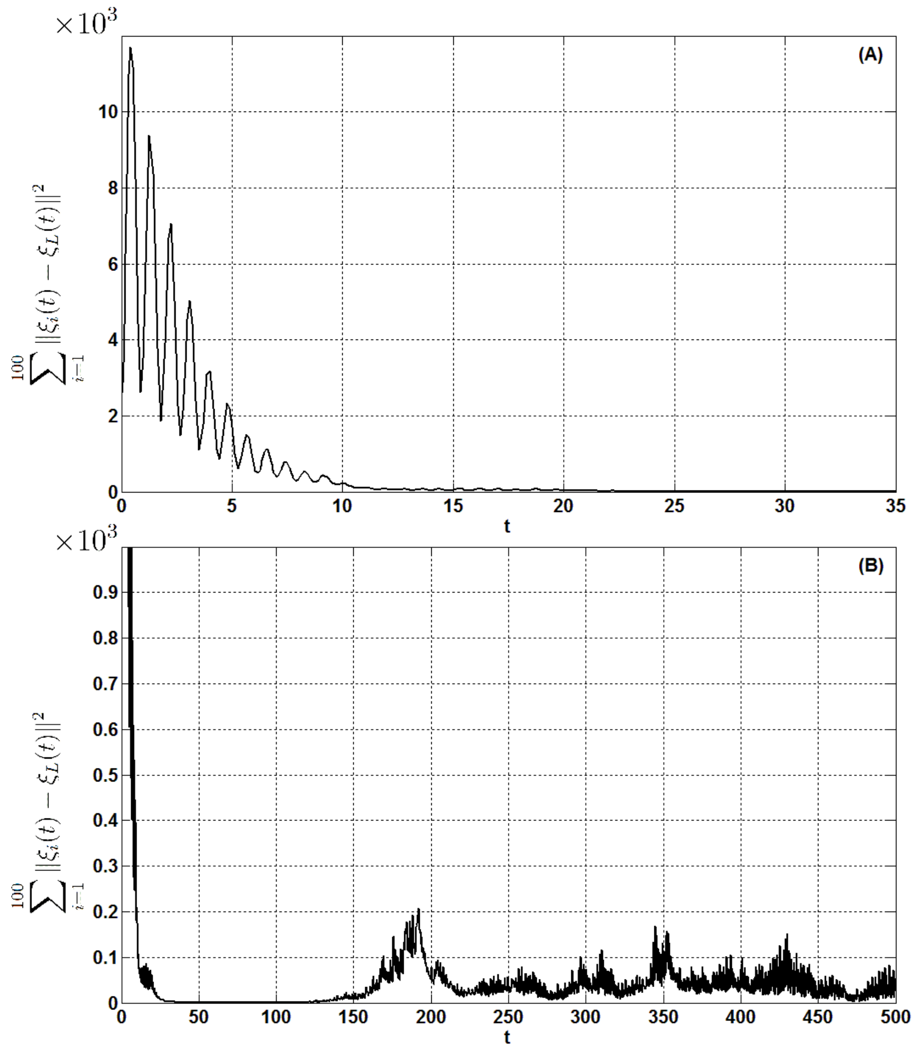

Figure 4: The value of $\sum_{i=1}^{100} \|\xi_i(t) - \xi_L(t)\|^2$: A — during 35 seconds of simulation; B — during 500 seconds of simulation

the topology of the network was chosen randomly such that $\overline{L} = 0.04$ and $\overline{M}_h = 0.04$. In this case Theorem 4 guarantees ultimate boundedness of the value $\sum_{i=1}^{100} \|\xi_i(t) - \xi_L(t)\|^2$.

7. Conclusion

We examined the problem of decentralized adaptive control for dynamical networks with instant and delayed nonlinear interconnections. In contrast to overwhelming majority of the previous results we proposed an adaptive control algorithm for both incomplete state measurements and incomplete control (the number of control variables is less than the number of the state variables). Controllability of the local dynamics is not required. Instead passifiability (hyper-minimum-phase property) of the linear part of local dynamics is assumed. Compared with a number of the previous works on decentralized control of interconnected systems [40, 41, 42, 43, 44] mainly dealing with Model Reference Adaptive Control, our passification based design provides more simple controllers. On the other hand, like in the previous designs, the proposed adaptive controllers (12), (22) are decentralized, and therefore, interconnections are required to be weak enough.

For the disturbance free case the convergence of each agent trajectory to the leader trajectory (synchronization) is proved. For the networks with disturbances ultimate boundedness of the trajectories is proved. Two types of agent nonlinearity φ_0 were considered. First, for Lipschitz continuous functions it is required that Lipschitz constant is small enough. Then for a special class of matched nonlinearity the monotonicity assumption (18) is imposed. All results are formulated for the case of slowly-varying time delay.

The proposed method is illustrated by numerical examples of 4 and 100 controlled Chua circuits. According to simulation results all adaptation parameters stay bounded and after a transient period are less than the parameters of the stabilizing static output feedback under the same uncertainty. Thus, the proposed adaptive output feedback controller allows to synchronize a network with smaller values of control gains that is more appropriate in practice.

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