Sampled-Data Control of Nonlinear Oscillations Based on LMIs and Fridman's Method ⋆

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Abstract: Emilia Fridman's method based on time-dependent Lyapunov-Krasovsii functionals is extended to nonlinear control systems with sector bounded nonlinearity. Furthermore, this paper considers sampled-data feedback control under uncertain sampling with the known upper bound on the sampling intervals, where the control function is multiplied by bounded scalar nonlinear function in system equation. This special case corresponds to many oscillator control systems, for example, cart-pendulum system. The problem is reduced to feasibility analysis of linear matrix inequalities based on classical results of V.A. Yakubovich about S-procedure.

Keywords: nonlinear control systems, oscillators, sampled-data control, LMI

1. INTRODUCTION

An important problem for design of computer controlled systems is the proper choice of the sampling interval providing stability and the desired performance of the control system. This problem is by no means trivial even for linear systems if one needs to evaluate nonconservative bounds for maximum admitted value of sampling interval. As for nonlinear systems the problem is not well studied despite its importance.

Recently in the literature an interest has grown up in a novel approach to the sampling time evaluation based on transformation of discrete-continuous system models to continuous delayed system with time-varying delay. The origin of the idea can be traced back to Mikheev et al. (1988); Fridman (1992). However being combined with the descriptor method of delayed systems analysis proposed by Fridman (2001) the idea has become equipped with a powerful calculation tools based on LMI and has become a powerful design method allowing one to dramatically reduce conservativity of the design Fridman et al. (2004); Fridman (2010). However the Fridman's method was previously applied only to linear systems. The only exception is the paper Teel et al. (1998) where the existence of a nonzero sampling interval for ISS nonlinear systems is shown.

In this paper nonlinear systems with sector bounded nonlinearities closed by sampled-time linear feedback are considered. Our aim is to evaluate the upper bound of the sampling interval below which the system is absolutely stable. Instead of the traditional reduction to discrete time system an alternative method is used: the effect of sampling is considered as delay followed by the construction and use of Lyapunov-Krasovskii functional Fridman (2010). With S-procedure Yakubovich et al. (2004) the estimation of sampling step is reduced to feasibility analysis of linear matrix inequalities. Nowadays there are increasing interests to use of LMI analysis in solving various problems. For example, in Chen et al. (2012); Zhu et al. (2010) the sampled-data control problem for master-slave synchronization of chaotic Lur'e systems is studied and the synchronization criterion is formulated as LMIs.

As example the cart-pendulum system is proposed. The system has a strongly nonlinear behavior, and it is broadly used in control education. The cart-pendulum system has attracted the attention of many researchers (for example Mazenc and Bowong (2003); Zhao and Spong (2001); Siuka and Schoberl (2009)). Because the cart is driven by traction force, the control function is multiplied by bounded scalar nonlinear function in system equation. The results on maximum lower bound of sampling intervals are compared with Matlab simulation.

2. PROBLEM STATEMENT

Consider the nonlinear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + q\xi(t) + (B + B_0\xi_0(t))\,u(t), \\ \sigma(t) &= r^T x(t), \qquad \sigma_0(t) = r_0^T x(t), \\ \xi(t) &= \varphi(\sigma(t), t), \quad \xi_0(t) = \varphi_0(\sigma_0(t), t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control function, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $B_0 \in \mathbb{R}^{n\times m}$ are constant matrices, $q \in \mathbb{R}^n, r \in \mathbb{R}^n, r_0 \in \mathbb{R}^n$ are constant vectors.

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Assume that $\xi(t) = \varphi(\sigma(t), t)$ is the nonlinear function (see Fig.1) satisfying

$$\mu_1 \sigma^2 \leqslant \sigma \xi \leqslant \mu_2 \sigma^2, \quad (2)$$

for all $t \geqslant 0$ where $\mu_1 < \mu_2$ are real numbers. Let nonlinear function $\xi_0(t) = \varphi_0(\sigma_0(t), t)$ be bounded for all $t \geqslant 0$

$$\varphi_0^- \leqslant \xi_0(t) \leqslant \varphi_0^+.$$

Given a sequence of sampling times $0 = t_0 < t_1 < \ldots <$

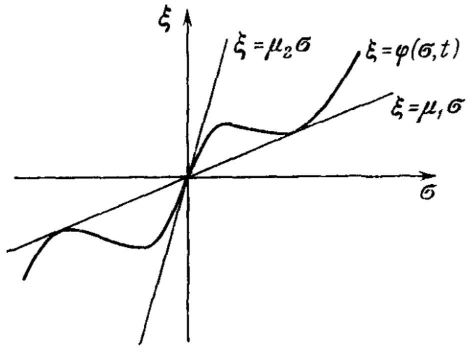

Fig. 1. Sector bounded nonlinearity

$t_k < \ \ldots$ and a piecewise constant control function

$$u(t) = u_d(t_k), \quad t_k \leqslant t < t_{k+1}, \quad (3)$$

where $\lim_{k\to\infty} t_k = \infty$.

Assume that $h \in \mathbb{R}\,(h > 0)$ and

$$t_{k+1} - t_k \leqslant h, \quad \forall k \geqslant 0 \quad (4)$$

and consider a sampled-time control law

$$u(t) = Kx(t_k), \ \ t_k \leqslant t < t_{k+1}, \quad (5)$$

where $K \in \mathbb{R}^{m\times n}$. The law (5) can be rewritten as follows:

$$u(t) = Kx(t - \tau(t)), \quad (6)$$

where $\tau(t) = t - t_k$, $t_k \leqslant t < t_{k+1}$.

It's required to analyze the influence of the upper bound h of sampling intervals on the closed-loop system exponential stability:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + (B + B_0\xi_0(t))\,Kx(t-\tau(t)) + q\xi(t),\\ &\sigma(t) = r^T x(t), \qquad \sigma_0(t) = r_0^T x(t),\\ &\xi(t) = \varphi(\sigma(t), t), \quad \xi_0(t) = \varphi_0(\sigma_0(t), t),\\ &\tau(t) = t - t_k, \ \ t \in [t_k, t_{k+1}). \end{aligned} \quad (7)$$

3. STABILITY ANALYSIS BASED ON LYAPUNOV FUNCTIONAL METHOD AND S-PROCEDURE

Definition 1. The space of absolutely continuous on $[-h, 0)$ functions $f : [-h, 0] \to \mathbb{R}^n$ having square integrable first-order derivatives is denoted by W with the norm

$$\|f\|_W = \max_{\theta\in[-h,0]} |f(\theta)| + \left[\int_{-h}^{0} |\dot{f}(s)|^2 ds\right]^{\frac{1}{2}}.$$

Denote $x_t(\theta) : [-h, 0] \to \mathbb{R}^n$ as $x_t(\theta) = x(t + \theta)$, where $x(\theta) \equiv 0$ if $\theta \in [-h, 0)$.

Definition 2. System (7) will be called exponentially stable with the decay rate $\alpha > 0$ if there exists $\beta > 0$ such that for solution $x(t)$ of (7) with initial condition x_{t_0} the following estimate holds

$$|x(t)|^2 \leqslant \beta e^{-2\alpha(t-t_0)}\,\|x_{t_0}\|_W^2, \ \forall t \geqslant t_0.$$

The proof of our main result is based on the following auxiliary statement that can be proved along the lines of Lemma 1 in Fridman (2010).

Lemma 1. Let there exist positive numbers β_1, β_2 and a functional $V : \mathbb{R} \times W \times L_2[-h, 0] \to \mathbb{R}$ such that

$$\beta_1\,|\phi(0)|^2 \leqslant V(t, \phi, \dot{\phi}) \leqslant \beta_2\,\|\phi\|_W^2. \quad (8)$$

Let the function $\bar{V}(t) = V(t, x_t, \dot{x}_t)$ be continuous from the right for $x(t)$ satisfying (7), absolutely continuous for $t \neq t_k$ and satisfies

$$\lim_{t\to t_k^-} \bar{V}(t) \geqslant \bar{V}(t_k). \quad (9)$$

Given $\alpha > 0$ if along $x(t)$

$$\dot{\bar{V}}(t) + 2\alpha\bar{V}(t) \leqslant 0 \quad (10)$$

almost for all t then (7) is exponentially stable with the decay rate α.

Consider the following functional on $\mathbb{R} \times W \times L_2[-h, 0]$:

$$V_0(t, x_t, \dot{x}_t) = x_t(0)^T P x_t(0) + + (h - \tau(t)) \int_{-\tau(t)}^{0} e^{2\alpha s}\,\dot{x}_t^T(s)\,Q\,\dot{x}_t(s) ds, \quad (11)$$

where P and Q are symmetric positive definite matrices.

Define function $\bar{V}_0(t) = V_0(t, x_t, \dot{x}_t)$, i.e.

$$\bar{V}_0(t) = x(t)^T P x(t) + V_Q(t, \dot{x}(t)), \quad (12)$$

where

$$V_Q(t, \dot{x}(t)) = (h - \tau(t)) \int_{-\tau(t)}^{0} e^{2\alpha s}\,\dot{x}^T(t+s)\,Q\,\dot{x}(t+s) ds.$$

To formulate the main result of the paper let us check the conditions of Lemma 1.

Note that $V_Q \geqslant 0$ and $\lim_{t\to t_k^+} V_Q(t, \dot{x}(t)) = V_Q(t_k, \dot{x}(t_k)) = 0$ because $\tau(t)_{|t=t_k} = 0$. Therefore, $\bar{V}_0(t)$ is continuous from the right and condition (9) holds.

Evaluate the left hand side of (10). Since $\frac{d}{dt}x(t - \tau(t)) = (1 - \dot{\tau}(t))\dot{x}(t - \tau(t)) = 0$, we obtain

$$\dot{\bar{V}}_0(t) + 2\alpha\bar{V}_0(t) \leqslant 2x^T(t)P\dot{x}(t) + 2\alpha x^T(t)Px(t) + +(h-\tau(t))\dot{x}^T(t)Q\dot{x}(t) - e^{-2\alpha h} \int_{-\tau(t)}^{0} \dot{x}^T(t+s)Q\dot{x}(t+s) ds. \quad (13)$$

Denote

$$v_1(t) = \frac{1}{\tau(t)} \int_{-\tau(t)}^{0} \dot{x}(t+s) ds, \quad (14)$$

where right hand side of (14) for $\tau(t) = 0$ is understood as $\lim_{\tau(t)\to 0} v_1 = \dot{x}(t)$.

From the Jensen's inequality Gu et al. (2003) we have

$$\int_{-\tau(t)}^{0} \dot{x}^T(t+s)\,Q\,\dot{x}(t+s) ds \geqslant \tau(t)\,v_1^T Q v_1. \quad (15)$$

Denote for brevity $\mathcal{B}(t) = B + B_0\xi_0(t)$, $\mathcal{B}^- = B + B_0\varphi_0^-$, $\mathcal{B}^+ = B + B_0\varphi_0^+$. If $x(t)$ is the solution of (7), then the following equality holds

$$0 = 2\left[x^T(t)P_2^T + \dot{x}^T(t)P_3^T\right] \times \times \left[(A + \mathcal{B}(t)K)\,x(t) - \tau(t)\mathcal{B}(t)Kv_1 + q\xi - \dot{x}(t)\right], \quad (16)$$

where $P_2 \in \mathbb{R}^{n\times n}, P_3 \in \mathbb{R}^{n\times n}$ are some matrices.

Denote $\eta_1 = col\,\{x, \dot{x}, \xi, v_1\}$, $\eta_1 \in \mathbb{R}^{3n+1}$ and $\eta_1(t) = col\,\{x(t), \dot{x}(t), \xi(t), v_1(t)\}$.

Adding (16) to the right-hand side of (13) and using (15) we obtain

$$\dot{\bar{V}}_0(t) + 2\alpha V_0(t) \leqslant \eta_1^T(t)\Psi_F(t)\eta_1(t), \quad (17)$$

where

$$\Psi_F(t) = \begin{bmatrix} \Phi_{F11}(t) & \Phi_{F12}(t) & \Phi_{F13} & -\tau(t)P_2^T\mathcal{B}(t)K \\ * & \Phi_{F22}(t) & \Phi_{F23} & -\tau(t)P_3^T\mathcal{B}(t)K \\ * & * & 0 & 0 \\ * & * & 0 & -\tau(t)Qe^{-2\alpha h} \end{bmatrix},$$

where "$*$" stands for corresponding block of the symmetric matrix and

$$\Phi_{F11}(t) = P_2^T(A + \mathcal{B}(t)K) + (A + \mathcal{B}(t)K)^T P_2 + 2\alpha P,$$
$$\Phi_{F12}(t) = P - P_2^T + (A + \mathcal{B}(t)K)^T P_3,$$
$$\Phi_{F13} = P_2^T q, \quad \Phi_{F23} = P_3^T q,$$
$$\Phi_{F22}(t) = -P_3 - P_3^T + (h - \tau(t))Q.$$

Thus, to check condition (10) it is sufficient to verify that the matrix $\Psi_F(t)$ is nonpositive for all $t \geqslant 0$. Consider the following linear matrix inequalities:

$$\Psi_{F0}^- = \begin{bmatrix} \Phi_{F11}^- & \Phi_{F12}^- & \Phi_{F13} \\ * & \Phi_{F22|\tau(t)=0} & \Phi_{F23} \\ * & * & 0 \end{bmatrix} < 0, \quad (18)$$

$$\Psi_{F0}^+ = \begin{bmatrix} \Phi_{F11}^+ & \Phi_{F12}^+ & \Phi_{F13} \\ * & \Phi_{F22|\tau(t)=0} & \Phi_{F23} \\ * & * & 0 \end{bmatrix} < 0, \quad (19)$$

$$\Psi_{F1}^- = \begin{bmatrix} \Phi_{F11}^- & \Phi_{F12}^- & \Phi_{F13} & -hP_2^T\mathcal{B}^-K \\ * & \Phi_{F22|\tau(t)=h} & \Phi_{F23} & -hP_3^T\mathcal{B}^-K \\ * & * & 0 & 0 \\ * & * & 0 & -hQe^{-2\alpha h} \end{bmatrix} < 0, \quad (20)$$

$$\Psi_{F1}^+ = \begin{bmatrix} \Phi_{F11}^+ & \Phi_{F12}^+ & \Phi_{F13} & -hP_2^T\mathcal{B}^+K \\ * & \Phi_{F22|\tau(t)=h} & \Phi_{F23} & -hP_3^T\mathcal{B}^+K \\ * & * & 0 & 0 \\ * & * & 0 & -hQe^{-2\alpha h} \end{bmatrix} < 0, \quad (21)$$

where

$$\Phi_{F11}^- = \Phi_{F11}(t)_{|\mathcal{B}(t)=\mathcal{B}^-}, \quad \Phi_{F11}^+ = \Phi_{F11}(t)_{|\mathcal{B}(t)=\mathcal{B}^+},$$
$$\Phi_{F12}^- = \Phi_{F12}(t)_{|\mathcal{B}(t)=\mathcal{B}^-}, \quad \Phi_{F12}^+ = \Phi_{F12}(t)_{|\mathcal{B}(t)=\mathcal{B}^+}.$$

Denote $\eta_0 = col\,\{x, \dot{x}, \xi\}$. Then (18), (19), (20) and (21) imply $\Psi_F(t) < 0\,\forall t > 0$ because

$$\frac{h-\tau(t)}{h}\,\frac{\varphi_0^+ - \varphi_0(t)}{\varphi_0^+ - \varphi_0^-}\,\eta_0^T\,\Psi_{F0}^-\,\eta_0 + + \frac{h-\tau(t)}{h}\,\frac{\varphi_0(t) - \varphi_0^-}{\varphi_0^+ - \varphi_0^-}\,\eta_0^T\,\Psi_{F0}^+\,\eta_0 + \frac{\tau(t)}{h}\,\frac{\varphi_0^+ - \varphi_0(t)}{\varphi_0^+ - \varphi_0^-}\,\eta_1^T\,\Psi_{F1}^-\,\eta_1 + + \frac{\tau(t)}{h}\,\frac{\varphi_0(t) - \varphi_0^-}{\varphi_0^+ - \varphi_0^-}\,\eta_1^T\,\Psi_{F1}^+\,\eta_1 = \eta_1^T\Psi_F(t)\,\eta_1 < 0, \ \ \forall\eta_1 \neq 0.$$

Denote

$$F_0^-(\eta_0) = \eta_0^T\,\Psi_{F0}^-\,\eta_0, \quad (22)$$
$$F_0^+(\eta_0) = \eta_0^T\,\Psi_{F0}^+\,\eta_0, \quad (23)$$
$$F_1^-(\eta_1) = \eta_1^T\,\Psi_{F1}^-\,\eta_1, \quad (24)$$
$$F_1^+(\eta_1) = \eta_1^T\,\Psi_{F1}^+\,\eta_1. \quad (25)$$

Thus, if

$$F_0^-(\eta_0) < 0, \ \ \forall\eta_0 \neq 0, \quad (26)$$
$$F_0^+(\eta_0) < 0, \ \ \forall\eta_0 \neq 0, \quad (27)$$
$$F_1^-(\eta_1) < 0, \ \ \forall\eta_1 \neq 0, \quad (28)$$
$$F_1^+(\eta_1) < 0, \ \ \forall\eta_1 \neq 0, \quad (29)$$

then condition (10) of Lemma 1 holds.

Introduce

$$G_0(\eta_0) = (\xi - \mu_1 r^T x)(\mu_2 r^T x - \xi), \quad (30)$$
$$G_1(\eta_1) = (\xi - \mu_1 r^T x)(\mu_2 r^T x - \xi). \quad (31)$$

Note that the forms $G_0(\eta_0)$ and $G_1(\eta_1)$ are defined on different spaces. From (2) the following inequalities holds along trajectories of system (7):

$$G_0(\eta_0) \geqslant 0, \quad G_1(\eta_1) \geqslant 0.$$

Therefore, we can require that (26) holds in the set $G_0(\eta_0) \geqslant 0$, i. e.

$$F_0^-(\eta_0) < 0 \text{ if } G_0(\eta_0) \geqslant 0, \ \ \forall\eta_0 \neq 0. \quad (32)$$

Similarly,

$$F_0^+(\eta_0) < 0 \text{ if } G_0(\eta_0) \geqslant 0, \ \ \forall\eta_0 \neq 0, \quad (33)$$
$$F_1^-(\eta_1) < 0 \text{ if } G_1(\eta_1) \geqslant 0, \ \ \forall\eta_1 \neq 0, \quad (34)$$
$$F_1^+(\eta_1) < 0 \text{ if } G_1(\eta_1) \geqslant 0, \ \ \forall\eta_1 \neq 0. \quad (35)$$

Let us transform (32) - (35) using S-procedure Yakubovich et al. (2004). Consider the following forms:

$$S_0^-(\eta_0) = F_0^-(\eta_0) + \varkappa_0^- G_0(\eta_0), \quad (36)$$
$$S_0^+(\eta_0) = F_0^+(\eta_0) + \varkappa_0^+ G_0(\eta_0), \quad (37)$$
$$S_1^-(\eta_1) = F_1^-(\eta_1) + \varkappa_1^- G_1(\eta_1), \quad (38)$$
$$S_1^+(\eta_1) = F_1^+(\eta_1) + \varkappa_1^+ G_1(\eta_1) \quad (39)$$

and require them to be negative for some non-negative $\varkappa_0^-$, $\varkappa_0^+$, $\varkappa_1^-$ and $\varkappa_1^+$ respectively:

$$\exists\varkappa_0^- \geqslant 0: \ S_0^-(\eta_0) < 0, \ \ \forall\eta_0 \neq 0, \quad (40)$$
$$\exists\varkappa_0^+ \geqslant 0: \ S_0^+(\eta_0) < 0, \ \ \forall\eta_0 \neq 0, \quad (41)$$
$$\exists\varkappa_1^- \geqslant 0: \ S_1^-(\eta_1) < 0, \ \ \forall\eta_1 \neq 0, \quad (42)$$
$$\exists\varkappa_1^+ \geqslant 0: \ S_1^+(\eta_1) < 0, \ \ \forall\eta_1 \neq 0. \quad (43)$$

By the theorem about a losslessness of S-procedure Yakubovich et al. (2004), condition (32) is equivalent to condition (40). Similarly, (33) is equivalent to (41), (34) is equivalent to (42) and (35) is equivalent to (43). Therefore, if conditions (40) - (43) hold, then (10) is fulfilled. Using (17), (22) - (25) we obtain the following inequalities:

$$S_0^-(\eta_0) \leqslant \eta_0^T\,\Psi_{S0}^-\,\eta_0, \quad (44)$$
$$S_0^+(\eta_0) \leqslant \eta_0^T\,\Psi_{S0}^+\,\eta_0, \quad (45)$$
$$S_1^-(\eta_1) \leqslant \eta_1^T\,\Psi_{S1}^-\,\eta_1, \quad (46)$$
$$S_1^+(\eta_1) \leqslant \eta_1^T\,\Psi_{S1}^+\,\eta_1, \quad (47)$$

where

$$\Psi_{S0}^- = \begin{bmatrix} \Phi_{S1}^- & \Phi_{F12}^- & \Phi_{S2}^- \\ * & \Phi_{F22|\tau(t)=0} & \Phi_{F23} \\ * & * & \Phi_{S3}^- \end{bmatrix}, \quad (48)$$

$$\Psi_{S0}^+ = \begin{bmatrix} \Phi_{S1}^+ & \Phi_{F12}^+ & \Phi_{S2}^+ \\ * & \Phi_{F22|\tau(t)=0} & \Phi_{F23} \\ * & * & \Phi_{S3}^+ \end{bmatrix}, \quad (49)$$

$$\Psi_{S1}^- = \begin{bmatrix} \Phi_{S4}^- & \Phi_{F12}^- & \Phi_{S5}^- & -hP_2^T\mathcal{B}^-K \\ * & \Phi_{F22|\tau(t)=h} & \Phi_{F23} & -hP_3^T\mathcal{B}^-K \\ * & * & \Phi_{S6}^- & 0 \\ * & * & 0 & -hQe^{-2\alpha h} \end{bmatrix}, \quad (50)$$

$$\Psi_{S1}^{+}=\begin{bmatrix}\Phi_{S4}^{+} & \Phi_{F12}^{+} & \Phi_{S5}^{+} & -hP_2^T\mathcal{B}^+K\\ * & \Phi_{F22|\tau(t)=h} & \Phi_{F23} & -hP_3^T\mathcal{B}^+K\\ * & * & \Phi_{S6}^{+} & 0\\ * & * & 0 & -hQe^{-2\alpha h}\end{bmatrix} \quad (51)$$

and

$$\Phi_{S1}^{-}=\Phi_{F11}^{-}-\varkappa_0^{-}\mu_1\mu_2 rr^T,\quad \Phi_{S1}^{+}=\Phi_{F11}^{+}-\varkappa_0^{+}\mu_1\mu_2 rr^T,$$
$$\Phi_{S2}^{-}=P_2^Tq+\frac{1}{2}\varkappa_0^{-}(\mu_1+\mu_2)r,\quad \Phi_{S3}^{-}=-\varkappa_0^{-},$$
$$\Phi_{S2}^{+}=P_2^Tq+\frac{1}{2}\varkappa_0^{+}(\mu_1+\mu_2)r,\quad \Phi_{S3}^{+}=-\varkappa_0^{+},$$
$$\Phi_{S4}^{-}=\Phi_{F11}^{-}-\varkappa_1^{-}\mu_1\mu_2 rr^T,\quad \Phi_{S4}^{+}=\Phi_{F11}^{+}-\varkappa_1^{+}\mu_1\mu_2 rr^T,$$
$$\Phi_{S5}^{-}=P_2^Tq+\frac{1}{2}\varkappa_1^{-}(\mu_1+\mu_2)r,\quad \Phi_{S6}^{-}=-\varkappa_1^{-},$$
$$\Phi_{S5}^{+}=P_2^Tq+\frac{1}{2}\varkappa_1^{+}(\mu_1+\mu_2)r,\quad \Phi_{S6}^{+}=-\varkappa_1^{+}.$$

Hence, if

$$\Psi_{S0^-}<0, \quad (52)$$
$$\Psi_{S0^+}<0, \quad (53)$$
$$\Psi_{S1^-}<0, \quad (54)$$
$$\Psi_{S1^+}<0, \quad (55)$$

then (10) holds.

Thus, we have proved the following

Theorem 1. Given $\alpha > 0$, let there exist matrices $P \in \mathbb{R}^{n\times n}$ $(P>0)$, $Q\in\mathbb{R}^{n\times n}$ $(Q>0)$, $P_2\in\mathbb{R}^{n\times n}$, $P_3\in\mathbb{R}^{n\times n}$ and real numbers $\varkappa_0^{-}\geqslant 0$, $\varkappa_0^{+}\geqslant 0$, $\varkappa_1^{-}\geqslant 0$, $\varkappa_1^{+}\geqslant 0$ such that LMIs (52) – (55) are feasible. Then system (7) is exponentially stable with decay rate α.

Consider the special case of the constant sampling intervals $t_{k+1}-t_k=h$, $k=0,1,\ldots$ In this case the result can be refined. Introduce the functional

$$V(t,x_t,\dot{x}_t)=V_0(t,x_t,\dot{x}_t)+V_1(t,x_t), \quad (56)$$

where

$$V_1(t,x_t)=(h-\tau(t))\,\zeta^T\begin{bmatrix}\frac{X+X^T}{2} & -X+X_1\\ * & -X_1-X_1^T+\frac{X+X^T}{2}\end{bmatrix}\zeta,$$

and $\zeta = col\left\{x_t(0), x_{t-\tau(t)}(0)\right\}$, $X\in\mathbb{R}^{n\times n}$, $X_1\in\mathbb{R}^{n\times n}$.

For fulfillment of (8) it is sufficient that

$$\Theta(h)>0, \quad (57)$$

where

$$\Theta(h)=\begin{bmatrix}P+h\frac{X+X^T}{2} & hX_1-hX\\ * & -hX_1-hX_1^T+h\frac{X+X^T}{2}\end{bmatrix}.$$

Indeed,

$$x_t(0)^TPx_t(0)+V_1(t,x_t)=\frac{h-\tau(t)}{h}\,\zeta^T\,\Theta(h)\,\zeta+ +\frac{\tau(t)}{h}\,\zeta^T\,\Theta(0)\,\zeta\geqslant\beta_1\,|x_t(0)|^2, \quad (58)$$

where $\beta_1=\min(\nu_1,\nu_2)$, ν_1 and ν_2 are minimum eigenvalues of P and $\Theta(h)$ respectively.

As before consider function

$$\bar{V}(t)=V(t,x_t,\dot{x}_t). \quad (59)$$

$\bar{V}(t)$ is continuous from the right, and condition (9) holds because $\bar{V}_0(t)$ is continuous from the right and $\lim\limits_{t\to t_k^-}V_1(t,x_t)=V_1(t_k,x_{t_k})=\lim\limits_{t\to t_k^+}V_1(t,x_t)=0$ (i. e. $\tau(t)=h$ at $t\to t_k^-$ and $\tau(t)=0$ at $t\to t_k^+$, hence, $x(t)=x(t-\tau(t))$).

Assume that $P_2\in\mathbb{R}^{n\times n}$, $P_3\in\mathbb{R}^{n\times n}$, $Y_1\in\mathbb{R}^{n\times n}$, $Y_2\in\mathbb{R}^{n\times n}$, $Y_3\in\mathbb{R}^{n\times n}$, $T\in\mathbb{R}^{n\times n}$ are some matrices. If $x(t)$ is the solution of (7), then the following equalities hold:

$$0=2\left[-x(t)+x(t-\tau(t))+\tau(t)v_1\right]\times \times\left[x^T(t)Y_1^T+\dot{x}^T(t)Y_2^T+x^T(t-\tau(t))T^T+\xi q^TY_3^T\right],$$

$$0=2\left[x^T(t)P_2^T+\dot{x}^T(t)P_3^T\right]\times \times\left[Ax(t)+\mathcal{B}(t)Kx(t-\tau(t))+q\xi-\dot{x}(t)\right].$$

Adding their right-hand sides to $\dot{\bar{V}}(t)$ and applying the Jensen's inequality we obtain

$$\dot{\bar{V}}(t)+2\alpha\bar{V}(t)\leqslant\eta^T(t)\Psi(t)\,\eta(t), \quad (60)$$

where $\eta(t)=col\left\{x(t),\dot{x}(t),x(t-\tau(t)),\xi(t),v_1(t)\right\}$, $\eta\in\mathbb{R}^{4n+1}$,

$$\Psi(t)=\begin{bmatrix}\Phi_{11}(t) & \Phi_{12}(t) & \Phi_{13}(t) & \Phi_{14} & \tau(t)Y_1^T\\ * & \Phi_{22}(t) & \Phi_{23}(t) & \Phi_{24} & \tau(t)Y_2^T\\ * & * & \Phi_{33}(t) & \Phi_{34} & \tau(t)T^T\\ * & * & * & 0 & \tau(t)q^TY_3^T\\ * & * & * & * & -\tau(t)Qe^{-2\alpha h}\end{bmatrix}. \quad (61)$$

In (61) the following notation is used:

$$\Phi_{11}(t)=A^TP_2+P_2^TA+2\alpha P-Y_1-Y_1^T- -(1-2\alpha(h-\tau(t)))\frac{X+X^T}{2},$$
$$\Phi_{12}(t)=P-P_2^T+A^TP_3-Y_2+(h-\tau(t))\frac{X+X^T}{2},$$
$$\Phi_{13}(t)=Y_1^T+P_2^T\mathcal{B}(t)K-T+(1-2\alpha(h-\tau(t)))(X-X_1),$$
$$\Phi_{22}(t)=-P_3-P_3^T+(h-\tau(t))Q,$$
$$\Phi_{23}(t)=Y_2^T+P_3^T\mathcal{B}(t)K-(h-\tau(t))(X-X_1),$$
$$\Phi_{33}(t)=T+T^T-(1-2\alpha(h-\tau(t)))\frac{X+X^T-2X_1-2X_1^T}{2},$$
$$\Phi_{14}=P_2^Tq-Y_3q,\quad \Phi_{24}=P_3^Tq,\quad \Phi_{34}=Y_3q.$$

Again using S-procedure (with parameters $\varkappa_0^{-}$, $\varkappa_0^{+}$, $\varkappa_1^{-}$, $\varkappa_1^{+}$) and considering four extreme cases instead of (61) we arrive at the following linear matrix inequalities:

$$\Psi_{H0}^{-}<0, \quad (62)$$
$$\Psi_{H0}^{+}<0, \quad (63)$$
$$\Psi_{H1}^{-}<0, \quad (64)$$
$$\Psi_{H1}^{+}<0, \quad (65)$$

where

$$\Psi_{H0}^{-}=\begin{bmatrix}\Phi_{H1|\tau(t)=0}^{-} & \Phi_{12|\tau(t)=0} & \Phi_{13|\tau(t)=0}^{-} & \Phi_{H2}^{-}\\ * & \Phi_{22|\tau(t)=0} & \Phi_{23|\tau(t)=0}^{-} & \Phi_{24}\\ * & * & \Phi_{33|\tau(t)=0} & \Phi_{34}\\ * & * & * & \Phi_{H3}^{-}\end{bmatrix}, \quad (66)$$

$$\Psi_{H0}^{+}=\begin{bmatrix}\Phi_{H1|\tau(t)=0}^{+} & \Phi_{12|\tau(t)=0} & \Phi_{13|\tau(t)=0}^{+} & \Phi_{H2}^{+}\\ * & \Phi_{22|\tau(t)=0} & \Phi_{23|\tau(t)=0}^{+} & \Phi_{24}\\ * & * & \Phi_{33|\tau(t)=0} & \Phi_{34}\\ * & * & * & \Phi_{H3}^{+}\end{bmatrix}, \quad (67)$$

$$\Psi_{H1}^{-}=\begin{bmatrix}\Phi_{H4|\tau(t)=h}^{-} & \Phi_{12|\tau(t)=h} & \Phi_{13|\tau(t)=h}^{-} & \Phi_{H5}^{-} & hY_1^T\\ * & \Phi_{22|\tau(t)=h} & \Phi_{23|\tau(t)=h}^{-} & \Phi_{24} & hY_2^T\\ * & * & \Phi_{33|\tau(t)=h} & \Phi_{34} & hT^T\\ * & * & * & \Phi_{H6}^{-} & hq^TY_3^T\\ * & * & * & * & -hQe^{-2\alpha h}\end{bmatrix}, \quad (68)$$

$$
\Psi^+_{H1} = \begin{bmatrix} \Phi^+_{H4|\tau(t)=h} & \Phi_{12|\tau(t)=h} & \Phi^+_{13|\tau(t)=h} & \Phi^+_{H5} & hY_1^T \\ * & \Phi_{22|\tau(t)=h} & \Phi^+_{23|\tau(t)=h} & \Phi_{24} & hY_2^T \\ * & * & \Phi_{33|\tau(t)=h} & \Phi_{34} & hT^T \\ * & * & * & \Phi^+_{H6} & hq^TY_3^T \\ * & * & * & * & -hQe^{-2\alpha h} \end{bmatrix}, \quad (69)
$$

and

$$
\begin{aligned}
&\Phi^-_{13}(t) = Y_1^T + P_2^T\mathcal{B}^-K - T + (1-2\alpha(h-\tau(t)))(X - X_1),\\
&\Phi^+_{13}(t) = Y_1^T + P_2^T\mathcal{B}^+K - T + (1-2\alpha(h-\tau(t)))(X - X_1),\\
&\Phi^-_{23}(t) = Y_2^T + P_3^T\mathcal{B}^-K - (h-\tau(t))(X - X_1),\\
&\Phi^+_{23}(t) = Y_2^T + P_3^T\mathcal{B}^+K - (h-\tau(t))(X - X_1),\\
&\Phi^-_{H1}(t) = \Phi_{11}(t) - \varkappa_0^-\mu_1\mu_2rr^T,\\
&\Phi^+_{H1}(t) = \Phi_{11}(t) - \varkappa_0^+\mu_1\mu_2rr^T,\\
&\Phi^-_{H2} = \Phi_{14} + \frac{1}{2}\varkappa_0^-(\mu_1+\mu_2)r, \quad \Phi^+_{H2} = \Phi_{14} + \frac{1}{2}\varkappa_0^+(\mu_1+\mu_2)r,\\
&\Phi^-_{H3} = -\varkappa_0^-, \quad \Phi^+_{H3} = -\varkappa_0^+,\\
&\Phi^-_{H4} = \Phi_{11}(t) - \varkappa_1^-\mu_1\mu_2rr^T, \quad \Phi^+_{H4} = \Phi_{11}(t) - \varkappa_1^+\mu_1\mu_2rr^T,\\
&\Phi^-_{H5} = \Phi_{14} + \frac{1}{2}\varkappa_1^-(\mu_1+\mu_2)r, \quad \Phi^+_{H5} = \Phi_{14} + \frac{1}{2}\varkappa_1^+(\mu_1+\mu_2)r,\\
&\Phi^-_{H6} = -\varkappa_1^-, \quad \Phi^+_{H6} = -\varkappa_1^+.
\end{aligned}
$$

To formulate our second result we need the following statement that can be proved along the lines of Lemma 2 in Fridman (2010).

Lemma 2. LMIs (57), (62) - (65) are convex in h: if they are feasible for h, then they are feasible for all $\bar{h} \in (0, h]$.

In view of Lemma 2 we obtain the following statement.

Theorem 2. Consider (7) with a constant sampling: $t_{k+1} - t_k = \bar{h} \leqslant h$. Given $\alpha > 0$, let there exist matrices $P \in \mathbb{R}^{n\times n}$ $(P > 0)$, $Q \in \mathbb{R}^{n\times n}$ $(Q > 0)$, $P_2 \in \mathbb{R}^{n\times n}$, $P_3 \in \mathbb{R}^{n\times n}$, $X \in \mathbb{R}^{n\times n}$, $X_1 \in \mathbb{R}^{n\times n}$, $T \in \mathbb{R}^{n\times n}$, $Y_1 \in \mathbb{R}^{n\times n}$, $Y_2 \in \mathbb{R}^{n\times n}$, $Y_3 \in \mathbb{R}^{n\times n}$ and real numbers $\varkappa_0^- \geqslant 0$, $\varkappa_0^+ \geqslant 0$, $\varkappa_1^- \geqslant 0$, $\varkappa_1^+ \geqslant 0$ such that LMIs (57), (62) - (65) are feasible. Then system (7) is exponentially stable with decay rate α.

Next we generalize this result to the case of variable sampling: $t_{k+1} - t_k = \bar{h}_k \leqslant h$, $k = 0, 1, \ldots$

Consider the following Lyapunov-Krasovskii functional:

$$
\begin{aligned}
V_{var}(t, x_t, \dot{x}_t) &= \bar{V}_{var}(t) = x^T(t)Px(t) + \\
&+ (t_{k+1} - t)\int_{t_k}^{t} e^{2\alpha(s-t)}\,\dot{x}^T(s)\,Q\,\dot{x}(s)ds + \\
&+ (t_{k+1} - t)\,\zeta^T(t)\begin{bmatrix} \frac{X+X^T}{2} & -X + X_1 \\ * & -X_1 - X_1^T + \frac{X+X^T}{2} \end{bmatrix}\zeta(t),\\
&\qquad t \in [t_k, t_{k+1}), \quad (70)
\end{aligned}
$$

where $\zeta(t) = col\{x(t), x(t_k)\}$. Note that in (70) the second and the third terms are equal to 0 for $t \to t_k^-$ and $t \to t_k^+$. Hence, $\bar{V}_{var}$ is continuous because $\lim_{t\to t_k} \bar{V}_{var}(t) = \bar{V}_{var}(t_k)$.

Applying arguments of the previous cases to $\bar{V}_{var}(t)$, we arrive at the main result:

Theorem 3. Consider (7) with variable sampling: $t_{k+1} - t_k \leqslant h$. Given $\alpha > 0$, let there exist matrices $P \in \mathbb{R}^{n\times n}$ $(P > 0)$, $Q \in \mathbb{R}^{n\times n}$ $(Q > 0)$, $P_2 \in \mathbb{R}^{n\times n}$, $P_3 \in \mathbb{R}^{n\times n}$, $X \in \mathbb{R}^{n\times n}$, $X_1 \in \mathbb{R}^{n\times n}$, $T \in \mathbb{R}^{n\times n}$, $Y_1 \in \mathbb{R}^{n\times n}$, $Y_2 \in \mathbb{R}^{n\times n}$, $Y_3 \in \mathbb{R}^{n\times n}$ and real numbers $\varkappa_0^- \geqslant 0$, $\varkappa_0^+ \geqslant 0$, $\varkappa_1^- \geqslant 0$, $\varkappa_1^+ \geqslant 0$ such that LMIs (57), (62) - (65) are feasible. Then system (7) is exponentially stable with decay rate α.

4. EXAMPLES

Let us illustrate the obtained results by examples.

4.1 Simple Pendulum

Consider the system, describing the simple computer controlled pendulum:

$$
\begin{aligned}
&\ddot{\varphi}(t) = -\frac{g}{l}sin(\varphi(t)) + \frac{1}{ml^2}u(t),\\
&u(t) = Kx(t - \tau(t)),\\
&\tau(t) = t - t_k, \;\; t \in [t_k, t_{k+1}), \;\; t_{k+1} - t_k = h, \;\; k = 0, 1 \ldots
\end{aligned} \quad (71)
$$

with parameter values $g = 9.8\,\text{m/s}^2$, $l = 1\,\text{m}$, $m = 2\,\text{kg}$, $K = [-23.6, -6]$.

Here φ is the deviation angle of the pendulum from vertical, u is control torque.

In this case the nonlinear function is $\sin(\varphi)$ (Fig. 2) and $\mu_1 = 1$, $\mu_2 = -0.2173$.

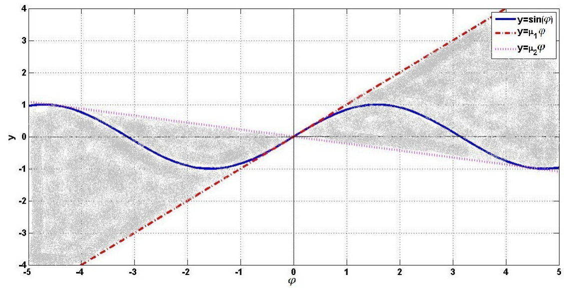

Fig. 2. Sector bounded nonlinearity

The feasibility of LMIs from Theorem 1 and Theorem 3 is checked in Matlab with package Yalmip. In Table 1 there are values of maximum upper bound h (for $\alpha = 0$) when (71) is exponentially stable with a small enough decay rate.

Theorem 1	Theorem 3	Simulation
$h = 0.192$	$h = 0.302$	$h = h_*$, $0.428 < h_* < 0.432$

Table 1. Upper bounds for the variable sampling

4.2 Cart-Pendulum System

Consider the pendulum with length l and deviation angle φ ($\varphi = 0$ at the upper position), rotating on the axle fixed on a cart (see Fig. 3). The cart with mass M is moving by external force u in a horizontal direction normal to the axle of pendulum rotation. Our goal is to stabilize the pendulum in the upper position and stabilize the cart (i.e. reach the state $s = 0, \dot{s} = 0$, where s is the cart motion) by changing the control force.

Consider the model of the cart-pendulum system (neglecting friction and pendulum reaction force on the cart):

$$
\begin{cases} \ddot{\varphi} = \dfrac{g}{l}\sin\varphi + \dfrac{\cos\varphi}{Ml}\,u, \\ \ddot{s} = \dfrac{1}{M}\,u. \end{cases} \quad (72)
$$

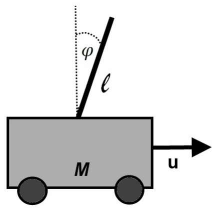

Fig. 3. Cart-Pendulum System

Seifullaev (2012) proposed the control algorithm of pendulum swing based on the *speed gradient method* Fradkov (2007). When the pendulum was close to the upper position, the stabilization algorithm had started. For system stabilization a state-feedback control law of the form

$$u(t) = Kx(t_k), \quad t_k \leqslant t < t_{k+1}, \tag{73}$$

was considered, where $x = col(\varphi, \dot{\varphi}, s, \dot{s})$, $t_{k+1} - t_k \leqslant h$, $\forall k \geqslant 0$.

For the system with the following parameter values

$$l = 1\,\text{m}, \; m = 0.1\,\text{kg}, \; M = 2\,\text{kg}, \; g = 9.8\,\text{m/s}^2,$$
$$K = [-21, -4.04, 0.001, 0.02]$$

with Theorem 1 of Fridman (2010) the maximum upper bound of sampling was found as 0.16.

However, to apply the results of E. Fridman the system should be linearized at the upper position. To avoid the linearization errors let us apply the results of Section 3. As in the previous example, the nonlinear function is $\sin(\varphi)$, and the nonlinearity sector is defined by (2) with $\mu_1 = 1$, $\mu_2 = -0.2173$. Assume that the pendulum stabilization mode (and hence initial conditions) occurs near the upper position in deviations up to 8 degrees. Therefore,

$$0.99 \leqslant \cos\varphi \leqslant 1.$$

Numerical results are illustrated by Table 2 and Fig. 4.

Theorem 1 at Fridman (2010) (for linearized system)	Theorem 1	Theorem 3	Simulation
$h = 0.160$	$h = 0.182$	$h = 0.331$	$h = h_*$, $0.401 < h_* < 0.403$

Table 2. Upper bounds for the variable sampling

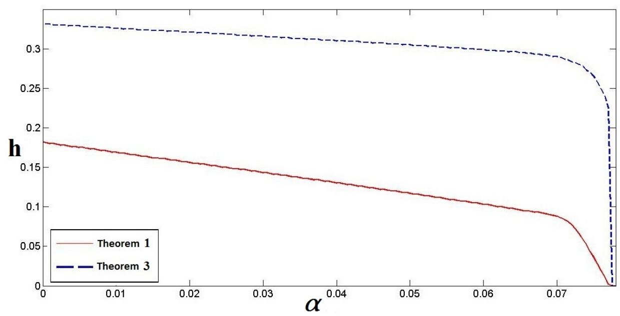

Fig. 4. The dependence of upper bound h on decay rate α

5. CONCLUSIONS

Using Yakubovich's S-procedure, Lyapunov-Krasovskii functional and Fridman's method the problem of the maximum lower bound estimation for the sampling interval providing exponential stability of the closed loop system is reduced to feasibility analysis of linear matrix inequalities. The obtained results are illustrated by examples of simple pendulum and cart-pendulum system. It is shown that the proposed method provides estimates for sampling interval not less that 70% (for simple pendulum) or even 82% (for cart-pendulum) of the value evaluated by simulation.

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