Adaptive synchronization in delay-coupled networks of Stuart-Landau oscillators

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We consider networks of delay-coupled Stuart-Landau oscillators. In these systems, the coupling phase has been found to be a crucial control parameter. By proper choice of this parameter one can switch between different synchronous oscillatory states of the network. Applying the speed-gradient method, we derive an adaptive algorithm for an automatic adjustment of the coupling phase such that a desired state can be selected from an otherwise multistable regime. We propose goal functions based on both the difference of the oscillators and a generalized order parameter and demonstrate that the speed-gradient method allows one to find appropriate coupling phases with which different states of synchronization, e.g., in-phase oscillation, splay, or various cluster states, can be selected.

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I. INTRODUCTION

The ability to control nonlinear dynamical systems has brought up a wide interdisciplinary area of research that has evolved rapidly in the past decades [1]. In particular, noninvasive control schemes based on time-delayed feedback [2–4] have been studied and applied to various systems ranging from biological and chemical applications to physics and engineering in both theoretical and experimental works [5–12]. Here we propose to use adaptive control schemes based on optimizations of cost or goal functions [13–15] to find appropriate control parameters. Besides isolated systems, control of dynamics in spatiotemporal systems and on networks has recently gained much interest [16–20]. The existence and control of cluster states was studied by Choe *et al.* [21,22] in networks of Stuart-Landau oscillators. This Stuart-Landau system arises naturally as a generic expansion near a Hopf bifurcation and is therefore often used as a paradigm for oscillators. The complex coupling constant that arises from the complex state variables in networks of Stuart-Landau oscillators consists of an amplitude and a phase. Similar coupling phases arise naturally in systems with all-optical coupling [6,23]. Such phase-dependent couplings have also been shown to be important in overcoming the odd-number limitation of time-delay feedback control [24,25] and in anticipating chaos synchronization [26]. Furthermore, it was shown in Refs. [21,22] that the value of the coupling phase is a crucial control parameter in these systems; by adjusting this phase one can deliberately switch between different synchronous oscillatory states of the network. In order to find an appropriate value of the coupling phase one could solve a nonlinear equation that involves the system parameters. However, in practice the exact values of the system parameters are unknown, and analytical conditions can be derived only for special values of the complex phase. An efficient way to avoid these limitations and find optimal values of the coupling phase is the use of adaptive control.

In this paper, we present an adaptive synchronization algorithm for delay-coupled networks of Stuart-Landau oscillators. To find an adequate coupling phase we apply the speed-gradient method [15], which was used previously in various nonlinear control problems, yet not for the control of dynamics in delay-coupled networks. By taking an appropriate goal function we derive an equation for the automatic adjustment of the coupling phase such that the goal function is minimized. At the same time the coupling phase converges to the theoretically predicted value. Our goal function is based on the Kuramoto order parameter and is able to distinguish the different states of synchrony in the Stuart-Landau networks irrespectively of the numbering of the nodes.

This paper is organized as follows. After this introduction, we describe the model system in Sec. II. Section III introduces the speed-gradient method and its application using the coupling phase in networks of Stuart-Landau oscillators. We present the main results for the control of in-phase synchronization in Sec. IV, and for cluster and splay states in Sec. V. Finally, Sec. VI contains some conclusions.

II. MODEL EQUATION

Consider a network of N delay-coupled oscillators,

$$\dot{z}_j(t) = f[z_j(t)] + Ke^{i\beta}\sum_{n=1}^{N} a_{jn}[z_n(t-\tau) - z_j(t)], \quad (1)$$

with $z_j = r_j e^{i\varphi_j} \in \mathbb{C}$, $j = 1,\ldots,N$. The coupling matrix $A = \{a_{ij}\}_{i,j=1}^{N}$ determines the topology of the network. The local dynamics of each element is given by the normal form of a supercritical Hopf bifurcation, also known as Stuart-Landau oscillator,

$$f(z_j) = [\lambda + i\omega - (1+i\gamma)|z_j|^2]z_j, \quad (2)$$

with real constants $\lambda,\omega \neq 0$, and γ. In Eq. (1), τ is the delay time. K and β denote the amplitude and phase of the complex coupling constant, respectively. Such kinds of networks are used in different areas of nonlinear dynamics, e.g., to describe neural activities [27].

Synchronous in-phase, cluster, and splay states are possible solutions of Eqs. (1) and (2). They exhibit a common amplitude $r_j \equiv r_{0,m}$ and phases given by $\varphi_j = \Omega_m t + j\Delta\varphi_m$ with a

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phase shift $\Delta\varphi_m = 2\pi m/N$ and collective frequency Ω_m. The integer m determines the specific state: in-phase oscillations correspond to $m = 0$, while splay and cluster states correspond to $m = 1,\ldots,N-1$. The cluster number d, which determines how many clusters of oscillators exist, is given by the least common multiple of m and N divided by m, and $d = N$ (e.g., $m = 1$), corresponds to a splay state.

The stability of synchronized oscillations in networks can be determined numerically, for instance, by the *master stability function* [28]. This formalism allows a separation of the local dynamics of the individual nodes from the network topology. In the case of the Stuart-Landau oscillators it was possible to obtain the Floquet exponents of different cluster states analytically with this technique [21]. By these means it has been demonstrated that the unidirectional ring configuration of Stuart-Landau oscillators exhibits in-phase synchrony, splay states, and clustering depending on the choice of the control parameter β. For $\beta = 0$, there exists multistability of the possible synchronous states in a large parameter range. However, when tuning the coupling phase to an appropriate value $\beta = \Omega_m\tau - 2\pi m/N$ according to a particular state m, this synchronous state is monostable for any values of the coupling strength K and the time delay τ. The main goal of this paper is to find adequate values of β by automatic adaptive adjustment. For this purpose, we make use of the speed gradient method [15], which is outlined in the next section.

III. SPEED-GRADIENT METHOD

In this section, we briefly review an adaptive control scheme called the speed-gradient (SG) method. Consider a general nonlinear dynamical system

$$\dot{x} = F(x,u,t) \tag{3}$$

with state vector $x \in \mathbb{C}^n$, input (control) variables $u \in \mathbb{C}^m$, and nonlinear function F. Define a control goal

$$\lim_{t\to\infty} Q(x(t),t) = 0, \tag{4}$$

where $Q(x,t) \geqslant 0$ is a smooth scalar goal function.

In order to design a control algorithm, the scalar function $\dot{Q} = \omega(x,u,t)$ is calculated, that is, the speed (rate) at which $Q(x(t),t)$ is changing along trajectories of Eq. (3):

$$\omega(x,u,t) = \frac{\partial Q(x,t)}{\partial t} + [\nabla_x Q(x,t)]^{\mathrm{T}} F(x,u,t). \tag{5}$$

Then we evaluate the gradient of $\omega(x,u,t)$ with respect to input variables:

$$\nabla_u \omega(x,u,t) = \nabla_u [\nabla_x Q(x,t)]^{\mathrm{T}} F(x,u,t).$$

Finally, we set up a differential equation for the input variables u,

$$\frac{du}{dt} = -\Gamma \nabla_u \omega(x,u,t), \tag{6}$$

where $\Gamma = \Gamma^{\mathrm{T}} > 0$ is a positive definite gain matrix. Algorithm (6) is called speed-gradient method, since it suggests to change u proportionally to the gradient of the speed of changing Q. There exist different versions of analytic conditions guaranteeing that the control goal (4) can be achieved in the system ((3), (6)); see [13,29]. The main condition is as follows: the existence of a constant value of the parameter u^*, ensuring attainability of the goal in the system $dx/dt = F(x,u^*,t)$. Details can be found in the control-related literature [13,29].

The idea of this algorithm is the following: The term $-\nabla_u \omega(x,u,t)$ points to the direction in which the value of $\dot{Q}$ decreases with the highest speed. Therefore, if one forces the control signal to "follow" this direction, the value of $\dot{Q}$ will decrease and finally be negative. When $\dot{Q} < 0$, then Q will decrease and, eventually, tend to zero.

We shall now apply the speed-gradient method to networks of Stuart-Landau oscillators. Since the coupling phase β is the crucial parameter that determines the stability of the possible in-phase, cluster, and splay states, we use this control parameter as the input variable u. Setting $u = \beta$ and $x = (z_1,\ldots,z_N)$, Eq. (1) takes the form of Eq. (3) with state vector $x \in \mathbb{C}^N$ and input variable $\beta \in \mathbb{R}$, and nonlinear function $F(x,\beta,t) = [f(z_1),\ldots,f(z_N)] + Ke^{i\beta}[Ax(t-\tau) - x(t)]$.

The SG control equation (6) for the input variable β then becomes

$$\frac{d\beta}{dt} = -\Gamma \frac{\partial}{\partial\beta}\omega(x,\beta,t) = -\Gamma \left(\frac{\partial F}{\partial\beta}\right)^{\mathrm{T}} \nabla_x Q(x,t), \tag{7}$$

where $\Gamma > 0$ is now a scalar. It follows from Eq. (7) that the control algorithm does not use the complete dynamical equations F, but only the partial derivative of F with respect to β. Because β is part of the coupling only, this involves solely the coupling term in Eq. (1) and not the local dynamics. In particular, in an experimental setup, there is no need to know the parameters of the local dynamics, i.e., λ, ω, and γ. Thus, our scheme is easy to implement also in experiments.

IV. IN-PHASE SYNCHRONIZATION

To apply the SG method for the selection of in-phase synchronization we need to find an appropriate goal function Q. It should satisfy the following conditions: The goal function must be zero for an in-phase synchronous state and larger than zero for other states. Hence, a simple goal function can be introduced by taking the distance of all oscillator phases to a reference oscillator's phase φ_1, assuming $|\varphi_k - \varphi_1| < 2\pi$:

$$Q_1(x(t),t) = \frac{1}{2}\sum_{k=2}^{N}(\varphi_k - \varphi_1)^2. \tag{8}$$

Taking the gradient of the derivative along the trajectories of the system (1) with local dynamics (2) one can derive an adaptive law of the following form by straightforward calculation. Using $\omega(x,\beta,t) = \dot{Q}_1$, Eq. (7) becomes

$$\dot{\beta} = -\Gamma K \sum_{k=2}^{N}(\varphi_k - \varphi_1)\left[\sum_{n=1}^{N} a_{kn}\left(\frac{r_{n,\tau}}{r_k}\cos(\beta + \varphi_{n,\tau} - \varphi_k) - \cos\beta\right) - \sum_{n=1}^{N} a_{1n}\left(\frac{r_{n,\tau}}{r_1}\cos(\beta + \varphi_{n,\tau} - \varphi_1) - \cos\beta\right)\right], \tag{9}$$

where we used the abbreviations $r_{n,\tau} = r_n(t-\tau)$ and $\varphi_{n,\tau} = \varphi_n(t-\tau)$ for notational convenience. Note that Eq. (9) has a singularity for $r_k \to 0$, but for small numbers N of oscillators we have never observed in the simulations that the amplitudes r_k become zero. Only for very large rings of oscillators this may happen, and in this case the speed-gradient algorithm may work only when appropriate measures to prevent divergence of the control signal are applied.

Figure 1 presents the results of a numerical simulation for an Erdős-Rényi random network with $N = 6$ nodes and row sum normalized to unity. Throughout this paper we use $\Gamma = 1$. According to the numerical simulations, decreasing Γ will yield a decrease of the speed of convergence. On the other hand, if Γ is too big, undesirable oscillations appear. The model parameters are chosen as in Ref. [21]. In Fig. 1(a) it can be seen that the absolute values $|z_j|$ of all nodes converge after about 60 time units. Figure 1(b) shows that the phase differences of the different oscillators approach zero, which corresponds to the in-phase synchronous state. Figure 1(c) depicts the evolution of β. The blue dashed line represents the value of the coupling phase $\beta = \Omega_0\tau = 0.48\pi$, for which stability was shown analytically in Ref. [21]. It can be seen that the adaptively adjusted phase comes close to this value. In other words, even without knowing the exact values of the system parameters, the SG algorithm yields an adequate value of β that stabilizes the target state of in-phase synchronization. Figure 1(d) shows that the goal function (8) indeed approaches zero.

Note that the above choice of the goal function Q is not the only possibility to generate a stable in-phase solution. Let us consider a function based on the order parameter

$$R_1 = \frac{1}{N}\left|\sum_{j=1}^{N} e^{i\varphi_j}\right|. \tag{10}$$

This global parameter can conveniently be measured in real-world systems, e.g., lasers. It is obvious that $R_1 = 1$ if and only if the state is in-phase synchronized. For other cases we have $R_1 < 1$. Using this observation we can introduce the following goal function:

$$Q_2 = 1 - \frac{1}{N^2}\sum_{j=1}^{N} e^{i\varphi_j}\sum_{k=1}^{N} e^{-i\varphi_k}. \tag{11}$$

From $\dot{\beta} = -\Gamma\frac{\partial}{\partial\beta}\dot{Q}_2$ we derive an alternative adaptive law:

$$\dot{\beta} = \Gamma\frac{2K}{N^2}\sum_{k=1}^{N}\sum_{j=1}^{N}\sin(\varphi_k - \varphi_j)\sum_{n=1}^{N} a_{jn} \times\left(\frac{r_{n,\tau}}{r_j}\cos(\beta + \varphi_{n,\tau} - \varphi_j) - \cos\beta\right). \tag{12}$$

Figure 2 shows the results of a numerical simulation. As before, the amplitude and phase approach appropriate values that lead to in-phase synchronization. This time, however, the obtained value of β does not converge to the one for which the analytical approach [10] has established stability of the in-phase oscillation (blue dashed line), but to another limit value. This can be explained as follows: There exists a whole interval of acceptable values of β around the value of the coupling phase for which an analytical treatment is possible, such that for any value from this interval an in-phase state is

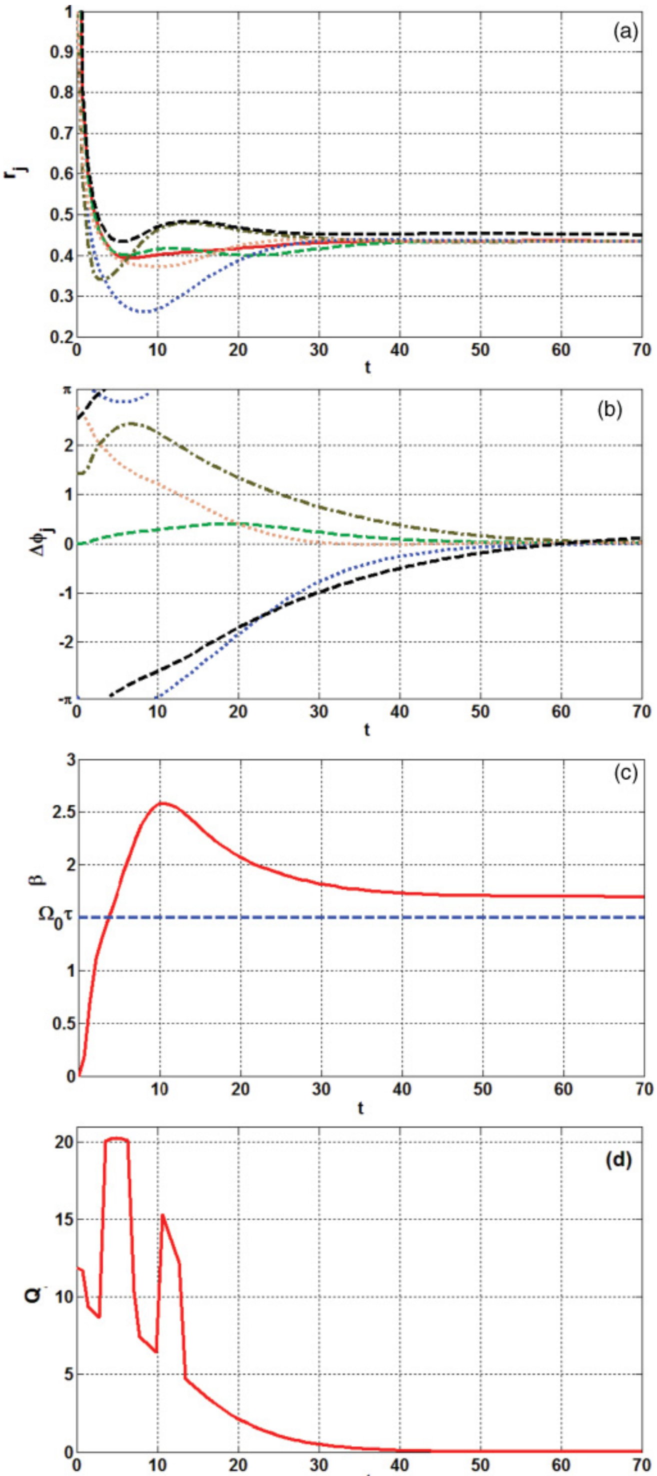

FIG. 1. (Color online) Adaptive control of in-phase oscillations with goal function Eq. (8). (a) Absolute values $r_j = |z_j|$ for $j = 1,\ldots,6$; (b) phase differences $\Delta\phi_j = \varphi_j - \varphi_{j+1}$ for $j = 1,\ldots,5$; (c) temporal evolution of β, blue dashed line: reference value for $\Omega_0 = 0.92$; (d) goal function. Parameters: $\lambda = 0.1, \omega = 1, \gamma = 0$, $K = 0.08$, $\tau = 0.52\pi$, $N = 6$. Initial conditions for r_j and φ_j are chosen randomly from [0,4] and [0,2π], respectively. The initial condition for β is zero.

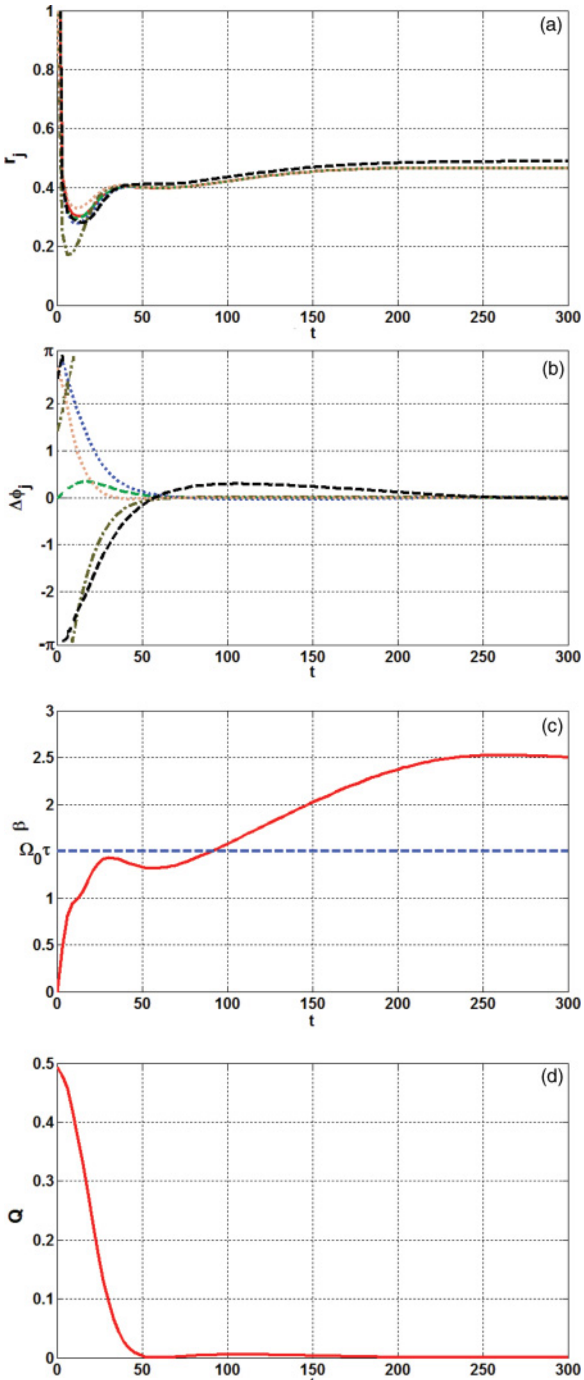

FIG. 2. (Color online) Adaptive control of in-phase oscillations with goal function Eq. (11). (a) Absolute values $r_j = |z_j|$; (b) phase differences $\Delta\phi_j = \varphi_j - \varphi_{j+1}$; (c) temporal evolution of β, blue dashed line: reference value for $\Omega_0 = 0.92$; (d) goal function. Other parameters as in Fig. 1.

stable. Our SG algorithm finds one of them, depending upon initial conditions.

V. SPLAY AND CLUSTER STATES STABILIZATION

In this section we will consider unidirectionally coupled rings with $N = 6$ nodes. That is, the coupling matrix has the following form:

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let $1 \leqslant m \leqslant N-1$. Then $d = \mathrm{LCM}(m,N)/m$, where LCM denotes the least common multiple, is the number of different clusters of a synchronized solution. A splay state corresponds to $d = N$ while cluster states yield $d < N$. Using similar arguments as those leading to Eq. (8) we could choose a goal function of the following form:

$$Q_3 = \frac{1}{2}\sum_{j=1}^{N}\left(\varphi_j - \varphi_{j+1} - \frac{2\pi}{d}\right)^2 \tag{13}$$

with $j = j \bmod N$.

The goal function Eq. (13) has a crucial disadvantage: We need to define an ordering of the system nodes. Since this is inconvenient for practical applications, we will extend the alternative goal function Eq. (11) such that we can stabilize splay and cluster states. First of all, note that the following condition holds for splay and cluster states:

$$\sum_{j=1}^{N} e^{i\varphi_j} = 0. \tag{14}$$

Indeed, if we have only three nodes and take $Q = \sum_{j=1}^{3} e^{i\varphi_j} \sum_{k=1}^{3} e^{-i\varphi_k}$ as a goal function, we will ensure stability of a splay state, as we have verified by numerical simulations. Note that this goal function does not need a fixed ordering of the nodes. Renumbering all nodes in a random way will yield the same goal function. One can define a generalized order parameter

$$R_d = \frac{1}{N}\left|\sum_{k=1}^{N} e^{di\varphi_k}\right| \tag{15}$$

with $d \in \mathbb{N}$. However, if we derive a goal function from this order parameter in an analogous way as in Eq. (11), this function will not have a unique minimum at the d-cluster state because $R_d = 1$ holds also for the in-phase state and for other p-cluster states where p are divisors of d.

For example, suppose that the system has six nodes. Then states for which conditions (14) and (15) with $R_d = 1$ for $d = 6$ hold are schematically depicted in Figs. 3(a)–3(c). In

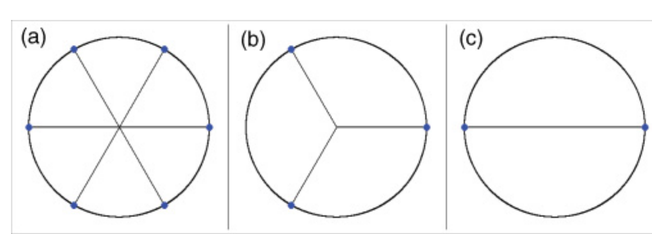

FIG. 3. Schematic diagrams of splay ($d = 6$), three-cluster ($d = 3$), and two-cluster ($d = 2$) states in panels (a)–(c), respectively ($N = 6$). Each cluster contains the same number of nodes.

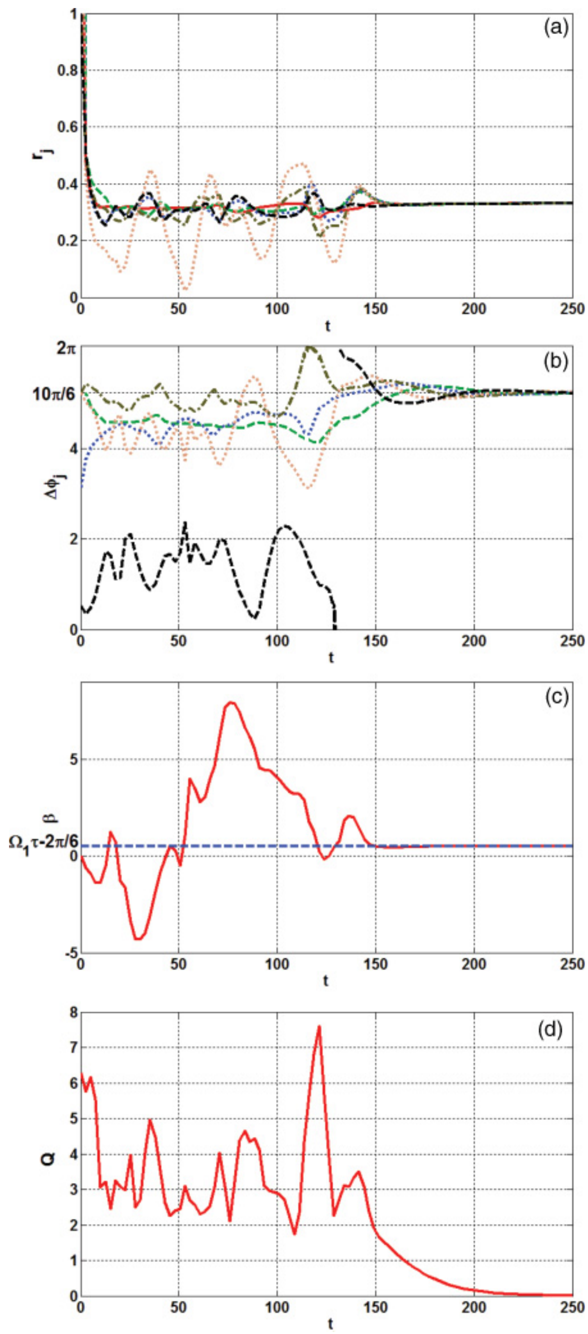

FIG. 4. (Color online) Adaptive control of splay state with goal function Eq. (17). (a) Absolute values $r_j = |z_j|$; (b) phase differences $\Delta\phi_j = \varphi_j - \varphi_{j+1}$; (c) temporal evolution of β, blue dashed line: reference value for $\Omega_1 = 0.96$; (d) goal function. Other parameters as in Fig. 1.

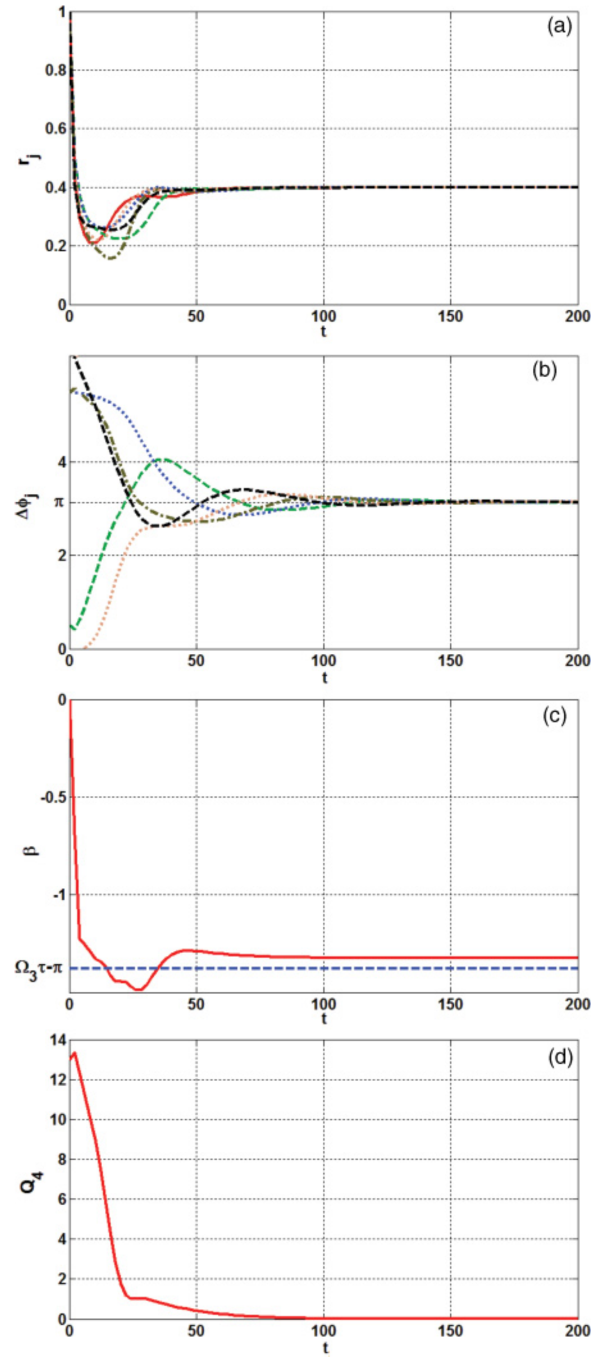

FIG. 5. (Color online) Adaptive control of two-cluster state ($m = 3$) with goal function Eq. (17). (a) Absolute values $r_j = |z_j|$; (b) phase differences $\Delta\phi_j = \varphi_j - \varphi_{j+1}$; (c) temporal evolution of β, blue dashed line: reference value for $\Omega_3 = 1.08$; (d) goal function. Other parameters as in Fig. 1.

order to distinguish between these three cases, let us consider the functions

$$f_p(\varphi) = \frac{1}{N^2}\sum_{j=1}^{N} e^{pi\varphi_j}\sum_{k=1}^{N} e^{-pi\varphi_k}. \tag{16}$$

A splay state [Fig. 3(a)] yields $f_1 = f_2 = f_3 = 0$, while in the three-cluster state displayed in Fig. 3(b) we have $f_1 = f_2 = 0$, $f_3 = 1$, and in the two-cluster state shown in Fig. 3(c), $f_1 = f_3 = 0,\ f_2 = 1$. Hence, we obtain $\sum_p f_p = 0$ if and only if there is a state with d clusters, where the sum is taken over all divisors of d.

Combining all previous results we adopt the following goal function:

$$Q_4 = 1 - f_d(\varphi) + \frac{N^2}{2}\sum_{p|d,1\leqslant p<d} f_p(\varphi), \tag{17}$$

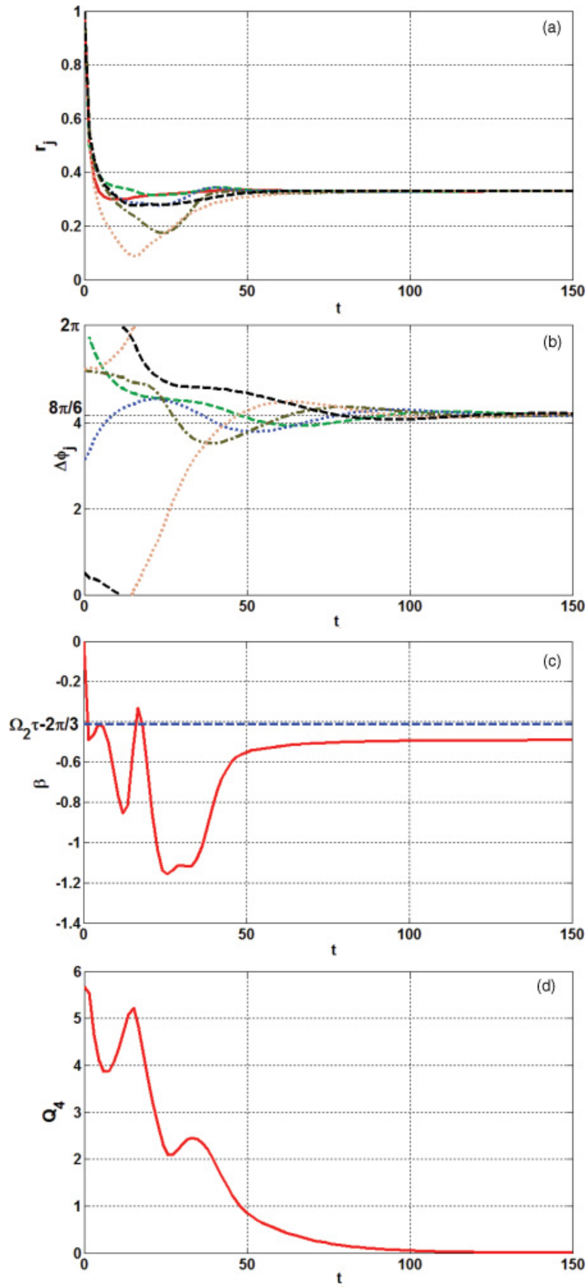

FIG. 6. (Color online) Adaptive control of three-cluster state ($m = 2{,}4$) with goal function Eq. (17). (a) Absolute values $r_j = |z_j|$; (b) phase differences $\Delta\phi_j = \varphi_j - \varphi_{j+1}$, blue dashed line: reference value for $\Omega_2 = 1.03$; (c) temporal evolution of β; (d) goal function. Other parameters as in Fig. 1.

where $p|d$ means that p is a factor of d. This goal function contains f_d as the primary contribution for the d-cluster state, but also a sum of penalty terms that counteract reaching other cluster states in which f_d is also unity. Whenever one of those unwanted cluster states is approached, the penalty term will lead to a gradient away from it. The prefactor $N^2/2$ is chosen for convenience to secure faster convergence of the algorithm. From $\dot{\beta} = -\Gamma \frac{\partial}{\partial \beta} \dot{Q}_4$ one can derive the adaptation law

$$\dot{\beta} = -\Gamma K \sum_{j=1}^{N} \sum_{k=1}^{N} \left\{ \sum_{p|d, 1 \leqslant p < d} p \sin[p(\varphi_k - \varphi_j)] - \frac{2d}{N^2} \sin[d(\varphi_k - \varphi_j)] \right\} \sum_{n=1}^{N} a_{jn} \times \left[\frac{r_{n,\tau}}{r_j} \cos(\beta + \varphi_{n,\tau} - \varphi_j) - \cos(\beta) \right]. \tag{18}$$

In Fig. 4 we show the results of a numerical simulation for splay state stabilization ($d = N = 6$, $m = 1$). The phase differences are $\Delta\phi_j = \varphi_j - \varphi_{j+1} = 2\pi - 2\pi/N$, which corresponds to the splay state. In Fig. 4(c) one can see that the adaptively obtained value of β converges to that for which stability was shown analytically in Ref. [21] (dashed blue line).

Figures 5 and 6 depict the results of numerical simulations for two clusters ($d = 2$, $m = 3$) and three clusters ($d = 3$, $m = 2{,}4$), respectively. Again we note that the obtained value of β comes close to the one for which stability was shown analytically in Ref. [21].

The above results indicate that the speed-gradient method is able to drive the network dynamics into the desired cluster or splay state by adaptively adjusting the coupling phase, where the goal function is chosen according to the corresponding target state. We have, however, used only exemplary values of the coupling parameters K and τ so far.

For the example of a splay state (four-cluster) in a network of four Stuart-Landau oscillators coupled in a unidirectional ring we have conducted a more exhaustive analysis of the (K,τ) plane. Figure 7 shows results in dependence on the coupling strength K and the coupling delay τ. According to Ref. [21] there exists an optimal value of the coupling phase that enables stability of this state for arbitrary values of K and τ. We ran simulations with 20 different initial conditions chosen randomly from the complex interval $[-1,1] \times [-i,i]$ for each oscillator z_j. Figure 7 shows the fraction f_c of those realizations that asymptotically approach a splay state after applying the speed-gradient method. We observe that the speed-gradient method is able to control the splay state in a wide parameter range. The range of possible coupling strengths K does, however, shrink considerably with increasing time delay τ. We conjecture several reasons for this shrinking. First, multistability of different splay and cluster

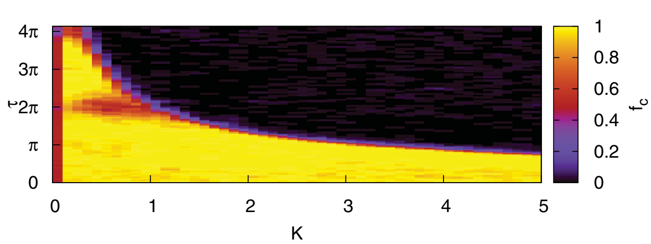

FIG. 7. (Color online) Success of the speed-gradient method in dependence on the coupling parameters K and τ for the splay state in a unidirectionally coupled ring of $N = 4$ Stuart-Landau oscillators. Other parameters as in Fig. 1. The color code shows the fraction of successful realizations.

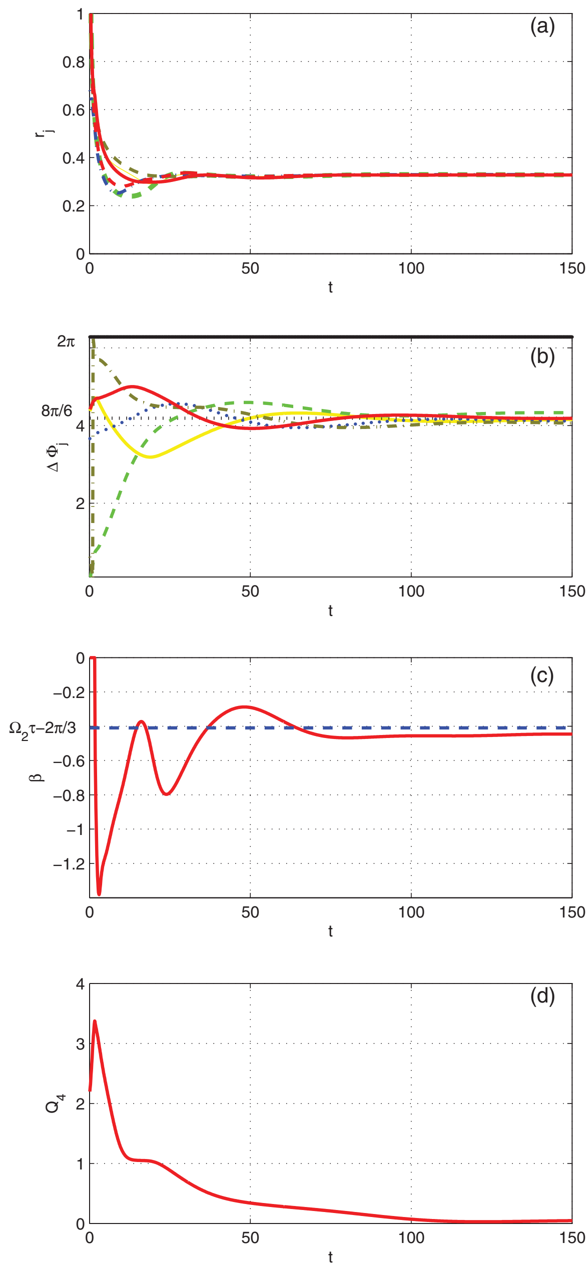

FIG. 8. (Color online) Adaptive control of 3-cluster state ($m = 2,4$) in a network with nonidentical oscillators with goal function Eq. (17). (a) Absolute values $r_j = |z_j|$; (b) phase differences $\Delta\phi_j = \varphi_j - \varphi_{j+1}$, blue dashed line: reference value for $\Omega_2 = 1.03$; (c) temporal evolution of β; (d) goal function. Parameters λ_j and ω_j are chosen from a Gaussian distribution with 1% standard deviation and mean values $\lambda = 0.1$ and $\omega = 1$, respectively. Other parameters as in Fig. 1.

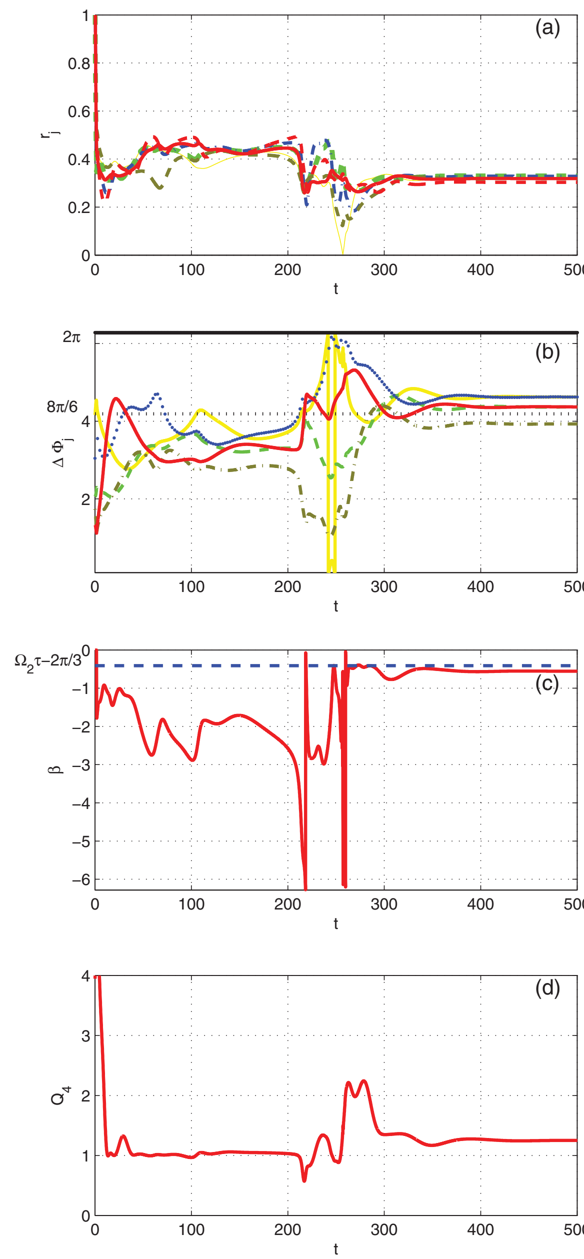

FIG. 9. (Color online) Same as in Fig. 8 for 5% standard deviation.

states is more likely for larger values of K and τ, which narrows down the basin of attraction for a given state. Second, Eq. (18), which describes the dynamics of the coupling phase under the adaptive control, is influenced by the time delay τ. Using large delay times, we observe overshoots of the control leading to a failure.

All the results presented so far were for identical oscillators in the network. It has been shown that control of cluster and splay states using an appropriate value of the phase β works even for slightly nonidentical frequencies ω of the oscillators [21]. Figures 8 and 9 show the adaptive control of a three-cluster state similar to Fig. 6, but with nonidentical parameters λ_j and ω_j of the individual oscillators. We choose them from a Gaussian distribution with mean value $\lambda = 0.1$ and $\omega = 1$, respectively, and standard deviation 1% (Fig. 8) and 5% (Fig. 9) for both. It is observed that the control goal is achieved after approximately the same time for the small standard deviation, but after a longer time for the larger standard deviation, and that the goal function Q_4 does not tend exactly to zero as in the case of identical parameters. The oscillators do not synchronize completely due to their amplitude and frequency mismatch, which can be seen in

Figs. 8 and 9 [parts (a) and (b)]. This explains why the goal function does not vanish completely.

VI. CONCLUSION

We have proposed a novel adaptive method for the control of synchrony on oscillator networks, which combines time-delayed coupling with the speed gradient method of control theory. Choosing an appropriate goal function, a desired state of generalized synchrony can be selected by the self-adaptive automatic adjustment of a control parameter, i.e., the coupling phase. This goal function, which is based on a generalization of the Kuramoto order parameter, vanishes for the desired state, e.g., in-phase, splay, or cluster states, irrespective of the ordering of the nodes. By numerical simulations we have shown that those different states can be stabilized, and the coupling phase converges to an appropriate value. We have elaborated on the robustness of the control scheme by investigating the success rates of the algorithm in dependence on the coupling parameters, i.e., the coupling strength and the time delay. We have further shown that the adaptive control is robust against mismatch of the parameters of the individual oscillators. In this work, we focused on the adaptive adjustment of the coupling phase while the other coupling parameters were fixed. The input variable u in Eq. (3) may in general contain all of the coupling parameters. Thus, as a generalization, our method might be applied to all coupling parameters including the coupling amplitude and the time delay. In this way control of cluster and splay synchronization might be possible without any *a priori* knowledge of the coupling parameters. Given the paradigmatic nature of the Stuart-Landau oscillator as a generic model, we expect broad applicability, for instance, to synchronization of networks in medicine, chemistry, or mechanical engineering. The mean-field nature of our goal function makes our approach accessible even for very large networks independently of the particular topology.

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