Adaptive time-delayed stabilization of steady states and periodic orbits

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We derive adaptive time-delayed feedback controllers that stabilize fixed points and periodic orbits. First, we develop an adaptive controller for stabilization of a steady state by applying the speed-gradient method to an appropriate goal function and prove global asymptotic stability of the resulting system. By an example we show that the advantage of the adaptive controller over the nonadaptive one is in a smaller controller gain. Second, we propose adaptive time-delayed algorithms for stabilization of periodic orbits. Their efficiency is confirmed by local stability analysis. Numerical examples demonstrate the applicability of the proposed controllers.

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I. INTRODUCTION

In recent times the stabilization of unstable periodic orbits and chaotic systems has received considerable interest in applied nonlinear science [1]. Starting with the work of Ott, Grebogi, and Yorke [2] a variety of methods have been developed in order to stabilize unstable periodic orbits embedded in a chaotic attractor. A particularly simple and efficient scheme is the time-delayed feedback control suggested by Pyragas [3]. The idea of this method is to apply a linear feedback $u(t) \in \mathbb{R}^n$ of the form

$$u(t) = k(x(t) - x(t - \tau)), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, τ is the period of the periodic orbit being stabilized, and k is the feedback gain. The advantage of this approach is that it requires only the knowledge of the period τ and an appropriate value of the control gain k, i. e., the periodic orbit itself need not to be known. Some theoretical stability conditions for the Pyragas method have been obtained in [4–9]. These results are based on the analysis of the Floquet exponents of the system linearized near the unstable periodic orbit of interest.

The feedback law given by Eq. (1) can also be used to stabilize fixed points of dynamical systems. As it has been shown in Ref. [10] time-delayed feedback control (1) with appropriate values of k and τ can stabilize the steady state of a linear two-dimensional system. Stabilizing unstable fixed points by means of time-delayed feedback control has been the subject of several theoretical and experimental studies: Rosenblum and Pikovsky discussed the stabilization of unstable fixed point in neural systems [11]. Experimental realizations include the control of electrochemical systems [12, 13] and lasers [14–16].

Thus, both the stabilization of periodic orbits as well as of fixed points requires the knowledge of appropriate values of k and τ. To find these stabilizing values of k and τ adaptive approaches are useful. In Ref. [17] the delay time to control an unstable periodic orbit was obtained by a gradient method and stability was investigated by calculating the Lyapunov exponents of the linearized system. In Ref. [18] the speed-gradient method (see Ref. [19]) was applied to a simple goal function, and an adaptive algorithm that finds an appropriate value of k was derived. By numerical simulations and a local stability analysis it was shown that this algorithm ensures local stability of the system. Another adaptive algorithm was applied to cluster synchronization in delay-coupled networks [20, 21].

The contribution of this work is the following. Firstly, we show that by choosing an appropriate goal function and applying the speed-gradient method one obtains a tuning algorithm for the control gain k that ensures global asymptotic stability of the origin of the closed-loop system. The only necessary condition for the algorithm is that an appropriate value of k exists. Secondly, we propose adaptive algorithms that stabilize periodic orbits in linear systems.

II. SPEED-GRADIENT METHOD

In this section we briefly review an adaptive control scheme called *speed-gradient method* [19]. Consider a general nonlinear dynamical system

$$\dot{x} = F(x, k, t) \tag{2}$$

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with a state vector $x \in \mathbb{R}^n$, control parameters $k \in \mathbb{R}^m$, and a nonlinear function F. Define a control goal

$$\lim_{t\to\infty} Q(x(t),t) = 0, \tag{3}$$

where $Q(x,t) \geq 0$ is a smooth scalar goal function.

In order to design a control algorithm the scalar function $\dot{Q} = \omega(x,k,t)$ is calculated, that is, the speed (rate) at which $Q(x(t),t)$ is changing along trajectories of Eq. (2):

$$\omega(x,k,t) = \frac{\partial Q(x,t)}{\partial t} + [\nabla_x Q(x,t)]^T F(x,k,t).$$

Then we tune k according to

$$\dot{k} = -\Gamma \nabla_k \omega(x,k,t), \tag{4}$$

where $\Gamma = \Gamma^T > 0$ is a positive definite gain matrix. The algorithm (4) is called *speed-gradient algorithm* since it suggests to change k proportionally to the gradient of the speed of change of Q. There exist different analytic conditions guaranteeing that the control goal (3) can be achieved in the system (2), (4) (see [22, 23] for details). The main condition is: existence of a constant value of the parameter k_* ensuring attainability of the goal in the system $dx/dt = F(x,k_*,t)$.

The idea of this algorithm is the following. The term $-\nabla_k \omega(x,k,t)$ points in the direction in which the value of $\dot{Q}$ decreases with the highest speed. Therefore, if one forces the control signal to "follow" this direction, the value of $\dot{Q}$ will decrease and finally be negative. If $\dot{Q} < 0$ then Q will decrease and, eventually, tend to zero.

III. STABILIZATION OF A STEADY STATE

In this section we consider a linear system

$$\dot{x}(t) = Ax(t) + u(t), \tag{5}$$

where $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ and $u \in \mathbb{R}^n$ is a control term. We derive an adaptive form of the time-delayed feedback control that ensures global asymptotical stability of the closed-loop system.

A. Adaptive controller synthesis

Suppose there exists $k_* \in \mathbb{R}$ such that the feedback control term

$$u(t) = -k_*(x(t) - x(t-\tau)) \tag{6}$$

makes the closed-loop system (5), (6) globally asymptotically stable. Denote $x_t(\theta) = x(t+\theta)$, where $\theta \in [-\tau, 0]$. It can be proven (see [24], Theorem 5.19) that if system (5) with control (6) is globally asymptotically stable then there exists a matrix

$$P = P^T \in \mathbb{R}^{n\times n},$$

where superscript T denotes *transposed*, and continuous matrix functions

$$Q(\xi), \quad R(\xi,\eta) = R^T(\eta,\xi), \quad S(\xi) = S^T(\xi) \in \mathbb{R}^{n\times n}$$

such that the functional

$$V_0(x_t(\cdot)) = x^T(t)Px(t) + 2x^T(t)\int_{t-\tau}^{t} Q(\xi - t)x(\xi)\,d\xi + \int_{t-\tau}^{t}\int_{t-\tau}^{t} x^T(\xi)R(\xi - t, \eta - t)x(\eta)\,d\eta d\xi + \int_{t-\tau}^{t} x^T(\xi)S(\xi)x(\xi)\,d\xi \tag{7}$$

satisfies

$$\alpha_1|x(t)|^2 \leq V_0(x_t(\cdot)) \leq \alpha_2\|x_t(\cdot)\|_C^2, \tag{8}$$

$$\dot{V}_0(x_t(\cdot), k_*) \leq -\varepsilon|x(t)|^2 \tag{9}$$

for some $\varepsilon > 0$. Here $\|x_t(\cdot)\|_C = max_{\theta\in[-\tau,0]}|x(t+\theta)|$. Equations (8) and (9) mean that Eq. (7) is a Lyapunov-Krasovskii functional for the system (5), (6).

Now consider a control term

$$u(t) = -k(t)(x(t) - x(t-\tau)). \tag{10}$$

To derive an adaptive algorithm for $k(t) \in \mathbb{R}$ we apply the speed-gradient method with a goal function given by Eq. (7):

$$\nabla_k \dot{V}_0 = -2(x(t) - x(t-\tau))^T \times \left\{ Px(t) + \int_{t-\tau}^{t} Q(\xi - t)x(\xi)\,d\xi \right\}.$$

This yields the following adaptive controller:

$$\begin{aligned} u(t) &= -k(t)(x(t) - x(t-\tau)), \\ \dot{k}(t) &= \gamma(x(t) - x(t-\tau))^T \\ &\quad \times \left\{ Px(t) + \int_{t-\tau}^{t} Q(\xi - t)x(\xi)\,d\xi \right\} \end{aligned} \tag{11}$$

with some scalar gain coefficient $\gamma > 0$.

B. Global stability analysis

Theorem 1 *Suppose there exists k_* such that the system (5), (6) is globally asymptotically stable. Then any solution of the system (5), (11) satisfies*

$$\lim_{t\to\infty} |x(t)| = 0$$

and $k(t)$ is a bounded function.

Proof. Consider

$$V(x_t(\cdot),k) = V_0(x_t(\cdot)) + V_k(k), \tag{12}$$

where

$$V_k(k) = \gamma^{-1}(k-k_*)^2$$

with k_* from Eq. (6). Then

$$\dot{V}(x_t(\cdot),k(t)) = \dot{V}_0(x_t(\cdot),k(t)) + \dot{V}_k(k(t)) = \dot{V}_0(x_t(\cdot),k_*) + 2\gamma^{-1}(k_*-k(t))(x(t)-x(t-\tau))^T \times \left\{Px(t) + \int_{t-\tau}^{t} Q(\xi-t)x(\xi)\,d\xi\right\} + \dot{V}_k(k(t)) = \dot{V}_0(x_t(\cdot),k_*) \le -\varepsilon\|x(t)\|^2.$$

Thus,

$$\dot{V}(x_t(\cdot),k(t)) \le -\varepsilon|x(t)|^2. \tag{13}$$

Since $V(t) = V(x_t(\cdot),k(t))$ is a non-negative decreasing function, there exists a finite limit for $V(t)$: $\lim_{t\to\infty} V(t) < \infty$. From the inequality (13) it follows $\varepsilon\int_{\tau}^{\infty}|x(s)|^2\,ds \le V(\tau) - \lim_{t\to\infty} V(t) < \infty$. By applying Barbalat's lemma [25, p. 323] we conclude:

$$\lim_{t\to\infty}|x(t)| = 0. \tag{14}$$

Boundedness of $k(t)$ follows from the boundedness of $V(t)$. ■

Remark 1 *In order to construct the adaptive controller* (11) *one needs to find P and Q such that* (7) *satisfies* (8), (9). *This can be done with the help of Discretized Lyapunov Functional approach (see [24]).*

Remark 2 *Note that both Eqs.* (6) *and* (11) *contain* $x(t-\tau)$. *Therefore, these controllers can only be applied for $t > \tau$. In other words, we suppose that $u(t) = 0$ for $t \in [0,\tau)$.*

C. Numerical example

Consider system (5) with

$$A = \begin{pmatrix} \lambda & \omega \\ -\omega & \lambda \end{pmatrix}. \tag{15}$$

As it has been shown in Ref. [10] the system (5), (6) with A given by Eq. (15) and $\lambda = 0.5$, $\omega = \pi$, $\tau = 1$ is globally asymptotically stable for $k_* = 0.3$. Thus, there should exist a Lyapunov-Krasovskii functional (7) that satisfies Eqs. (8) and (9). With the help of Discretized Lyapunov Functional approach (see [24]) we find appropriate values of P and $Q(\xi)$:

$$P = I, \quad Q(\xi) = -\xi Q_0 + (1+\xi)Q_1, \quad \forall\xi\in[-1,0], \tag{16}$$

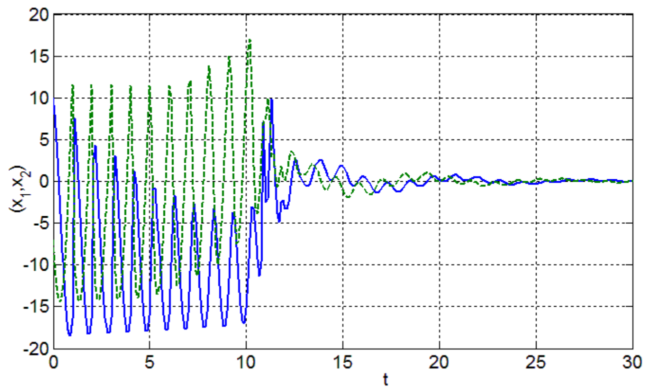

Figure 1: Solutions $x_1(t)$ (blue solid line) and $x_2(t)$ (green dashed line) of the closed-loop system (5), (11) with A given by Eq. (15), P, Q given by Eq. (16). The parameters are: $\lambda = 0.5$, $\omega = \pi$, $\tau = 1$, $\gamma = 1$. Initial conditions are: $x_1(0) = 10$, $x_2(0) = -7$, $k(\tau) = 0$.

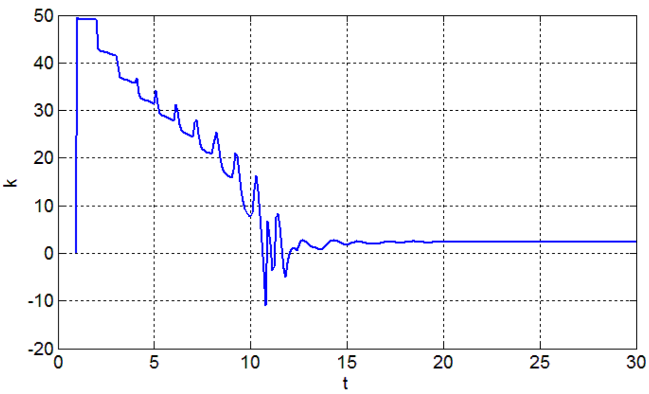

Figure 2: Evolution of $k(t)$ for the system from Fig. 1. The limit value is $\lim_{t\to\infty} k \approx 2.4$.

where

$$Q_0 = \begin{pmatrix} -1 & -0.5 \\ 0.5 & -1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}.$$

It follows from Theorem 1 that any solution of the system (5), (11) with A given by Eq. (15) and P, $Q(\xi)$ given by Eq. (16) is such that $\lim_{t\to\infty}\|x(t)\| = 0$ and $k(t)$ is a bounded function.

Figure 1 shows a time series of the stabilization of the origin. The blue solid and the green dashed lines refer to $x_1(t)$ and $x_2(t)$, respectively. The evolution of $k(t)$ is depicted in Fig. 2.

Note that any nonlinear system close to a fixed point can be reduced to the linear system given by Eq. (5). Thus, we expect our results to be applicable in a variety of different systems where time-delayed feedback control is used to stabilize unstable steady states (fixed points). In the case of a two-variable system the focus is the only type of fixed point which can be stabilized by time-delayed feedback control of the form given by Eq. (6), which fails for saddle-points and unstable nodes [10]. Thus, Eq. (15) represents the most general case, as the linear stability equation of any two-variable nonlinear system close to an unstable focus can be linearly transformed to the linear system given by Eqs. (5) and (15). The Van der Pol oscillator [26] which is a common model for nonlinear oscillations occurring in a variety of physical systems, e.g., nonlinear electronic circuits, can be used to demonstrate this. The Van der Pol equation

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= -y_1 - \varepsilon(y_1^2 - 1)y_2, \end{aligned}$$

where ε is the bifurcation parameter, has a fixed point $(y_1, y_2) = (0, 0)$ of unstable focus type for $0 < \varepsilon < 2$. Hence, the linear stability equation can be reduced to the linear normal form in Eqs. (5), (15) with $\omega = \frac{\sqrt{4-\varepsilon^2}}{2}$, $\lambda = \frac{\varepsilon}{2}$ by using the linear transformation

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = J \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad J = \begin{pmatrix} \omega & \lambda \\ 0 & 1 \end{pmatrix}.$$

Now we consider the Van der Pol equation with time-delayed feedback control $u = (u_1, u_2)^T$:

$$\begin{aligned} \dot{y}_1 &= y_2 + u_1, \\ \dot{y}_2 &= -y_1 - \varepsilon(y_1^2 - 1)y_2 + u_2. \end{aligned} \tag{17}$$

For $\varepsilon = 0.2$, $\tau = \frac{\pi}{\omega} \approx 3.16$ using the Discretized Lyapunov Functional approach we find that

$$P = I,$$

$$Q(\xi) = \begin{cases} \left(-\frac{2\xi}{\tau} - 1\right) Q_0 + \left(2 + \frac{2\xi}{\tau}\right) Q_1, & \xi \in [-\tau, -\frac{\tau}{2}), \\ -\frac{2\xi}{\tau} Q_1 + \left(1 + \frac{2\xi}{\tau}\right) Q_2, & \xi \in [-\frac{\tau}{2}, 0], \end{cases} \tag{18}$$

where

$$Q_0 \approx \begin{pmatrix} -0.11 & -0.12 \\ 0.12 & -0.11 \end{pmatrix}, Q_1 \approx \begin{pmatrix} 0.01 & -0.15 \\ 0.15 & 0.01 \end{pmatrix},$$

$$Q_2 \approx \begin{pmatrix} 0.26 & -0.03 \\ 0.03 & 0.26 \end{pmatrix}.$$

Since $x(t) = J^{-1}y(t)$, from (11) we obtain

$$\begin{aligned} u(t) &= -k(t)(y(t) - y(t-\tau)), \\ \dot{k}(t) &= \gamma(y(t) - y(t-\tau))^T \Big\{ (J^{-1})^T P J^{-1} y(t) \\ &\quad + \int_{t-\tau}^{t} (J^{-1})^T Q(\xi - t) J^{-1} y(\xi)\, d\xi \Big\}. \end{aligned} \tag{19}$$

The results of numerical simulations for the system (17), (18), (19) for 5 randomly chosen initial conditions $(y_1(0), y_2(0))^T \in [-2, 2] \times [-2, 2]$ and $k(\tau) = 0$ are presented in Figs. 3 and 4. In Fig. 3 one can see that the norm of the system state $y(t) = (y_1(t), y_2(t))^T$ converges to zero, i.e., asymptotical stability is achieved for all initial conditions. From Fig. 4 it can be seen that the adaptation gain tends to a constant value.

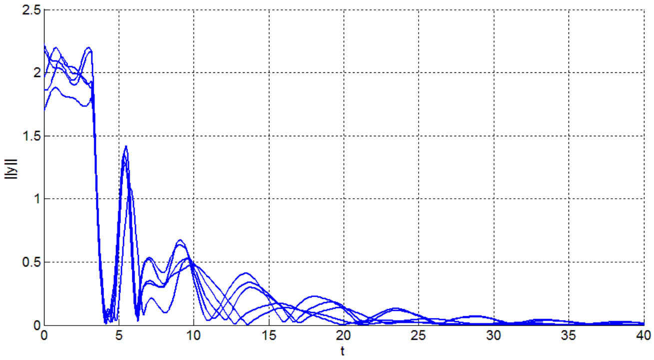

Figure 3: Norms of 5 different solutions $y(t)$ of the closed-loop system (17), (18), (19) with randomly chosen initial conditions $(y_1(0), y_2(0))^T \in [-2, 2] \times [-2, 2]$ and $k(\tau) = 0$. The parameters are: $\varepsilon = 0.2$, $\tau = \frac{\pi}{\omega} \approx 3.16$, $\gamma = 1$.

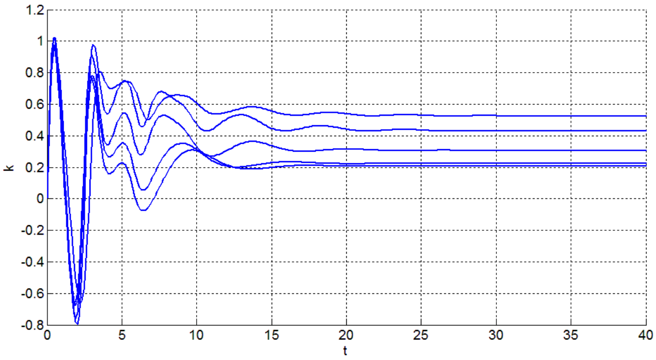

Figure 4: Evolution of adaptation gain k(t) for 5 different initial conditions as in Fig. 3.

D. Uncertain systems

In order to construct the adaptive controller (11) for the system (5) appropriate values for the matrix P and the matrix function $Q(\xi)$ have to be calculated. To find these values the Discretized Lyapunov Functional approach is used which requires the knowledge of all system parameters. On the other hand one could argue that if all system parameters were known it would be possible to construct a static feedback of the form (6). It turns out that sometimes it is possible to find appropriate values of P and $Q(\xi)$ for entire class of uncertain systems. In this case the advantage of the adaptive approach compared to nonadaptive one is in a smaller controller gain.

In order to demonstrate this point we consider the case when the system matrix A has an uncertain parameter $\lambda \in [0, 1]$. Then $k_* \in [0.5, 3.22]$ stabilizes the system (5), (6) for any value of $\lambda \in [0, 1]$ (see Ref. [10]). In Fig. 5 the blue shaded region marks the stability region of time-delayed feedback control without adaptive tuning of k. The red-dotted line $k_* = 0.5$ denotes the lower boundary of the interval $[0.5, 3.22]$. For these values of k_*, with the help of the Discretized Lyapunov Functional approach, it is possible to find P and $Q(\xi)$ such that (7) is a Lyapunov-Krasovskii functional for the system (5), (6). Taking these values of P and $Q(\xi)$ and substituting them into Eq. (11) we derive an adaptive controller that, according to Theorem 1, ensures global asymptotic stability of the system (5), (11) with respect to $x(t)$ for any value of $\lambda \in [0, 1]$. The limit value of the control gain is $k(t) = 0.153$ which is obviously much smaller than 0.5. Figure 6 shows the evolution of $k(t)$ and the smallest static gain $k_* = 0.5$.

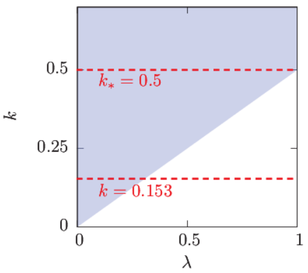

Figure 5: (k-λ)-plane: blue shaded region – region of stability of (non-adaptive) time-delayed feedback control [10]; k_* – critical coupling strength above which the system is stable for all $\lambda \in [0, 1]$; $k = 0.153$ – value reached by adaptive control according to Eq. (11).

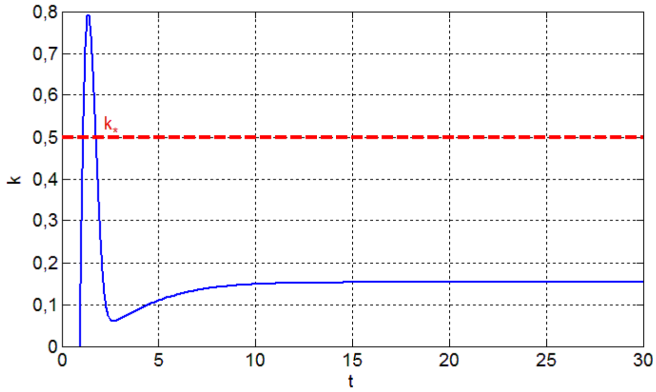

Figure 6: Evolution of $k(t)$ (blue solid line) for the system (5), (11) with A given by Eq. (15). Value $k_* = 0.5$ (red dashed line). $\lambda = 0.1$. Other parameters as in Fig. 1. Initial conditions are: $x(0) = 2$, $y(0) = 0$, $k(\tau) = 0$.

IV. STABILIZATION OF A PERIODIC ORBIT

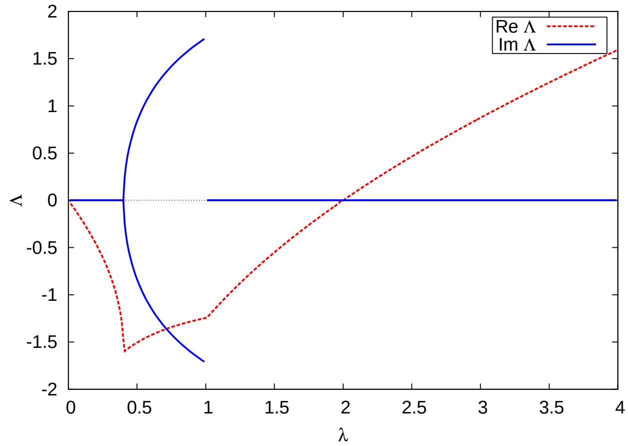

Figure 7: Maximum real part (red dashed line) and corresponding imaginary part (blue solid line) of Floquet exponent Λ obtained by numerically solving Eq. (23). Parameters as in Fig. 1.

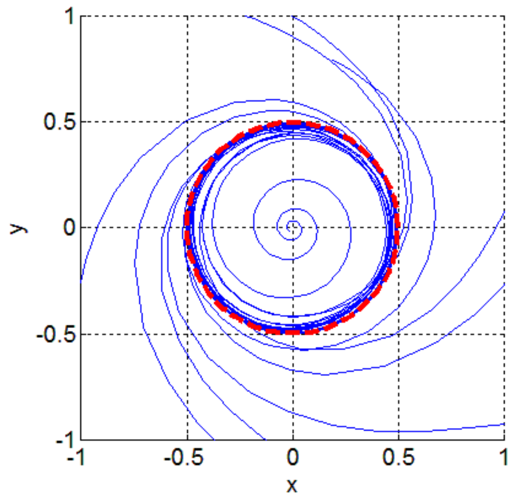

Figure 8: Phase portrait of the system (20), (21) for $t \geq \tau$. Blue solid lines: Trajectories for 10 different initial conditions chosen randomly from $[-1, 1] \times [-1, 1]$. Red dashed line corresponds to the periodic solution $(x_*(t), y_*(t))$ given by Eq. (22). Parameters as in Fig. 1.

In this section we address the problem of adaptive stabilization of a periodic orbit of a linear system. Consider the system

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \lambda & \omega \\ -\omega & \lambda \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} - k(t)\begin{bmatrix} x(t) - x(t-\tau) \\ y(t) - y(t-\tau) \end{bmatrix} \tag{20}$$

with $\lambda > 0$. We recall that for a constant feedback gain $k(t) = k$ the linear time-delayed system (20) has a periodic orbit if and only if at least one of its eigenvalues has

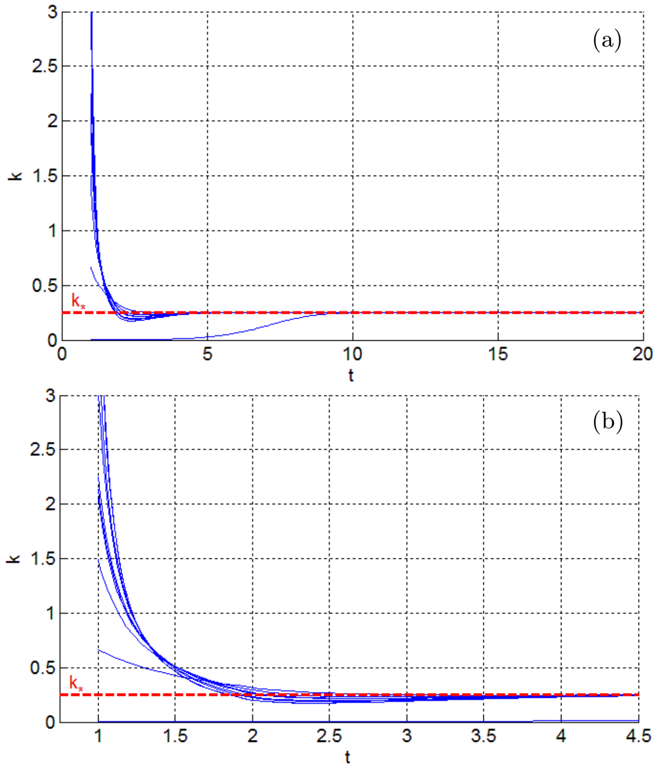

Figure 9: (a) Evolution of $k(t)$ (blue solid line) for 10 different initial conditions chosen randomly from $[-1,1]\times[-1,1]$; the value $k_*=\frac{\lambda}{2}$ (red dashed line). Parameters as in Fig. 1. (b) zoom of (a).

a zero real part and the real parts of all other eigenvalues are smaller than zero. Moreover, as it has been shown in Ref. [10], in this particular case the fixed point $x=y=0$ is an unstable focus for $k<k_*=\frac{\lambda}{2}$, a stable focus for $k_*<k<\bar{k}$ and a center for $k_*=\frac{\lambda}{2}$. Nevertheless, a periodic orbit of the system (20) with a constant $k(t)=k$ will never be *asymptotically* stable since a linear system cannot have a limit cycle. However, implementation of an adaptive tuning algorithm for $k(t)$ makes the closed-loop system nonlinear and admits the existence of a limit cycle.

We propose the following adaptive law:

$$k(t)=\gamma(x^2(t)+y^2(t)) \tag{21}$$

with $\gamma>0$. Note that the algorithm (21) has the so-called finite form that is it determines the value of $k(t)$, not $\dot{k}(t)$ as above. This algorithm introduces a nonlinearity in the system equation that transforms it into a limit cycle oscillator.

Next we perform a local asymptotic stability analysis for a periodic solution of the system (20), (21). Linearization near the periodic orbit

$$x_*(t)=\sqrt{\frac{\lambda}{2\gamma}}\cos(-\omega t+\theta_0),\quad y_*(t)=\sqrt{\frac{\lambda}{2\gamma}}\sin(-\omega t+\theta_0) \tag{22}$$

and the ansatz $(\delta r,\delta\theta)^T=e^{\Lambda t}q$ yields a characteristic equation:

$$\left(-\frac{\lambda}{2}\left(3+e^{-\Lambda\tau}\right)-\Lambda\right)\left(\frac{\lambda}{2}\left(1-e^{-\Lambda\tau}\right)-\Lambda\right)=0, \tag{23}$$

where Λ is the Floquet exponent. Note that Eq. (23) does not depend on $\gamma>0$. The solution of Eq. (23) with the maximum real part of Λ (red dashed line) and the corresponding imaginary part (blue solid line) is depicted in Fig. 7. For $\lambda\in(0,2)$ the maximum real part of Λ is negative, therefore the system (20) with the control (21) is asymptotically stable. Figure 8 shows a phase portrait. Blue solid lines show the trajectories for different initial conditions; the red dashed line depicts the limit cycle. It can be seen that all trajectories approach the limit cycle as can be expected from the linear stability analysis. Figure 9 (a) presents the evolution of $k(t)$ as a blue solid line. Clearly, $k(t)$ tends to $k_*=\frac{\lambda}{2}$ (red dashed line) for which the fixed point of system (20) with constant k is a center. Fig. 9 (b) is a zoom of Fig. 9 (a).

A. Stabilization of a given periodic orbit

It follows from Eq. (22) that in order to stabilize a periodic orbit of system (20), (21) with a given radius a one should choose $\gamma=\gamma_*=\frac{\lambda}{2a^2}$. If the value of λ is unknown the following adaptive algorithm can be applied to stabilize an orbit of radius a:

$$\begin{aligned}k(t)&=\gamma(t)(x^2(t)+y^2(t)),\\ \dot{\gamma}(t)&=\alpha(x^2(t)+y^2(t)-a^2)\end{aligned} \tag{24}$$

with some $\alpha>0$. The periodic orbit is then given by

$$\begin{aligned}x_*(t)&=a\cos(-\omega t+\theta_0),\\ y_*(t)&=a\sin(-\omega t+\theta_0),\\ \gamma_*(t)&=\frac{\lambda}{2a^2}.\end{aligned} \tag{25}$$

For numerical simulations we choose initial conditions randomly from $[-5,5]\times[-5,5]$. Figure 10 depicts the trajectories (blue solid line) for the different initial conditions and the asymptotically stable limit cycle (red dashed line). A time series of $\gamma(t)$ (blue solid line) is shown in Fig. 11. The values of $\gamma(t)$ always tend to $\gamma_*=\frac{\lambda}{2a^2}\approx 0.028$ (red dashed line) which is the fixed point value of γ as given by Eq. (25).

Next we analyze the local asymptotic stability of the periodic solution of the system (20), (24). Linearization near the periodic solution (25) and the ansatz

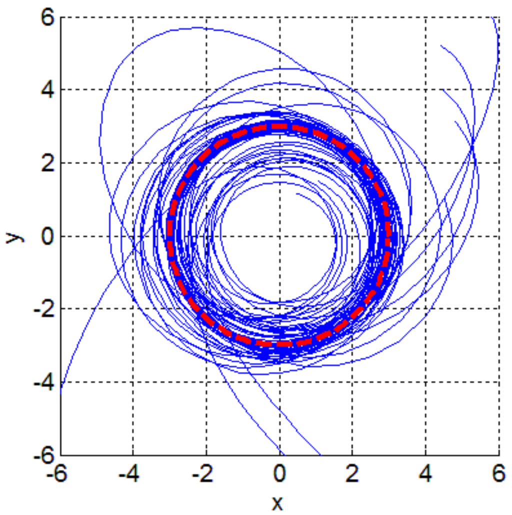

Figure 10: Phase portrait of the system (20), (24) for $t \geq \tau$. Blue solid lines: Trajectories for 10 different initial conditions chosen randomly from $[-5, 5] \times [-5, 5]$. Red dashed line corresponds to a cycle of radius $a = 3$. $\alpha = 0.01$. Other parameters as in Fig. 1.

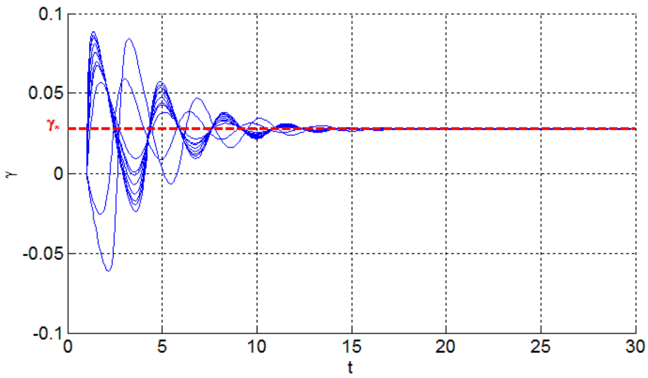

Figure 11: Evolution of $\gamma(t)$ (blue solid line) for the system (20), (24). Red dashed line is the value of $\gamma_* = \frac{\lambda}{2a^2}$. Parameters and initial conditions as in Fig. 10.

$(\delta r, \delta\theta, \delta\gamma)^T = e^{\Lambda t} q$ yield the characteristic equation:

$$\left(\frac{\lambda}{2}\Lambda\left(3 + e^{-\Lambda\tau}\right) + \Lambda^2 + 4\alpha a^4\right) \times \left(\frac{\lambda}{2}\left(1 - e^{-\Lambda\tau}\right) - \Lambda\right) = 0, \quad (26)$$

where Λ is the Floquet exponent. The maximum real part (red dashed line) and the corresponding imaginary part (blue solid line) of Λ are depicted in Fig. 12. As one can see the maximum real part of Λ is negative for $\lambda \in (0, 2)$. Therefore the solution $(x_*(t), y_*(t), \gamma_*(t))^T$ of the closed-loop system (20), (24) is asymptotically stable for λ from this interval.

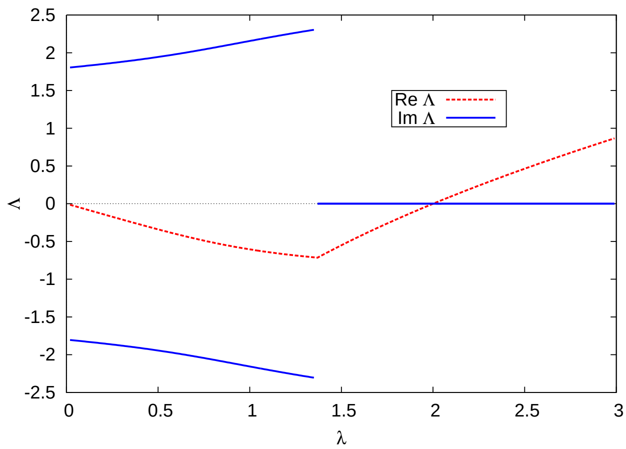

Figure 12: Maximum real part (red dashed line) and corresponding imaginary part (blue dashed line) of Floquet exponent Λ obtained by numerically solving Eq. (26). $\alpha = 0.01$, $a = 3$. Other Parameters as in Fig. 1

V. CONCLUSION

We have proposed three different adaptive controllers to tune the feedback gain in time-delayed feedback control to appropriate values.

The first adaptive controller is based on the speed-gradient method and a Lyapunov-Krasovskii functional of the system without adaptation. This adaptive control can be applied in any linear system to stabilize the equilibrium in its origin with the only requirement that an appropriate value of the feedback gain exists. Global stability of this method is proven with an extended Lyapunov-Krasovskii functional for the system with adaptation. By the example of the normal form of an unstable focus (see Eq. (15)) it is demonstrated that the advantage of the adaptive controller over the nonadaptive one is in a smaller controller gain. Note that the linear stability equation of any two-variable nonlinear system close to an unstable focus can be reduced, by means of a linear transformation, to the linear system given by Eqs. (5), and (15). This allows for applying our method to a variety of different nonlinear systems, which we have demonstrated with the example of the Van der Pol oscillator [26].

The second and third adaptive controllers stabilize periodic orbits of a linear systems. The advantage of the third controller compared to the second one is that the radius of the limit orbit can be chosen even in cases where some system parameters are unknown. Simulations and a local stability analysis demonstrate the usefulness of both controllers.

Time delayed feedback control is a widely used control method. The adaptive controller presented here can even widen its applicability as they allow for time-delayed feedback control in cases where appropriate control or system parameters are unknown.

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