International Journal of Modern Physics B
Vol. 26, No. 25 (2012) 1246007 (11 pages)

DOI: 10.1142/S0217979212460071

CONTROL OF SYNCHRONIZATION IN DELAY-COUPLED NETWORKS

E. SCHÖLL*,¶, A. SELIVANOV†, J. LEHNERT*, T. DAHMS*,
P. HÖVEL*,‡ and A. FRADKOV†,§

*Institut für Theoretische Physik, Technische Universität Berlin,
Hardenbergstr. 36, 10623 Berlin, Germany
†Department of Theoretical Cybernetics, Saint-Petersburg State University,
Universitetskii pr.28, St.Petersburg, 198504 Russia
‡Bernstein Center for Computational Neuroscience, Humboldt-Universität zu Berlin,
Philippstraße 13, 10115 Berlin, Germany
§Institute for Problems of Mechanical Engineering, Russian Academy of Sciences,
Bolshoy Ave, 61, V. O., St. Petersburg, 199178 Russia
¶schoell@physik.tu-berlin.de

Received 15 March 2012
Accepted 8 April 2012
Published 30 August 2012

We consider synchronization in networks of delay-coupled oscillators. In these systems, the coupling phase has been found to be a crucial control parameter. By proper choice of this parameter one can switch between different synchronous oscillatory states of the network, e.g., in-phase oscillation, splay or various cluster states. Applying the speed-gradient method, we derive an adaptive algorithm for an automatic adjustment of the coupling phase, coupling strength, and delay time such that a desired state can be selected from an otherwise multistable regime.

Keywords: Synchronization; delay; networks; Hopf normal forms.

1. Introduction

The control of nonlinear dynamical systems has evolved into a wide interdisciplinary area of research over the past decades.[1] In particular, noninvasive control schemes based on time-delayed feedback have been studied and applied to various systems ranging from biological and chemical applications to physics and engineering.[2–4] Here we propose to use adaptive control schemes based on optimizations of cost or goal functions to find appropriate control parameters.[5–8] Besides isolated systems, control of dynamics in spatio-temporal systems and on networks has recently gained much interest.[9–13] The existence and control of cluster states was studied by Choe *et al.*[14] in networks of Stuart-Landau oscillators. This Stuart-Landau system arises naturally as a generic expansion near a Hopf bifurcation and is therefore often used as a paradigm for oscillators. The complex coupling constant that arises from the complex state variables in networks of Stuart-Landau oscillators consists of an amplitude and a phase. Similar coupling phases arise naturally in systems with all-optical coupling.[16,17] Such phase-dependent couplings have also been shown to be important in overcoming the odd-number limitation of time-delay feedback control,[18–21] and on the other hand in anticipating chaos synchronization.[22] Furthermore, it was shown in Refs. 14, 15 that the value of the coupling phase is a crucial control parameter in these systems, and by adjusting this phase one can deliberately switch between different synchronous oscillatory states of the network.

In this paper, we review synchronization in delay-coupled networks.[23,24] For the example of networks of Stuart-Landau oscillators we apply the speed-gradient method to find optimal values of the control parameters.[7] By taking an appropriate goal function we derive equations for the automatic adjustment of the coupling phase, the coupling strength, and the delay time such that the goal function is minimized. Our goal function is based on the Kuramoto order parameter and is able to distinguish the different states of synchrony in the Stuart-Landau networks irrespectively of the numbering of the nodes.

2. Network Model

First consider a network of N delay-coupled oscillators

$$\dot{z}_j(t) = f[z_j(t)] + Ke^{i\beta} \sum_{n=1}^{N} a_{jn}[z_n(t-\tau) - z_j(t)], \tag{1}$$

with $z_j = r_j e^{i\varphi_j} \in \mathbb{C}$, $j = 1, \ldots, N$. The coupling matrix $A = \{a_{ij}\}_{i,j=1}^N$ determines the topology of the network. The local dynamics of each element is given by the normal form of a supercritical Hopf bifurcation, also known as Stuart-Landau oscillator,

$$f(z_j) = [\lambda + i\omega - (1 + i\gamma)|z_j|^2]z_j, \tag{2}$$

with real constants $\lambda, \omega \neq 0$, and γ. In Eq. (1), τ is the delay time. The amplitude and phase of the complex coupling constant are denoted by K and β respectively. Such kinds of networks are used in different areas of nonlinear dynamics, e.g., to describe neural activities.[25]

Synchronous in-phase, cluster, and splay states are possible solutions of Eqs. (1) and (2). They exhibit a common amplitude $r_j \equiv r_{0,m}$ and phases given by $\varphi_j = \Omega_m t + j\Delta\varphi_m$ with a phase shift $\Delta\varphi_m = 2\pi m/N$ and collective frequency Ω_m. The integer m determines the specific state: in-phase oscillations correspond to $m = 0$, while splay and cluster states correspond to $m = 1, \ldots, N-1$. The cluster number d, which determines how many clusters of oscillators exist, is given by the least common multiple of m and N divided by m, and $d = N$ (e.g., $m = 1$), corresponds to a splay state.

The stability of synchronized oscillations in networks can be determined numerically, for instance, by the *master stability function*.[26] This formalism allows for a separation of the local dynamics of the individual nodes from the network topology. In the case of the Stuart-Landau oscillators it was possible to obtain the Floquet exponents of different cluster states analytically with this technique.[14] By these means it has been demonstrated that the unidirectional ring configuration of Stuart-Landau oscillators exhibits in-phase synchrony, splay states, and clustering depending on the choice of the control parameter β. For $\beta = 0$, there exists multistability of the possible synchronous states in a large parameter range. However, when tuning the coupling phase to an optimal value $\beta = \Omega_m\tau - 2\pi m/N$ according to a particular state m, this synchronous state is monostable for any values of the coupling strength K and the time delay τ. In the following, an adaptive algorithm is used to find optimal values of the control parameters K, β, and τ.

3. Speed-Gradient Method

In this section, we briefly review an adaptive control scheme called *speed-gradient* (SG) *method*.[7] Consider a general nonlinear dynamical system

$$\dot{x} = F(x, u, t), \tag{3}$$

with state vector $x \in \mathbb{C}^n$, input (control) variables $u \in \mathbb{C}^m$, and nonlinear function F. Define a control goal

$$\lim_{t\to\infty} Q(x(t), t) = 0, \tag{4}$$

where $Q(x, t) \geq 0$ is a smooth scalar goal function.

In order to design a control algorithm, the scalar function $\dot{Q} = \omega(x, u, t)$ is calculated, that is, the speed (rate) at which $Q(x(t), t)$ is changing along trajectories of Eq. (3):

$$\omega(x, u, t) = \frac{\partial Q(x, t)}{\partial t} + [\nabla_x Q(x, t)]^{\mathrm{T}} F(x, u, t). \tag{5}$$

Then we evaluate the gradient of $\omega(x, u, t)$ with respect to input variables:

$$\nabla_u \omega(x, u, t) = \nabla_u [\nabla_x Q(x, t)]^{\mathrm{T}} F(x, u, t).$$

Finally, we set up a differential equation for the input variables u

$$\frac{du}{dt} = -\Gamma \nabla_u \omega(x, u, t), \tag{6}$$

where $\Gamma = \Gamma^{\mathrm{T}} > 0$ is a positive definite gain matrix. The algorithm (6) is called *speed-gradient* (SG) *algorithm*, since it suggests to change u proportionally to the gradient of the speed of changing Q.

The idea of this algorithm is the following. The term $-\nabla_u \omega(x, u, t)$ points to the direction in which the value of $\dot{Q}$ decreases with the highest speed. Therefore, if one forces the control signal to "follow" this direction, the value of $\dot{Q}$ will decrease and finally be negative. When $\dot{Q} < 0$, then Q will decrease and, eventually, tend to zero.

We shall now apply the speed-gradient method to networks of Stuart-Landau oscillators. Since the coupling phase β is the crucial parameter that determines stability of the possible in-phase, cluster, and splay states, we use this control parameter as the input variable u. Setting $u = \beta$, $x = (z_1, \ldots, z_N)$ and $\Gamma = \Gamma_\beta$, Eq. (1) takes the form of Eq.(3) with state vector $x \in \mathbb{C}^N$ and input variable $\beta \in \mathbb{R}$, and nonlinear function $F(x, \beta, t) = [f(z_1), \ldots, f(z_N)] + Ke^{i\beta}[Ax(t-\tau) - x(t)]$.

The SG control equation (6) for the input variable β then becomes

$$\frac{d\beta}{dt} = -\Gamma_\beta \frac{\partial}{\partial \beta}\omega(x, \beta, t) = -\Gamma_\beta \left(\frac{\partial F}{\partial \beta}\right)^{\mathrm{T}} \nabla_x Q(x, t), \tag{7}$$

where $\Gamma_\beta > 0$ is now a scalar.

4. Zero-Lag Synchronization

To apply the SG method for the selection of in-phase (zero-lag) synchronization we need to find an appropriate goal function Q. It should satisfy the following conditions: the goal function must be zero for an in-phase synchronous state and larger than zero for other states. Hence, a simple goal function can be introduced by considering a function based on the order parameter

$$R_1 = \frac{1}{N}\left|\sum_{j=1}^{N} e^{i\varphi_j}\right|. \tag{8}$$

This global parameter can conveniently be measured in real-world systems, e.g., lasers. It is obvious that $R_1 = 1$ if and only if the state is in-phase synchronized. For other cases we have $R_1 < 1$. Using this observation we can introduce the following goal function

$$Q_0 = 1 - \frac{1}{N^2}\sum_{j=1}^{N} e^{i\varphi_j}\sum_{k=1}^{N} e^{-i\varphi_k}. \tag{9}$$

From $\dot{\beta} = -\Gamma_\beta \frac{\partial}{\partial\beta}\dot{Q}_0$ we derive an adaptive law:

$$\dot{\beta} = \Gamma_\beta \frac{2K}{N^2}\sum_{k=1}^{N}\sum_{j=1}^{N}\sin(\varphi_k - \varphi_j)\sum_{n=1}^{N} a_{jn}\left(\frac{r_{n,\tau}}{r_j}\cos(\beta + \varphi_{n,\tau} - \varphi_j) - \cos\beta\right). \tag{10}$$

Fig. 1 shows the results of a numerical simulation for an Erdős-Rényi random network with $N = 6$ nodes and row sum normalized to unity. Unless otherwise stated, we use $\Gamma_\beta = 1$. According to the numerical simulations decreasing Γ_β will yield a decrease of the speed of convergence. On the other hand, if Γ_β is too big, undesirable oscillations appear. The model parameters are chosen as in Ref. 14. The amplitude and phase approach appropriate values that lead to in-phase synchronization. Note that the obtained value of β does not converge to the one for which the analytical

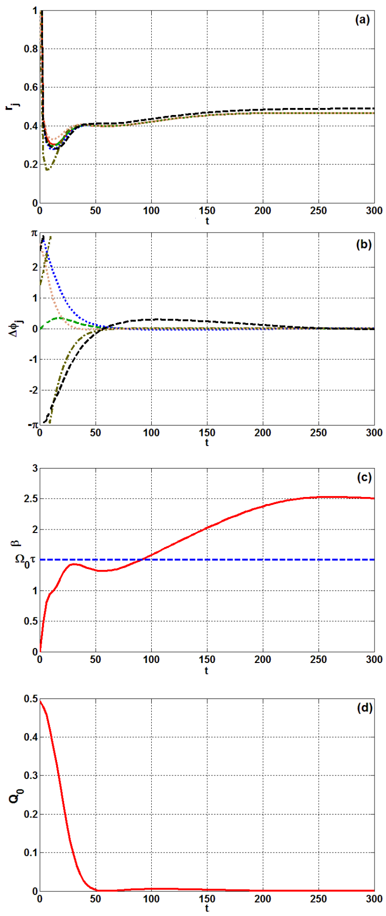

Figure 1. (Color online) Adaptive control of in-phase oscillations with goal function Eq. (9). (a): absolute values $r_j = |z_j|$ for $j = 1, ..., 6$; (b): phase differences $\Delta\phi_j = \varphi_j - \varphi_{j+1}$ for $j = 1, ..., 5$; (c) temporal evolution of β, blue dashed line: reference value for $\Omega_0 = 0.92$; (d): goal function. Parameters: $\lambda = 0.1, \omega = 1, \gamma = 0$, $K = 0.08$, $\tau = 0.52\pi$, $N = 6$. Initial conditions for r_j and φ_j are chosen randomly from $[0, 4]$ and $[0, 2\pi]$, respectively. The initial condition for β is zero.

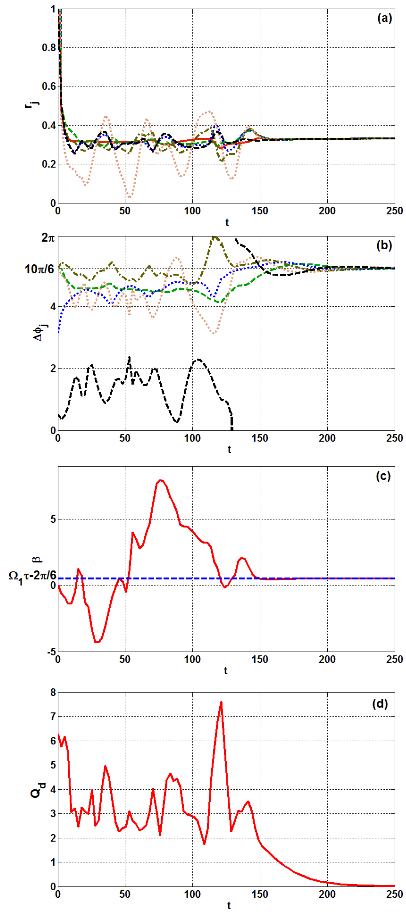

Figure 2. (Color online) Adaptive control of splay state with goal function Eq. (13). (a): absolute values $r_j = |z_j|$; (b): phase differences $\Delta\phi_j = \varphi_j - \varphi_{j+1}$; (c): temporal evolution of β, blue dashed line: reference value for $\Omega_1 = 0.96$; (d): goal function. Other parameters as in Fig. 1.

approach has established stability of the in-phase oscillation (blue dashed line),[14] but to another limit value. This can be explained as follows: There exists a whole interval of acceptable values of β around the value of the coupling phase for which an analytical treatment is possible, such that for any value from this interval an in-phase state is stable. Our SG algorithm finds one of them, depending upon initial conditions.

5. Splay State and Cluster Synchronization

In this section we will consider unidirectionally coupled rings with $N = 6$ nodes. That is, the coupling matrix has the following form:

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let $1 \leq m \leq N-1$. Then $d = \mathrm{LCM}(m, N)/m$, where LCM denotes the least common multiple, is the number of different clusters of a synchronized solution. A splay state corresponds to $d = N$ while cluster states yield $d < N$. In order to extend the goal function Eq. (9) such that we can stabilize splay and cluster states, we define a generalized order parameter

$$R_d = \frac{1}{N}\left|\sum_{k=1}^{N} e^{di\varphi_k}\right|. \tag{11}$$

with $d \in \mathbb{N}$. However, if we derive a goal function from this order parameter in an analogous way as in Eq. (9), this function will not have a unique minimum at the d-cluster state because $R_d = 1$ holds also for the in-phase state and for other p-cluster states where p are divisors of d.

For example, suppose that the system has six nodes. Then states for which Eq. (11) yields $R_d = 1$ for $d = 6$, 3, and 2 hold are schematically depicted in Fig. 3(a),(b),(c), respectively. In order to distinguish between these three cases, let us consider the functions

$$f_p(\varphi) = \frac{1}{N^2}\sum_{j=1}^{N} e^{pi\varphi_j}\sum_{k=1}^{N} e^{-pi\varphi_k}. \tag{12}$$

A splay state (Fig. 3(a)) yields $f_1 = f_2 = f_3 = 0$, while in the 3-cluster state displayed in Fig. 3(b) we have $f_1 = f_2 = 0$, $f_3 = 1$, and in the 2-cluster-state shown in Fig. 3(c) $f_1 = f_3 = 0$, $f_2 = 1$. Hence, we obtain $\sum_p f_p = 0$ if and only if there is a state with d clusters, where the sum is taken over all divisors of d.

Combining all previous results we adopt the following goal function:

$$Q_d = 1 - f_d(\varphi) + \frac{N^2}{2}\sum_{p|d,1\leq p<d} f_p(\varphi), \tag{13}$$

where $p|d$ means that p is a factor of d. This goal function contains f_d as the primary contribution for the d-cluster state, but also a sum of penalty terms that

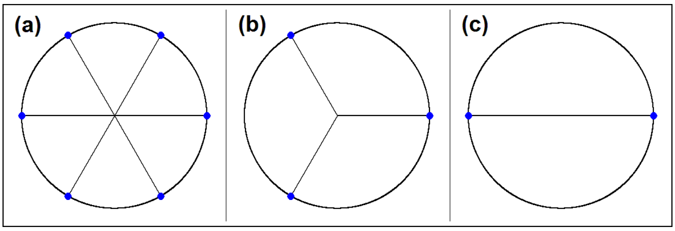

Figure 3. Schematic diagrams of splay ($d = 6$), three-cluster ($d = 3$), and two-cluster ($d = 2$) states in panels (a), (b), and (c), respectively ($N = 6$). Each cluster contains the same number of nodes.

counteract reaching other cluster states in which f_d is also unity. Whenever one of those unwanted cluster states is approached, the penalty term will lead to a gradient away from it. The prefactor $N^2/2$ is chosen for convenience to secure faster convergence of the algorithm. From $\dot{\beta} = -\Gamma_\beta \frac{\partial}{\partial\beta}\dot{Q}_d$ one can derive the adaptation law

$$\dot{\beta} = -\Gamma_\beta K \sum_{j=1}^{N}\sum_{k=1}^{N}\left\{\sum_{p|d,1\leq p<d} p\sin[p(\varphi_k - \varphi_j)] - \frac{2d}{N^2}\sin[d(\varphi_k - \varphi_j)]\right\} \times \sum_{n=1}^{N} a_{jn}\left[\frac{r_{n,\tau}}{r_j}\cos(\beta + \varphi_{n,\tau} - \varphi_j) - \cos(\beta)\right]. \quad (14)$$

In Fig. 2 we show the results of a numerical simulation for splay state stabilization ($d = N = 6$, $m = 1$). The phase differences are $\Delta\phi_j = \varphi_j - \varphi_{j+1} = 2\pi - 2\pi/N$, which corresponds to the splay state. In Fig. 2(c) one can see that the adaptively obtained value of β converges to that for which stability was shown analytically in Ref. 14 (dashed blue line).

Figures 4 and 5 depict the results of numerical simulations for two clusters ($d = 2$, $m = 3$) and three clusters ($d = 3$, $m = 2, 4$), respectively. Again we note that the obtained value of β comes close to the one for which stability was shown analytically in Ref. 14.

The above results indicate that the speed-gradient method is able to drive the network dynamics into the desired cluster or splay state by adaptively adjusting the coupling phase, where the goal function is chosen according to the corresponding target state. We have, however, used only exemplary values of the coupling parameters K and τ so far.

For the example of a splay state (4-cluster) in a network of 4 Stuart-Landau oscillators coupled in a unidirectional ring we have conducted a more exhaustive analysis of the (K, τ) plane. Figure 6 shows results in dependence on the coupling strength K and the coupling delay τ. According to Ref. 14 there exists an optimal value of the coupling phase that enables stability of this state for arbitrary values of K and τ. We ran simulations with 20 different initial conditions chosen randomly from the complex interval $[-1, 1] \times [-i, i]$ for each oscillator z_j. Figure 6 shows the

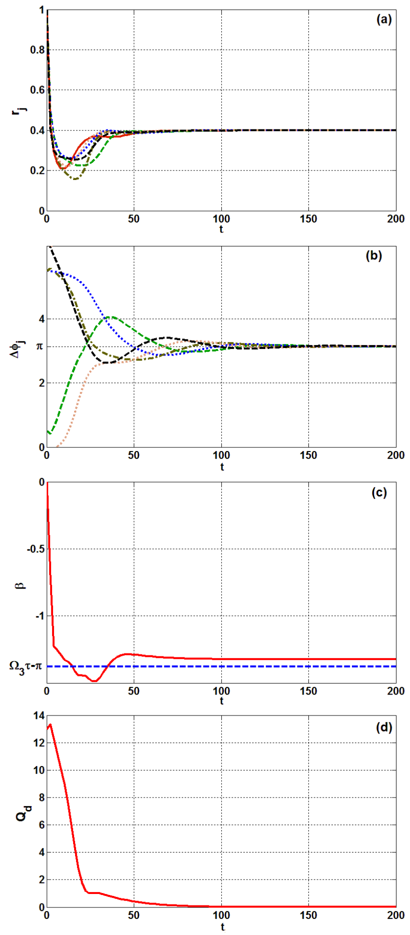

Figure 4. (Color online) Adaptive control of 2-cluster state ($m = 3$) with goal function Eq. (13). (a): absolute values $r_j = |z_j|$; (b): phase differences $\Delta\phi_j = \varphi_j - \varphi_{j+1}$; (c): temporal evolution of β, blue dashed line: reference value for $\Omega_3 = 1.08$; (d): goal function. Other parameters as in Fig. 1.

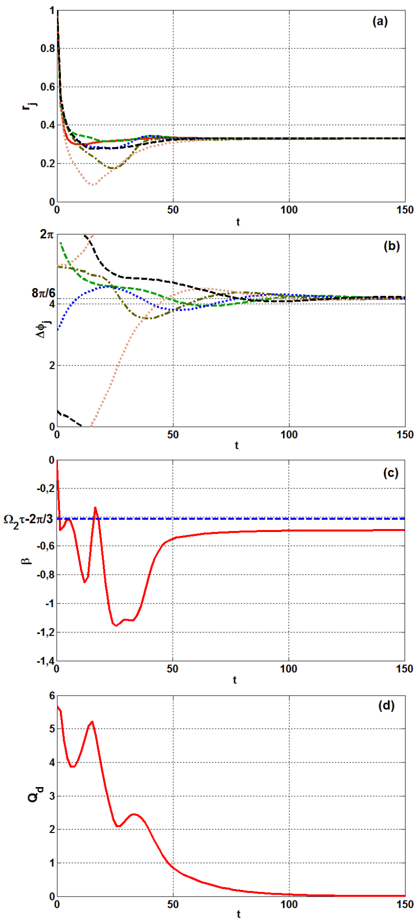

Figure 5. (Color online) Adaptive control of 3-cluster state ($m = 2, 4$) with goal function Eq. (13). (a): absolute values $r_j = |z_j|$; (b): phase differences $\Delta\phi_j = \varphi_j - \varphi_{j+1}$, blue dashed line: reference value for $\Omega_2 = 1.03$; (c): temporal evolution of β; (d): goal function. Other parameters as in Fig. 1.

fraction f_c of those realizations that asymptotically approach a splay state after applying the speed-gradient method. We observe that the speed-gradient method is able to control the splay state in a wide parameter range. The range of possible coupling strengths K does, however, shrink considerably with increasing time delay τ. We conjecture several reasons for this shrinking. Firstly, multistability of different splay and cluster states is more likely for larger values of K and τ, which narrows down the basin of attraction for a given state. Secondly, Eq. (14), which describes the dynamics of the coupling phase under the adaptive control, is influenced by the

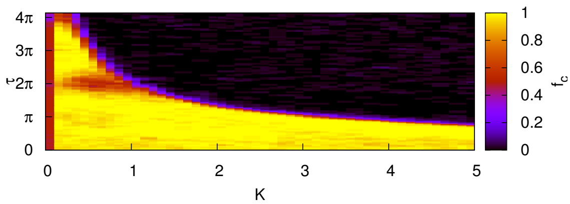

Figure 6. (Color online) Success of the speed-gradient method in dependence on the coupling parameters K and τ for the splay state in a unidirectionally coupled ring of $N = 4$ Stuart-Landau oscillators. Other parameters as in Fig. 1. The color code shows the fraction of successful realizations.

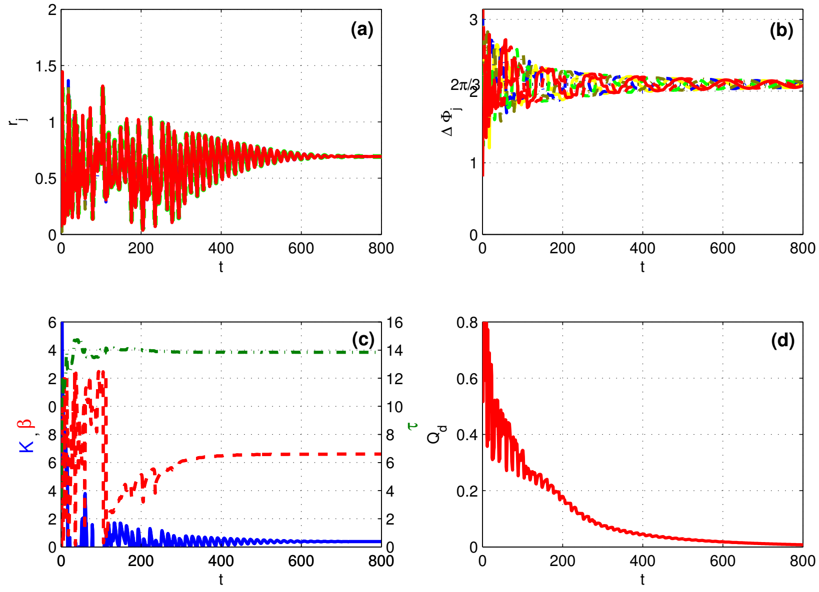

Figure 7. (Color online) Adaptive control of 3-cluster state in a network of 6 nodes by simultaneously tuning K, β and τ according to Eqs. (14), (16), and (15), respectively. (a): absolute values $r_j = |z_j|$; (b): phase differences $\Delta\phi_j = \varphi_j - \varphi_{j+1}$; (c): temporal evolution of control parameters β (dashed red line), K (solid blue line), τ (dashed-dotted green line); (d): goal function. $\Gamma_\beta = \Gamma_K = 10$, $\Gamma_\tau = 0.05$. Other parameters as in Fig. 1.

time delay τ. Using large delay times, we observe overshoots of the control leading to a failure.

6. Controlling Several Parameters Simultaneously

The general form of the SG method as given in Eq. (6) is also suitable for controlling more than one parameter. In this section, this is demonstrated by controlling β,

K, and τ simultaneously. The vector u in Eq. (6) is then given by $u = (\beta, K, \tau)$. We choose Γ as a diagonal matrix with the diagonal elements $\Gamma_{11} \equiv \Gamma_\beta$, $\Gamma_{22} \equiv \Gamma_K$ and $\Gamma_{33} \equiv \Gamma_\tau$. Using the goal function Q_d of Eq. (13) we obtain for β the adaptive algorithm given by Eq. (14). For $\dot{K} = -\Gamma_K \frac{\partial}{\partial K} \dot{Q}_d$ we obtain:

$$\dot{K} = -\Gamma_K \sum_{j=1}^{N} \sum_{k=1}^{N} \left\{ \sum_{p|d, 1 \leq p < d} p \sin[p(\varphi_k - \varphi_j)] - \frac{2d}{N^2} \sin[d(\varphi_k - \varphi_j)] \right\} \times \sum_{n=1}^{N} a_{jn} \left[\frac{r_{n,\tau}}{r_j} \sin(\beta + \varphi_{n,\tau} - \varphi_j) - \sin(\beta) \right]. \tag{15}$$

and for $\dot{\tau} = -\Gamma_\tau \frac{\partial}{\partial \tau} \dot{Q}_d$

$$\dot{\tau} = -\Gamma_\tau \sum_{j=1}^{N} \sum_{k=1}^{N} \left\{ \sum_{p|d, 1 \leq p < d} p \sin[p(\varphi_k - \varphi_j)] - \frac{2d}{N^2} \sin[d(\varphi_k - \varphi_j)] \right\} \times \sum_{n=1}^{N} a_{jn} \left[-\frac{\dot{r}_{n,\tau}}{r_j} \sin(\beta + \varphi_{n,\tau} - \varphi_j) - \dot{\varphi}_{n,\tau} \frac{r_{n,\tau}}{r_j} \cos(\beta + \varphi_{n,\tau} - \varphi_j) \right]. \tag{16}$$

Figure 7 shows the successful control of a 3-cluster state in a network consisting of 6 nodes where appropriate values of β, K, and τ are found adaptively.

7. Conclusion

We have proposed a novel adaptive method for the control of synchrony on oscillator networks, which combines time-delayed coupling with the speed gradient method of control theory. Choosing an appropriate goal function, a desired state of generalized synchrony can be selected by the self-adaptive automatic adjustment of a control parameter, i.e., the coupling phase. This goal function, which is based on a generalization of the Kuramoto order parameter, vanishes for the desired state, e.g., in-phase, splay, or cluster states, irrespectively of the ordering of the nodes. By numerical simulations we have shown that those different states can be stabilized, and the coupling phase converges to an optimum value. We have elaborated on the robustness of the control scheme by investigating the success rates of the algorithm in dependence on the coupling parameters, i.e., the coupling strength and the time delay. The input variable u in Eq. (3) may in general contain all of the coupling parameters. As an example, we have applied our method simultaneously to the coupling phase, amplitude, and the time delay. In this way control of cluster and splay synchronization is possible without any a priori knowledge of the coupling parameters. Given the paradigmatic nature of the Stuart-Landau oscillator as a generic model, we expect broad applicability, for instance to synchronization of networks in medicine, chemistry or mechanical engineering. The mean-field nature of our goal function makes our approach accessible even for very large networks independently of the particular topology.

Acknowledgments

This work is supported by the German-Russian Interdisciplinary Science Center (G-RISC) funded by the German Federal Foreign Office via the German Academic Exchange Service (DAAD). JL, TD, PH, and ES acknowledge support by Deutsche Forschungsgemeinschaft (DFG) in the framework of SFB 910. PH acknowledges support by the BMBF under the grant no. 01GQ1001B (Förderkennzeichen).

References

1. E. Schöll and H. G. Schuster, *Handbook of Chaos Control*, (Wiley-VCH, Weinheim, 2008).
2. K. Pyragas, *Phys. Lett. A* **170**, 421 (1992).
3. J. E. S. Socolar, D. W. Sukow and D. J. Gauthier, *Phys. Rev. E* **50**, 3245 (1994).
4. V. Pyragas and K. Pyragas, *Phys. Rev. E* **73**, 036215 (2006).
5. A. L. Fradkov, *Autom. Remote Control* **40**, 1333 (1979).
6. A. L. Fradkov, *Physics-Uspekhi* **48**, 103 (2005).
7. A. L. Fradkov, *Cybernetical Physics: From Control of Chaos to Quantum Control* (Springer, Heidelberg, Germany, 2007).
8. J. Lehnert, P. Hövel, V. Flunkert, P. Y. Guzenko, A. L. Fradkov and E. Schöll, *Chaos* **21**, 043111 (2011).
9. M. Kehrt, P. Hövel, V. Flunkert, M. A. Dahlem, P. Rodin and E. Schöll, *Eur. Phys. J. B* **68**, 557 (2009).
10. P. Hövel, M. A. Dahlem and E. Schöll, *Int. J. Bifur. Chaos* **20**, 813 (2010).
11. V. Flunkert, S. Yanchuk, T. Dahms and E. Schöll, *Phys. Rev. Lett.* **105**, 254101 (2010).
12. I. Omelchenko, Y. L. Maistrenko, P. Hövel and E. Schöll, *Phys. Rev. Lett.* **106**, 234102 (2011).
13. G. Brown, C. M. Postlethwaite and M. Silber, *Physica D* **240**, 859 (2011).
14. C. U. Choe, T. Dahms, P. Hövel and E. Schöll, *Phys. Rev. E* **81**, 025205(R) (2010).
15. C. U. Choe, T. Dahms, P. Hövel and E. Schöll, in *Proc. 8th AIMS Int. Conf. on Dyn. Systems, Diff. Eqs. and Appl.* (American Institute of Mathematical Sciences, Springfield, MO, USA, 2011), pp. 292–301, DCDS Supplement Sept. 2011.
16. S. Schikora, P. Hövel, H. J. Wünsche, E. Schöll and F. Henneberger, *Phys. Rev. Lett.* **97**, 213902 (2006).
17. V. Flunkert and E. Schöll, *Phys. Rev. E* **76**, 066202 (2007).
18. B. Fiedler, V. Flunkert, M. Georgi, P. Hövel and E. Schöll, *Phys. Rev. Lett.* **98**, 114101 (2007).
19. S. Schikora, H. J. Wünsche and F. Henneberger, *Phys. Rev. E* **83**, 026203 (2011).
20. V. Flunkert and E. Schöll, *Phys. Rev. E* **84**, 016214 (2011).
21. V. Flunkert and E. Schöll, *New. J. Phys.* (2012), arXiv:1111.5131v1.
22. K. Pyragas and T. Pyragiene, *Phys. Rev. E* **78**, 046217 (2008).
23. J. Lehnert, T. Dahms, P. Hövel and E. Schöll, *EPL* **96**, 60013 (2011).
24. A. A. Selivanov, J. Lehnert, T. Dahms, P. Hövel, A. L. Fradkov and E. Schöll, *Phys. Rev. E* **85**, 016201 (2012).
25. C. Hauptmann, O. Omel‘chenko, O. V. Popovych, Y. L. Maistrenko and P. A. Tass, *Phys. Rev. E* **76**, 066209 (2007).
26. L. M. Pecora and T. L. Carroll, *Phys. Rev. Lett.* **80**, 2109 (1998).