Passification-based adaptive control: uncertain input and output delays

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Abstract

For a class of uncertain systems we analyze passification-based adaptive controller in the presence of small, unavoidable input and output time-varying delays as may be present in controller implementation. We derive upper bounds for time delays such that in some domain of initial conditions the states of the closed-loop system tend to zero, whereas an adaptive controller gain tends to a constant value. The results are semi-global, that is the domain of initial conditions is bounded but can be made arbitrary large by tuning an appropriate controller parameter. For the first time, we apply an adaptive controller to linear uncertain networked control systems, where sensors, controllers, and actuators exchange their information through communication networks. The efficiency of the results is demonstrated by the example of adaptive network-based control of an aircraft.

Key words: Adaptive control; passification method; input and output delay; networked control systems.

1 Introduction

In this paper we consider passification-based adaptive controller, which proved to be efficient for stabilization of delay-free systems. As it has been shown in [5], any hyper-minimum-phase linear time-invariant system can be stabilized by a static output feedback $u(t) = -ky(t)$ if $k$ is large enough (for more established description see [1]). For the case of uncertain systems an adaptive version of this controller has been derived via the speed gradient method [7].

While applying adaptive controller it is important to take into account unknown unavoidable input/output delays, which is a challenging problem [24,15]. Most of the existing works on adaptive control deal only with state delays, e.g. [17,18,3,25] to name a few. Adaptive controllers for linear systems with full state measurements and a constant input delay have been proposed and analyzed in [23,4]. Passification-based adaptive output-feedback controller with a constant input delay has been studied in [19].

Note that for linear time-invariant systems with constant time-delays there is almost no difference between an input and output delay, since the transfer function is the same. A more challenging problem is adaptive stabilization with time-varying delays, where input and output delays should be treated separately. A possible way to approach this problem is to assume that the difference between current and delayed signal is small enough [2,20], but this assumption is restrictive and difficult to verify.

In the present paper we suggest a simple adaptive output-feedback controller that stabilizes hyper-minimum-phase systems with input and output time-varying delays. Namely, we derive upper bounds on the time-delays such that in a given domain of initial conditions the states of the closed-loop system tend to zero, whereas an adaptive controller gain tends to a constant value. By changing a particular controller parameter the domain of acceptable initial conditions can be made arbitrary large leading to semi-global stability (see Remark 2). Moreover, we consider fast-varying delays (without any constraints on the delay-derivatives). This allows to apply, for the first time, an adaptive controller to linear uncertain networked control systems, where variable sampling intervals and communication delays...
are taken into account (see Section 4). Some preliminary results (without input delays) have been presented in [22].

**Notations:** Throughout the paper the superscript “T” stands for matrix transposition, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with vector norm $\| \cdot \|$, $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices, the notation $P > 0$ for $P \in \mathbb{R}^{n \times n}$ means that $P$ is symmetric and positive definite, $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ stand for the minimum and maximum eigenvalues of the matrix $P$, respectively. The symmetric elements of the symmetric matrix will be denoted by $\ast$. The set $\{0, 1, 2, \ldots \}$ is denoted by $\mathbb{Z}_+$. 

2 Preliminaries and problem formulation

2.1 Preliminaries: Passification method

For non-delay linear time-invariant systems passification method and the corresponding design of an adaptive controller are based on Passification lemma that we state below.

**Definition 1** For given matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$, $g \in \mathbb{R}^{1 \times 1}$ a transfer function $g^T W(s) = g^T C(sI - A)^{-1} B$ is called hyper-minimum-phase (HMP) if the polynomial $\phi(s) = g^T W(s) \det(sI - A)$ is Hurwitz and $g^T CB$ is a positive number.

**Lemma 1 (Passification lemma) [6]** Let the matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$, $g \in \mathbb{R}^{1 \times 1}$ be given. Then for existence of $P \in \mathbb{R}^{n \times n}$ and $k_+ \in \mathbb{R}$ such that

$$P > 0, \quad PA_* + A_*^T P < 0, \quad PB = C^T g,$$

where $A_* = A - Bk_* g^T C$, it is necessary and sufficient that the function $g^T W(s) = g^T C(sI - A)^{-1} B$ is HMP.

An appropriate value for $k_+$ in Lemma 1 is any positive number such that

$$k_+ > - \inf_{\omega \in \mathbb{R}} \text{Re} \left\{ (g^T W(i\omega))^{-1} \right\}. \quad (2)$$

See [1] for more details on Passification method.

2.2 Problem formulation

Consider an uncertain linear system

$$\dot{x}(t) = A_\xi x(t) + Bu(t - r_1(t)), \quad x(0) = x_0,$$
$$y(t) = Cx(t - r_2(t)),$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, $y \in \mathbb{R}^l$ is the measurable output; $A_\xi$ is an uncertain matrix that resides in the polytope

$$A_\xi = \sum_{i=1}^N \xi_i A_i, \quad 0 \leq \xi_i \leq 1, \quad \sum_{i=1}^N \xi_i = 1. \quad (4)$$

The delays $r_1(t), r_2(t)$ are supposed to be unknown and bounded:

$$0 \leq r_1(t) \leq h_1, \quad 0 \leq r_2(t) \leq h_2.$$

We set $x(t) = 0$ for $t < 0$. This does not affect the solution $x(t)$ and implies that $y(t) = 0$ if $t - r_2(t) < 0$.

Denote

$$r(t) = r_1(t) + r_2(t - r_1(t)). \quad (5)$$

The quantity $r(t)$ is the overall delay of the closed-loop system. Clearly

$$r(t) \leq h_1 + h_2 \leq h.$$

If $t - r(t) < 0$ then the system (3) is in the open-loop since it has not received a signal from the controller. Therefore, a special analysis is needed on the intervals where $t - r(t) < 0$. Following [16] we assume

**Assumption 1** There exists a unique $t_\ast > 0$ such that

$$\begin{cases}
    t - r(t) < 0, \quad t < t_\ast, \\
    t - r(t) \geq 0, \quad t \geq t_\ast.
\end{cases}$$

Assumption 1 has a simple physical meaning: the system (3) starts to receive signals from the controller at time $t_\ast$. It is clear that $t_\ast \leq h$. Assumption 1 is always satisfied for slowly-varying delays with $\dot{r}(t) \leq 1$ (since $t - r(t)$ is increasing) and for networked control systems as considered in Section 4.

Similar to [1,6] we assume

**Assumption 2** There exists a known $g \in \mathbb{R}^l$ such that $g^T C(sI - A_\xi)^{-1} B$ is HMP for all $A_\xi$ from (4).

For a given $g$ satisfying Assumption 2 we consider the adaptive controller

$$u(t) = -k(t) g^T y(t),$$
$$\dot{k}(t) = \gamma^{-2} \left( g^T y(t) \right)^2,$$

where $k, \gamma \in \mathbb{R}, \gamma > 0$.

For $r_1(t) = r_2(t) \equiv 0$ under Assumption 2 it has been shown in [1] that solutions of the closed-loop system (3), (6) satisfy the following property: for all $k(0) \in \mathbb{R}$

$$\lim_{t \to \infty} \|x(t)\| = 0, \quad \lim_{t \to \infty} k(t) = \text{const}. \quad (7)$$
Our objective is to derive conditions ensuring (7) for non-zero delays and for a certain choice of $k(0)$.

3 Main results

The closed-loop system (3), (6) can be presented in the form

$$
\dot{x}(t) = A_x x(t) - k_* B g^T C x(t - r(t)) + (k_* - k(t - r_1(t))) B g^T C x(t - r(t)),
\dot{k}(t) = \gamma^{-2} \left( g^T C x(t - r_2(t)) \right)^2
$$

with $k(t) = k(0)$ for $t < 0$. Note that here $\dot{x}(0)$ and $\dot{k}(.)$ denote right-hand side derivatives.

The idea of passification-based approach is the following. Under Assumption 2 there exist $P > 0, k_*$ that satisfy (1). Consider a Lyapunov-like function

$$
V_0(x, k) = x^T P x + \gamma^2 (k - k_*)^2.
$$

Its derivative along the trajectories of (8) has the form

$$
\dot{V}_0 = 2 x^T (t) P \left[ A_x x(t) - k_* B g^T C x(t - r(t)) \right]
+ 2 (k_* - k(t - r_1(t))) x^T (t) P B g^T C x(t - r(t))
+ 2 (k(t) - k_*) \left( g^T C x(t - r_2(t)) \right)^2.
$$

For $r_1(t) = r_2(t) \equiv 0$ the last two terms can be canceled because $P B = C^T g$. Hence, (1) guarantees that $\dot{V}_0 \leq -\varepsilon ||x||^2$ for some $\varepsilon > 0$. The latter implies (7) (see [1]).

Remark 1 Note that the above arguments can be easily extended to systems with state delays. Consider the system

$$
\dot{x}(t) = A_x x(t) + A_1 x(t - r(t)) + B u(t),
y(t) = C x(t).
$$

Here $x, u, y$ are defined as previously and all matrices are constant with appropriate dimensions. Calculating $\dot{V}_0$ we obtain:

$$
\dot{V}_0 = 2 x^T (t) P \left[ A_x x(t) + A_1 x(t - r(t)) \right]
- k_* B g^T C x(t)] + 2 (k_* - k(t)) x^T (t) P B g^T C x(t)
+ 2 (k(t) - k_*) (g^T y(t))^2.
$$

Since $P B = C^T g$, the last two terms can be canceled and the Lyapunov-based analysis of (10) is reduced to the standard one for linear time-delay systems under $u(t) = -k_* g^T y(t)$. In the case of an input/output time-varying delay such a cancelation is not possible since the controller does not measure the current value of the state. Therefore, adaptive control of systems with input/output delays is much more challenging than the one under state delays.

As already mentioned, if $r_1(t) \neq 0$ or $r_2(t) \neq 0$ the cancelation of the last two terms in (9) is not possible because $x$ and $k$ depend on different time instants. Note that the right-hand side of (9) can be considered as a quadratic form in $x(t), x(t - r(t)),$ and $x(t - r_i(t)) (i = 1, 2)$, where matrices depend on the following time-varying parameters:

$$
a = k_* - k(t),
b = k_* - k(t - r_1(t)),
c = k(t) - k(t - r_1(t)).
$$

Consider the following Lyapunov-Krasovskii functional:

$$
V(x, \dot{x}, k) = V_0(x, k) + V_S(\dot{x}) + V_R(\dot{x}),
$$

where $x_t(\theta) = x(t + \theta), \theta \in [-h, 0]$, $h = h_1 + h_2$, and

$$
V_S(x_i) = \int_{-h}^0 x_i^T (s) S x_i(s) ds, \quad S > 0,
V_R(\dot{x}_i) = h \int_{-h}^0 \dot{x}_i^T (s) R \dot{x}_i(s) ds d\theta, \quad R > 0.
$$

Here $V_S$ and $V_R$ are standard terms for systems with fast-varying delays [11,13]. Our goal is to derive conditions that ensure $V \leq -\varepsilon ||x||^2$ for some $\varepsilon > 0$ if $\left| a \right| \leq M_* \cdot \left| b \right| \leq M_* \cdot \left| c \right| \leq M_1$ (13) for $t \geq 0$, where $M_* \cdot M_1$ are some fixed bounds. Further we will show that (13) is valid for $t \geq 0$ if one choose appropriate values of $k(0)$ and $\gamma$.

We are in a position to formulate our main result.

Theorem 1 Let Assumptions 1 and 2 hold. Given $h > 0$ and tuning parameters $M_* > 0, M_1 > 0, k_* > 0$, let there exist $n \times n$ matrices $P > 0, S > 0, R > 0, G_1, G_2, G_3$ such that the following relations hold:

$$
H_i(a, b, c) \bigg|_{a \pm M_* \cdot b \pm M_* \cdot c \pm M_1} < 0, \quad i = 1, \ldots, N,
$$

$$
P B = C^T g, \quad \begin{pmatrix} R & G_j \\ G_j^T & R \end{pmatrix} \geq 0 (j = 1, 2, 3),
$$

where

$$
H_i(a, b, c) = 
\begin{pmatrix}
H_1(a) & H_2(a) & H_3(a) \\
H_1^T & H_3^T & H_4(a) \\
H_2(a) & H_4(a) & H_5(a)
\end{pmatrix}
$$
\[ H_1' = P[A_i - Bk_sg T C] + [A_i - Bk_sg T C]^T P + S, \]
\[ H_2(c) = cPBg T C, \]
\[ H_3 = k_hPBg T C, \]
\[ H_4(a) = k_hPBg T C - ahPBg T C, \]
\[ H_5(a) = ah^2PBg T C - h^2(G_2), \]
\[ H_7(b) = bb^2PBg T R - hh^2 G_3. \]

Assume additionally that
\[ h_1 \leq M_2 \lambda_{min}(P) \]
\[ M_2 ||g T C||^2 \quad (15) \]

Then for any \( \delta > 0 \) there exists \( \gamma > 0 \) such that for all initial conditions \( x_0, k(0) \) subject to
\[ ||x_0|| < \delta, \quad k(0) \in [k_s - M_s, k_s], \]
\[ \text{(16)} \]
solutions of the system (3), (6) satisfy the property (7).

**Proof.** See Appendix A.

**Remark 2** Conditions of Theorem 1 ensure semi-global results, where (7) is guaranteed for any \( \delta > 0 \) and \( x_0 \) with \( ||x_0|| < \delta \). It follows from the proof of Theorem 1 that an appropriate \( \gamma \) can be chosen in the inequality
\[ \delta^2 \leq \gamma^2 \min \left\{ \frac{M_2^2 \epsilon^{-2A_s t_s}}{\lambda_{min}(P)}, \frac{M_s}{c_1}, \frac{M_s^2 - c_s^2}{c_x} \right\}, \]
\[ \text{(17)} \]
where \( \Lambda_A = \max_i ||A_i|| \) and \( c_1, c_s, c_x \) are given in (A.2), (A.3), (A.6), correspondingly. If \( t_s \) in (17) is unknown, one should substitute a known upper bound for \( t_s \).

**Remark 3** Under Assumptions 1, 2 due to Passification lemma there exist \( P \) and \( k_s \) that satisfy (1). With these \( P \) and \( k_s \), relations (14) are feasible for \( M > 0 \) and \( M_s > 0 \) if \( h \) is small enough. Indeed, due to (A.8) \( H_5 \rightarrow 0 \) for \( h \rightarrow 0 \). The same is true for \( H_3, ..., H_7 \). Then by Schur complement theorem [12, p.318] it can be shown that \( \mathcal{H}_i < 0 \) for \( R = I, S = I, G_j = 0 \) \((j = 1, 2, 3) \). When \( h \rightarrow 0 \) allowable \( M_s \) and \( M_1 \) tend to infinity, therefore, our results recover the global results from [1] for delay-free case. Relations (14) give acceptable bounds \( h_1 \) and \( h_2 \) such that (7) holds for the closed-loop time-delay system (3), (6).

**Remark 4** There is a trade-off between enlarging of \( M_s \), \( M_1 \) and enlarging of the delay bounds \( h_1, h_2 \). The smaller \( h \) is, the larger \( M_s \) can be taken such that the LMIs (14) are feasible, i.e. a wider choice of adaptive gain is possible. Furthermore, given \( M_1, k_s \) that (14) are feasible for \( M_s \), they remain feasible with the same decision variables for all \( M_1 < M_s \). The latter means that the stability is guaranteed for the same \( h \) but for larger \( h_1 \) due to (15).

### 4 Network-based adaptive control

#### 4.1 Case study: adaptive control of networked control systems

In this section we apply passification-based adaptive control to networked control systems. Consider the uncertain system
\[ \dot{x}(t) = A_x x(t) + B u(t), \]
\[ y(t) = C x(t) \]
with several nodes (a sensor, a controller, and an actuator) that are connected via two communication networks: a sensor network (from the sensor to the controller) and a control network (from the controller to the actuator). Let \( s_k \) be the sequence of sampling instants:
\[ 0 = s_0 < s_1 < \ldots < s_k < \ldots, \quad k \in \mathbb{N}, \quad \lim_{k \to \infty} s_k = \infty. \]

At each sampling instant \( s_k \) the output \( y(t) \) is sampled and transmitted via the network to the controller with a variable delay \( \tau_{sc}^a \). Therefore, the updating instant time of the controller is \( \sigma_k = s_k + \tau_{sc}^a \). For simplicity, we assume that \( \sigma_k > \sigma_{k+1} \), that is the old sample cannot get to the destination after the most recent one. Then the controller has the form
\[ u(t) = -k(t)g T y(\sigma_k - \tau_{sc}^a), \]
\[ \dot{k}(t) = \gamma^2 (g T y(\sigma_k - \tau_{sc}^a))^2, \]
\[ t \in [\sigma_k, \sigma_{k+1}) \quad (19) \]

At the sampling instants \( \sigma_k \) the control signal is sampled and transmitted through the network to the Zero-Order Hold (ZOH) with a variable delay \( \tau_{zoh}^a \). Therefore, the updating instant time of the ZOH is \( t_k = s_k + \tau_{zoh}^a + \tau_{sc}^a \). We assume that \( t_k < t_{k+1} \) and there is a known MAD (maximum allowable delay) such that \( \tau_{sc}^a + \tau_{zoh}^a \leq \text{MAD} \).

Following the time-delay approach to sampled-data control [10,9], the resulting closed-loop system can be presented in the form
\[ \dot{x}(t) = A_x x(t) + B(k(t) - r_1(t))g T C x(t - r(t)), \]
\[ \dot{k}(t) = \gamma^2 (g T C x(t - r_2(t)))^2, \quad (20) \]
where
\[
\begin{align*}
r_1(t) &= t - t_k + \tau_{ca}^k, \quad t \in [t_k, t_{k+1}),
r_2(t) &= t - \sigma_k + \tau_{sc}^k, \quad t \in [\sigma_k, \sigma_{k+1}),
r(t) &= t - t_k + \tau_{ca}^k + \tau_{sc}^k, \quad t \in [t_k, t_{k+1}).
\end{align*}
\]
Here \( r(t) = r_1(t) + r_2(t - r_1(t)) \). Note that \( r(t) \) satisfies Assumption 1 with
\[
t_* = t_0 = \tau_{sc}^0 + \tau_{ca}^0 \leq \text{MAD}.
\]

Assume that
\[
\begin{align*}
t_{k+1} - t_k + \tau_{ca}^k &\leq h_1, \\
t_{k+1} - t_k + \tau_{ca}^k + \tau_{sc}^k &\leq h, \quad \forall k \in \mathbb{Z}^+.
\end{align*}
\]
Since (20) coincides with (8), the results of Theorem 1 provide bounds for the sampling intervals and network-induced delays. We illustrate this below by an example of network-based adaptive control of an aircraft.

### 4.2 Example: yaw angle control

As an example we apply our results to the following model of a lateral motion of an aircraft [8]:
\[
\begin{align*}
\dot{x}_1(t) &= a_1 x_1(t) + x_2(t) + b_1 u(t - r_1(t)), \\
\dot{x}_2(t) &= a_2 x_1(t) + a_3 x_2(t) + b_2 u(t - r_1(t)), \\
\dot{x}_3(t) &= x_2(t), \\
y_1(t) &= x_2(t - r_2(t)), \\
y_2(t) &= x_3(t - r_2(t)),
\end{align*}
\]
where \( x_2 \) and \( x_3 \) are the yaw angle and the yaw rate, respectively, and \( x_1 \) denotes the sideslip angle; \( u \) is the rudder angle; \( y_i \) are measurable outputs; \( a_i \) and \( b_i \) denote the aircraft model parameters. We suppose that the aircraft is controlled through a network, that is the closed-loop system has the form (20) with \( r_1(t), r_2(t) \) given by (21). Then Assumption 1 is satisfied with \( t_* \leq \text{MAD} \). Following [8] we take \( a_3 = 1.3, b_1 = 19/15, b_2 = 19 \) and suppose that \( a_1 \in [0.1, 1.5], a_2 \in [27, 52] \) are uncertain system parameters. Then for \( g = (1,1)^T \) the transfer function
\[
g^T W(s) = \frac{b_2 s^2 + (b_1 a_2 - b_2 a_1 + b_2) s + b_1 a_2 - b_2 a_1}{s(s^2 - (a_1 + a_3) s + a_1 a_3 - a_2)}
\]
is HMP, since for all \( a_1, a_2 \) from the given sets its numerator is a stable polynomial and \( b_2 > 0 \). Therefore, Assumption 2 is true. For \( M_0 = 5, M_1 = 0.4, k_* = 4.61 \), conditions of Theorem 1 are satisfied with \( b_1 = 4 \times 10^{-4}, h = 10^{-3}, \gamma = 25, \delta = 20 \). To illustrate Remark 4 we take \( M'_0 = 4 < M_0 \) with the same \( M_1, k_* \). This leads to the same \( h \) but larger \( h_1 = 6.4 \times 10^{-4} \).

### 5 Conclusion

For a class of uncertain hyper-minimum-phase systems we analyzed passification-based adaptive controller in the presence of unknown time-varying delays in the measurements and control input. If a delay-free system under the controller is such that the state tends to zero, whereas the adaptive controller gain tends to a constant value, then our results give an acceptable bound for time-delay such that this property is preserved within a given
domain of initial conditions. This domain of stability can be made arbitrary large by changing an appropriate parameter in the adaptation law. The results are applicable to networked control systems and provide acceptable bounds for transmission intervals and network-induced delays. This important application was demonstrated by an example of an aircraft that is adaptively controlled through a network. One of the directions for the future research is extension of the obtained results to the adaptive control of networks.

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References


A Proof of Theorem 1

We analyze the dynamics of (8) separately for $t \in [0, t_\ast]$ (A.1) and $t \in [t_\ast, \infty)$ (A.2).

A.1 State bound for $t \in [0, t_\ast]$

Assumption 1 implies $t - r(t) < 0$, therefore $\dot{x}(t) = A_\xi x(t)$. Thus,

$$\|x(t)\| \leq e^{\lambda_\xi t}\|x_0\|,$$  \quad (A.1)
where $\Lambda_A = \max_\xi \|A_\xi\|$. Note that
\[
\|A_\xi\| \leq \sum_i \xi_i \|A_i\| \leq \max_i \|A_i\|
\]
and for appropriate $\xi$, $\|A_\xi\| = \max_i \|A_i\|$. Therefore,
\[
\max_\xi \|A_\xi\| = \max_i \|A_i\|.
\]
Note that (A.1) is preserved for $t < 0$, hence $\|x(t - r_2(t))\| \leq e^{\Lambda_A(t-r_2(t))}\|x_0\| \leq e^{\Lambda_A t}\|x_0\|$. As a result, we have
\[
k(t) - k(0) = \gamma^{-2} \int_0^t \left( g^TCx(s - r_2(s)) \right)^2 ds \leq \frac{c_1\|x_0\|^2}{\gamma^2},
\]
where
\[
c_1 = \frac{\Lambda_G^2}{2\Lambda_A} (e^{2\Lambda_A t_*} - 1), \quad (A.2)
\]
with $\Lambda_G = \|g^TC\|$. By conditions of the theorem $k(0) \leq k_*$, therefore $k(t) - k_* \leq k(t) - k(0) \leq c_1 \gamma^{-2}\|x_0\|^2$. Equation (8) implies $k(t) \geq k(0)$, thus $k_* - k(t) \leq k_* - k(0)$. Finally, for $t \in [0, t_*)$ we have
\[
|k(t) - k_*| \leq c_* = \max\left\{k_* - k(0), c_1 \gamma^{-2}\delta^2\right\}, \quad (A.3)
\]
where the inequality $\|x(0)\| \leq \delta$ was used. Since the right-hand side of (8) is piecewise continuous, functions $x(t)$ and $k(t)$ are continuous for $t > 0$, therefore, (A.1) and (A.3) are valid for $t = t_*$. 

### A.2 The bound on $\dot{V}$ for $t \in [t_*, \infty)$ under (13)

Assumption 1 implies $t - r(t) \geq 0$, therefore, $\dot{x}(t)$ and $\dot{k}(t)$ do not depend on $x(t)$ with $t < 0$. Thus, we set $x(t) = x_0$ for $t < 0$ and consider $V$ given by (12) (see [16] for details).

Now we calculate the derivative of $V$ along the trajectories of (8) for $t \in [t_*, \infty)$. Denote
\[
\mu(t) = \frac{1}{h} \int_{t-r(t)}^{t-r_2(t)} \dot{x}(s) ds, \quad \nu(t) = \frac{1}{h} \int_{t-r_2(t)}^t \dot{x}(s) ds.
\]

Then
\[
\dot{V}_0 = x^T(t) [PA_* + A^T_* P] x(t) + 2x(t)^T k_* PB^T Ch(\mu(t) + \nu(t)) + 2\{k_* - k(t)\} x^T(t - r_2(t)) PB^T Cx(t - r_2(t)) - 2\{k_* - k(t)\} x^T(t - r_2(t)) PB^T Ch(\mu(t)) + 2\{k_* - k(t)\} h_\nu(t) x^T(t) PB^T Cx(t - r(t)) + 2\{k_* - k(t)\} \{g^T Cx(t - r_2(t))\}^2,
\]
where $A_* = A_k - k_* B g^TC$. Using the relation $PB = C^T g$ we find that
\[
\dot{V}_0 = x^T(t) [PA_* + A^T_* P] x(t) + 2x(t)^T k_* PB^T C(\mu(t) + \nu(t)) - 2\{k_* - k(t)\} h_\nu(t) x^T(t - r_2(t)) PB^T C(\mu(t)) + 2\{k_* - k(t)\} h_\nu(t) x^T(t) PB^T Cx(t - r(t)) + 2\{k_* - k(t)\} \{g^T Cx(t - r_2(t))\}^2.
\]

Further
\[
\dot{V}_S = x^T(t) S x(t) - x^T(t - h) S x(t - h), \quad \dot{V}_R = h^2 \dot{x}^T(t) R \dot{x}(t) - h \int_{t-h}^t \dot{x}^T(s) R \dot{x}(s) ds.
\]

Denote
\[
\alpha_1 = \frac{h - r(t)}{h}, \quad \alpha_2 = \frac{r(t) - r_2(t)}{h}, \quad \alpha_3 = \frac{r_2(t)}{h},
\]
\[
f_1(t) = \int_{t-h}^{t-r(t)} \dot{x}^T(s) ds R \int_{t-r(t)}^{t-r_2(t)} \dot{x}(s) ds, \quad f_2(t) = h^2 \mu^T(t) R \mu(t), \quad f_3(t) = h^2 \nu^T(t) R \nu(t).
\]

Using Jensen inequality [12, p.322] we have
\[
- h \int_{t-r(t)}^{t-r_2(t)} \dot{x}^T(s) R \dot{x}(s) ds = - h \int_{t-r(t)}^{t-r_2(t)} \dot{x}^T(s) R \dot{x}(s) ds - h \int_{t-r_2(t)}^t \dot{x}^T(s) R \dot{x}(s) ds \leq - \left[\frac{1}{\alpha_1} f_1(t) + \frac{1}{\alpha_2} f_2(t) + \frac{1}{\alpha_3} f_3(t)\right].
\]

Since $G_j$ ($j = 1, 2, 3$) are such that
\[
\begin{pmatrix} R & G_j \\ \ast & R \end{pmatrix} \geq 0
\]

7
it follows from Park’s theorem [21, Theorem 1] that

\[- \left[ \frac{1}{\alpha_1} f_1(t) + \frac{1}{\alpha_2} f_2(t) + \frac{1}{\alpha_3} f_3(t) \right] \leq - \left[ f_1(t) + f_2(t) + f_3(t) + 2g_1(t) + 2g_2(t) + 2g_3(t) \right],

where

\[
g_1(t) = h \int_{t-h}^{t-r(t)} \dot{x}^T(s) ds G_1 \mu(t),
\]

\[
g_2(t) = h \int_{t-h}^{t-r(t)} \dot{x}^T(s) ds G_2 \nu(t),
\]

\[
g_3(t) = h^2 \mu^T(t) G_3 \nu(t).
\]

Using representation \(x(t-r_2(t)) = x(t) - h \nu(t)\) we finally arrive at

\[
\dot{V} \leq \eta^T(t) W \eta(t) + h^2 \dot{x}^T(t) R \dot{x}(t),
\]

where

\[
\eta(t) = (x^T(t), x^T(t-r(t)), x^T(t-h)^T, \nu^T(t), \mu^T(t))^T,
\]

\[
W = \begin{pmatrix}
H_1 & H_2 & H_3 & H_4 \\
R & R & H_5 & -h G_1 \\
-R & -(S+R) & h G_2 & h G_1 \\
-R & -h^2 R & -h^2 R & H_6
\end{pmatrix},
\]

with \(a = k_s - k(t), c = k(t) - k(t-r_1(t))\), \(H_1 = P[A_{s} - B_{s} g^T C] + [A_{s} - B_{s} g^T C]^T P + S\). Substituting right-hand side of (8) instead of \(\dot{x}(t)\) into (A.4) and applying Schur complement theorem [12, p.318] we find that, if for \(i = 1, \ldots, M\)

\[
H_i(a, b, c) < 0,
\]

with \(a, b, c\) given by (11), then \(\exists \epsilon > 0: \dot{V}(t) \leq -\epsilon \|\eta(t)\|^2\), where \(V(t) = V(x_t, \dot{x}_t, k(t))\).

**A.3 Proof of (13) for \(t \geq 0\)**

Now we show that \(|a| \leq M_s, |b| \leq M_s, \text{ and } |c| \leq M_1\), what will guarantee negative definiteness of \(\dot{V}\). Using estimates for \(|k_s - k(t)|\) and \(|x(t)|\) on \(t \in [0,t_s]\), we calculate

\[V(t_s) \leq c_* \|x_0\|^2 + \gamma^2 c_*^2,
\]

where \(c_*\) is from (A.3) and

\[
c_* = \|P[e^{2A_{s} t_s} + S] \left( h - t_s + \frac{1}{2A_{s}} (e^{2A_{s} t_s} - 1) \right) + \frac{h \Lambda_{ARA}}{2A_{s}} \left( e^{2A_{s} t_s} t_s + \frac{1}{2A_{s}} (1 - e^{2A_{s} t_s}) \right) + (h - t_s) (e^{2A_{s} t_s} - 1) \right],
\]

\[\Lambda_{ARA} = \max_{\xi} \|A_{s}^T R A_{s}\|.
\]

By conditions of the theorem \(|k_* - k(0)| < M_s\). If \(\gamma\) is large enough then

\[\|x_0\|^2 \leq \frac{\gamma^2 M_s}{c_*^2}.
\]

Hence, \(c_* < M_s\). By increasing \(\gamma\) one can ensure that

\[\|x_0\|^2 \leq \gamma^2 (M_s^2 - c_*^2) c_*^{-1},
\]

what will guarantee

\[V(t_*^s) < \gamma^2 M_s^2.
\]

Now we show that \(V(t) < \gamma^2 M_s^2\) for \(t \in [t_s, \infty)\). Let \(t_1 = \min \{t \in [t_*^s, \infty) | V(t) = \gamma^2 M_s^2\}\). Then for \(s \in [t_*^s, t_1]\) we have

\[V(s) \leq \gamma^2 M_s^2 \Rightarrow \begin{cases}
|k_* - k(s)| \leq M_s \\
\|x(s)\|^2 \leq \gamma^2 M_s^2 \lambda_{\min}^{-1}(P).
\end{cases}
\]

Since \(c_* < M_s\), (A.3) implies \(|k_* - k(t)| \leq M_s\) for \(t \leq t_1\). We require \(\gamma\) to be large enough to ensure

\[\|x_0\|^2 \leq \gamma^2 M_s^2 e^{-2A_{s} t_1} \lambda_{\min}^{-1}(P).
\]

In this case (A.1) and (A.7) guarantee that \(|x(t)|^2 \leq \gamma^2 M_s^2 \lambda_{\min}^{-1}(P)\) for \(t \leq t_1\). Thus, for \(t \leq t_1\)

\[|c| = |k(t - r_1(t)) - k(t)| \leq \int_{t-r_1(t)}^{t} \gamma^{-2} (g^T C x(s - r_1(s)))^2 ds \leq h_1 M_s^2 A_{s}^2 \lambda_{\min}^{-1}(P).
\]

As a result, for \(t \leq t_1\) we have:

\[|k_* - k(t)| \leq M_s, \quad |k_* - k(t - r_1(t))| \leq M_s, \quad |k(t - r_1(t)) - k(t)| \leq M_1.
\]

In this case conditions of the theorem guarantee (A.5) for \(t \leq t_1\). Therefore \(\exists \epsilon > 0: \dot{V}(t) \leq -\epsilon \|\eta(t)\|^2\). Since \(V(t_1) = V(t_1) + \int_{t_1}^{t} \dot{V}(s) ds\), we have \(V(t_1) \leq V(t_1) \leq \gamma^2 M_s^2\). The latter contradicts to \(V(t_1) = \gamma^2 M_s^2\), that is \(t_1\) does not exist and, therefore, \(V(t) < \gamma^2 M_s^2\) for \(t \in [t_*^s, \infty)\) and (A.9) are valid for \(t \geq 0\).

**A.4 Proof of (7)**

We have proved that \(\exists \epsilon > 0: \dot{V}(t) \leq -\epsilon \|\eta(t)\|^2\) for \(t \geq t_*\). Since \(V(t)\) is a nonnegative decreasing function,
it has a finite limit: \( \lim_{t \to \infty} V(t) < \infty \). Thus,

\[
\lim_{t \to \infty} V(t) = V(t_*) + \int_{t_*}^{\infty} \dot{V}(s) \, ds \\
\leq V(t_*) - \varepsilon \int_{t_*}^{\infty} \|\eta(s)\|^2 \, ds.
\]

Therefore, \( \varepsilon \int_{t_*}^{\infty} \|\eta(s)\|^2 \, ds < \infty \), i.e. \( \int_{t_*}^{\infty} \|x(s)\|^2 \, ds < \infty \). Boundedness of \( V \) implies boundedness of \( x(t) \) and \( k(t) \). Therefore, \( \dot{x}(t) \) given by (8) is bounded and \( x(t) \) is uniformly continuous. Then from Barbalat’s lemma [14, Lemma 8.2] we have \( \|x(t)\| \to 0 \). Moreover, \( \int_{t_*}^{\infty} \|\eta(s)\|^2 \, ds < \infty \) implies that there exists a finite limit

\[
\lim_{t \to \infty} k(t) = k(t_*) + \gamma^{-2} \int_{t_*}^{\infty} (g^T C x(s - r_2(s)))^2 \, ds,
\]

that is \( k(t) \) tends to a constant value.