MODELLING, SIMULATION AND EXPERIMENT WITH DOUBLE PENDULUM CHAOTIC TOY

B.R. Andrievskii, A.L. Fradkov, V.A. Konoplev, A.P. Konjukhov

Institute of Problems of Mechanical Engineering Russian Ac. Sc.,
61 Bolshoy V.O., Saint-Petersburg, 199178, Russia.
Fax: +7 (812) , e-mail: andr@alf.ipme.ru

Keywords : oscillations, control, state estimation, identification.

Abstract

In the article a problem of mathematical models construction for oscillatory mechanical systems (OMS) with complex (chaotic) dynamics is considered. The mathematical model of system is constructed with application of the aggregative method. The device for OMS control with use of the computer is described. Verification of the obtained model on the basis of laboratory experiments results is made. Pulse-number algorithm of OMS control is proposed and checked with use of laboratory set-up. State estimation recursive algorithm is proposed.

1 Introduction

The problem of oscillatory mechanical systems control has the significant theoretical interest and groving value in practice. For the purposes of the control engineering education it would be necessary to build up appropriate laboratory equipment and software, to work out approaches for investigation of these systems.

This article deals with well-known mechanical toy, consisting of two coupled pendulums. In the sections 2, 3 the brief description of the construction and its mathematical model is given [1, 2, 3].

The simplified model was used for the purposes of the energy-based oscillations amplitude control [2, 3, 4] (Sec. 4), state estimation (Sec. 5) and parameters identification (Sec. 6). The case of the very poor measurements is considered – namely, only period of oscillations is assumed be measurable and only pulsed control torque can be applied to the plant.

2 Controlled plant

The controlled plant is the oscillatory mechanical system (OMS) [1, 3, 4] (see Fig. 1) which consists of the double links pendulum with displaced centres of parts weight and sliding support of an external link. An external part of system is a metal ring with located on it a massive ball and two cylindrical magnets. The ball and magnets displace the centre of weight of ring. Magnets, in addition, provide transfer of control efforts to both links. On an outside surface of a ring two opposite directed half-axes, ensuring its support on two flat platforms with terminators of a course, are located. System is fixed on the massive basis, in centre of which the inductive sensor of the first link zero state and electromagnet, transmitting control efforts on cylindrical magnet are located.

The second part is presented by two cylindrical loads with magnets which established on axis symmetric concerning its centre. This part rotates inside an external ring with an axis of rotation fixed on it with the help of cylindrical hinges. These hinges are on an axis which is turned by 45 degrees with respect to the axis of an external ring rotation.

The specified mechanical system has three degrees of freedom:

1. Parallel carry of an external ring on flat support of the basis;

2. Rotation of an external ring with the above-stated half-axes;

3. Rotation of the second link of a rather external ring.

On the specified mechanical system the following control efforts are apply:

1. Influence of an electromagnet of the basis on one of magnets, established on an external ring;

2. Influence of the second magnet, established on an external ring on magnets of second (internal) link.

Measuring value is an interval of time between transitions of an external ring magnet above an electromagnet of the basis.

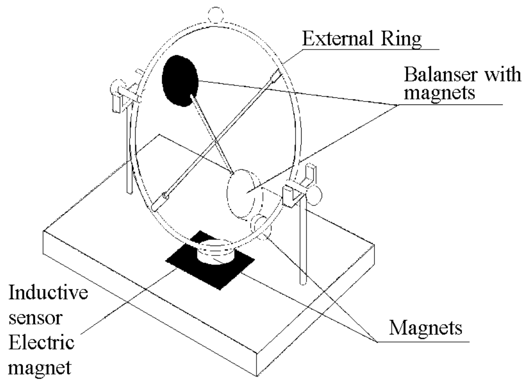

Figure 1: Oscillatory mechanical system

3 Mathematical model of the mechanical system

The equations of a mechanical system motion are written in aggregative form [3, 8, 9]:

$$\begin{aligned} \mathbf{A}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{B}(\dot{\mathbf{q}},\mathbf{q}) &= \mathbf{SP} + \mathbf{g} + \mathbf{u}, \qquad (1)\\ \mathbf{A}(\mathbf{q}) &= \mathbf{SAS}^T,\\ \mathbf{B}(\dot{\mathbf{q}},\mathbf{q}) &= \mathbf{SAS}^{T\bullet} + \mathbf{S\Phi AS}^T, \end{aligned}$$

where $\mathbf{S} - (\mathbf{3}\times\mathbf{12})$ - structural matrix of system; $\mathbf{A} = diag(\mathbf{A}_1^1, \mathbf{A}_2^2)$- block-diagonal (12×12) - matrix of inertia with (6×6) - blocks of a kind $\mathbf{A}_1^1 = \sum_{S=1}^{10} \mathbf{L}_{1S}^1 \mathbf{\Theta}_{1S}^{1S} \mathbf{L}_{1S}^{1,T}$ and $\mathbf{A}_2^2 = \sum_{S=1}^{4} \mathbf{L}_{2S}^2 \mathbf{\Theta}_{2S}^{2S} \mathbf{L}_{2S}^{2,T}$, where $\mathbf{\Theta}_{LS}^{LS}$ - inertia matrix of an S-th body in the system of coordinate which connected to it, $\mathbf{L}_{LS}^L$ - matrix of transformation from system of coordinates E_L to coordinate system E_S which connected to a body; $\mathbf{\Phi} = (\mathbf{\Phi}_1^{1,0}\mathbf{\Phi}_2^{0,2})$ - block-diagonal (12×12) - matrix of quasivelocities with (6×6) - blocks on a main diagonal,

$$\mathbf{\Phi} = \begin{bmatrix} <\omega_L^0>^L & 0 \\ <\nu_L^0>^L & <\omega_L^0>^L \end{bmatrix},$$

where $L = 1, 2$; $<\cdot>$ - mark of a skew-symmetric matrix; $\omega_L^{0L}, \nu_L^{0L}$ - elements of a quasivelocities vector $\mathbf{V} = [\mathbf{V}_1^{01}, \mathbf{V}_2^{02}] = \mathbf{S}^T\dot{\mathbf{q}}, \mathbf{V}_L^{0L} = [\nu_L^{0L}, \omega_L^{0L}]$; $\mathbf{G} - (12\times 1)$ - vector of gravitation forces dynamic screws; $\mathbf{p} - (\mathbf{3}\times\mathbf{1})$ - column of friction efforts on axes of system mobility; $\mathbf{u} - (\mathbf{3} * \mathbf{1})$ - column of magnets interactions from which first element is control effort.

The dependence of magnets interaction force amplitude was approximated by density-of- Gauss-distribution-like equation

$$F_{mi} = A_{mi}\exp -\frac{(\Theta_5^i)^2}{\sigma_{mi}^2}, \quad i = 1, 2, \qquad (2)$$

where Θ_5^i - corner of i-th body rotation. The numerical meanings of factors A_{mi} and σ_{mi} were determined with use of experimental data and looks as follows $A_m = 10, A_m = 3$, $\sigma_{m1} = 0,05; \sigma_{m2} = 0,01$.

The obtained system of equations is realized in environment of a program package MATLAB^R-ADAM^R [7].

In the considered plant the inner pendulum exerts weak influence on the motion of the external ring. Therefore, for the sake of simplifying the stage of the algorithms synthesis, the reduced plant model can be used.

This model is obtained by means of the Euler-Lagrange equation. By means of transforms and simplified suggestions, it is reduced to the following model

$$\begin{aligned} &J_e\ddot{\varphi} = -m_1 g(R\sin\varphi - (\rho + r)\cos\varphi) - \\ &m_2 g(f\sin\varphi - (\rho + r)\cos\varphi) + m_p g(\rho + r)\cos\varphi + \qquad (3)\\ &m_3 g(R_m\sin\varphi + (\rho + r)\cos\varphi) + A\exp(-(\tfrac{\varphi}{\sigma})^2), \end{aligned}$$

where $J_e = M\rho^2 + J_0$ is *the equivalent moment of inertia* of the plant. The coefficients m_i, r, R_m, ρ denote mass-geometric parameters of the system, the last term describes the torque of electromagnetic interaction. The value of A can be changed by the controller and is used as a control action.

4 OMS control by means of the computer

4.1 The laboratory facilites for the experiments on the real-time control and estimation.

For realization of experiments on OMS the specialized controller was developed. The more detailed description of this controller can be found in [1].

The development of control algorithm was made on the basis of speed-gradient algorithm see [5, 6], where the control aim was stabilization of a given total energy level of a mechanical system [1, 2]. Control aim can be written as follows

$$H \to H_{ref}, \quad when \quad t \to \infty, \qquad (4)$$

where H_{ref} is the given energy level.

As was shown in [1, 2], such algorithm allows to achieve various types of systems behaviour at regulation of whole one parameter. For considered system the development of control algorithm complicated by absence of the complete information about its state (period of external link oscillations is measured only). However, as was shown by additional investigation, development of control algorithm, realizing the control aim, close to considered in the specified work, is possible in this case also. So, it is possible to show, that for a simple nonlinear pendulum the period of fluctuations is not constant, and is determined by a formula:

$$T = \sqrt{2}\int_{A_1}^{A_2} \frac{dx}{\sqrt{h - P(x)}}, \qquad (5)$$

where A_1 and A_2 - deviation of a pendulum in moment, when its speed is equal to zero; h - total energy of system;

$P(x)$ - potential function, i.e. a function, proportional to potential energy of the system see [10].

4.2 The control law based on energy level stabilization via speed-gradient approach.

The control of a simple nonlinear pendulum oscillation period is at the same time lead to the control of a pendulum total energy. The considered system is a double pendulum, however the influence of a second link movement on a movement of a first link is small and can be considered as revolting disturbance. This assumption used at development of a control algorithm. Other peculiarity of considered system is limitation of the control action application zone - it is limited by sector, in which a large magnet of the first link is induced registered by sensitive element EMF. Suppose that input signal is fixed and the control action is applied at discreet moments of time t_k, where $k = 1, 2, \ldots$. Also suppose that maximal magnitude of output control voltage is equal to U_{max}. In this case the elementary pulsing control algorithm is formulated as follows:

$$U_{out}(t_k) = \begin{cases} 0, & when \quad U_{inp}(t_k) = 0, \\ U_{max}, & when \quad U_{inp}(t_k) \neq 0. \end{cases} \qquad (6)$$

This algorithm can be rewritten also for a case of feedback control. In this case the problem of period stabilization of the first link oscillations is solved. It is offered thus to determine number of pulses in a package (i.e. in group of consequent pulses, appropriate to a Eq. (5)) to provide the given value of oscillations period, and thus the given total energy of the system. The algorithm can be presented in the form

$$\begin{aligned} n &= n_{base} + \Delta n, \\ n_{min} &\leq n \leq n_{max}, \\ \Delta n &= \mathbf{F}\{(T_{base} - T_{ref})\}, \end{aligned} \qquad (7)$$

where n_{min} - minimum number of impulses in package, n_{max} - maximum number of impulses in package (equal to package's lenght), n_{base} - base impulses number in the package (propotional to the base oscillation period T_{base}), T_{ref} - reference oscillation period; $\mathbf{F}$ - piece-line function of a kind $\mathbf{F}(x) = const$ where $x_i \leq x < x_{i+1}$, $i = 1, \ldots, 5$, $\mathbf{F}(x) = n_{min}$ when $x < x_1$ and $\mathbf{F}(x) = n_{max}$ when $x > x_6$. Values of x_i were determined with use of experimental data. Results of numerical and laboratory experiments with the specified control algorithms considered below.

4.3 Results of the modelling and laboratory experiments

The results of numerical experiments are presented on Fig. 2 and 3. The following values of experimentally determined parameters were accepted: $n_{base} = 10$, $n_{max} = 10$, $n_{min} = 4$, $T_{base} = 1300ms$, $U_{max} = 9V$. Initial conditions were taken as follows: $\Theta_1 = \pi/6\ rad$, $\Theta_2 = 0\ rad$, $\omega_1 = 0\ rad/s$, $\omega_2 = 0\ rad/s$.

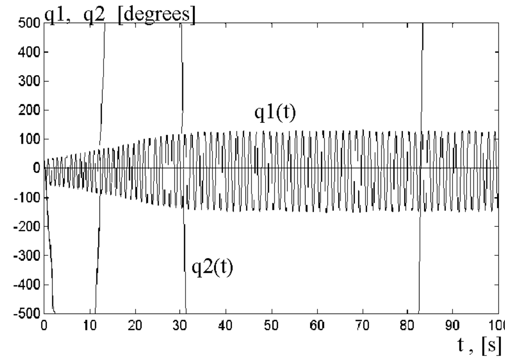

Figure 2: Transient process (numerical experiment)

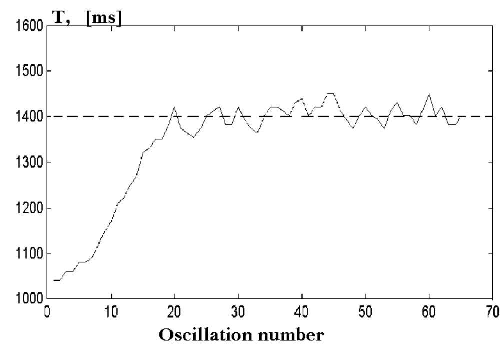

Figure 3: Period-oscillation number plot for numerical experiment

The results of laboratory research with use of the elementary controller and the same parameter values are presented on Fig. 4.

As it is visible from figures, numerical and the experimental research give similar results. As it seen, curve on Fig. 3 and Fig. 4 are approximated by exponent equation with equal parameters and disturbances both for first and second samples restricted in the same interval. It says about high adequacy of mathematical model to an OMS. Besides it is possible to note high quality of process

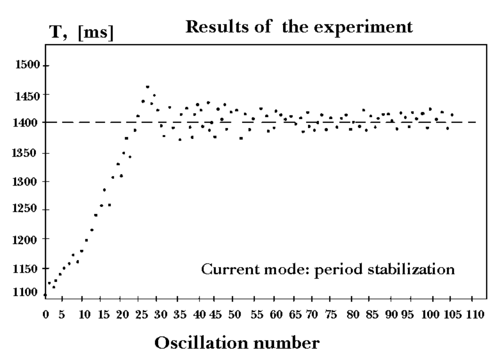

Figure 4: Period-oscillation number plot for laboratory experiment

of regulation, that testifies to efficiencies of offered control algorithm. However it is necessary to note, that the considered system has an essential lack, namely - narrow range of the regulated variable: 1100-1500ms. Nevertheless, the obtained result is interesting and important, as it is shown the opportunity of non-traditional methods use for control algorithm design.

5 State estimation

This section is devoted to the problem of the real-time state estimation, based the accessible flow data and the system model. At first (Sec. 5.1), more simple problem is considered, when the current (maybe – sampled) measurements of the pendulum rotation angle can be used for the velocity estimation.

The case of nonmeasurable flow rotations is considered in the Sec. 5.2. It is assumed, that only time intervals between the crossings the external ring the lowest point can be measured, and both the rotation angle and the velocity, are required to be estimated.

5.1 State vector estimation in the case of the current angles measurement

The simplified model (3) of free motion is as follows

$$J\ddot{\varphi}(t) + mgl_e \sin(\varphi(t) - \varphi_0) = 0, \tag{8}$$

where $J_e \stackrel{\text{def}}{=} ml_e^2$, l_e is the *effective length of the pendulum*, φ_0 is caused by the pendulum centroid displacement from the surface of the external ring (see Fig. 1). The nonlinear full-order observer is as follows

$$\begin{cases} \dot{\hat{\varphi}}(t) = l_1(\varphi - \hat{\varphi}) + \hat{\omega}(t), \\ \dot{\hat{\omega}}(t) = l_2(\varphi - \hat{\varphi}) - \beta^2 \sin(\varphi(t) - \varphi_0), \end{cases} \tag{9}$$

where $\omega \stackrel{\text{def}}{=} \dot{\varphi}$ is the angular velocity, $\beta \stackrel{\text{def}}{=} mgl_e J_e^{-1} = gl_e^{-1}$ denotes *the fundamental frequency* of the linear oscillator, l_1, l_2 are *the gain coefficients*, which make up *the gain vector* $L \stackrel{\text{def}}{=} \text{col}\{l_1, l_2\}$. To find the vector L, the standard Butterworth form was applied to the characteristic polynomial of the observer equations (9), previously linearized near the steady state. It leads to the following expressions: $l_1 = \sqrt{2}\beta_0,$, $l_2 = \beta_0^2 - \beta^2$, where parameter β_0 determins the estimation rate in the linear case.

Some experimental results for $J_e = 0.63 \cdot 10^{-3}$ kg·m^2, $\varphi_0 = 7.3$ deg., $\beta = 6.6$ s^{-1}, $\beta_0 = 8s^{-1}$ are given in the Fig. 5.

In the case when the viscous friction and the external disturbing torque plays essential role in the system dynamic, the equation (8) has to be changed to

$$J\ddot{\varphi}(t) + \rho\dot{\varphi}(t) + mgl_e \sin(\varphi(t) - \varphi_0) = M(t), \tag{10}$$

where ρ is the friction coefficient, M is an external torque. (Note, that the last term can be used to approximately describe the influence of the internal pendulum.) The laboratory experiments with the considered system let to take

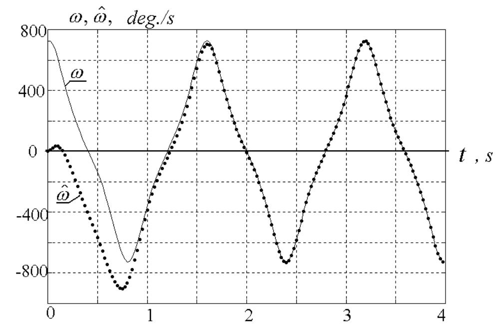

Figure 5: Angular velocity estimation based on the angles measurement.

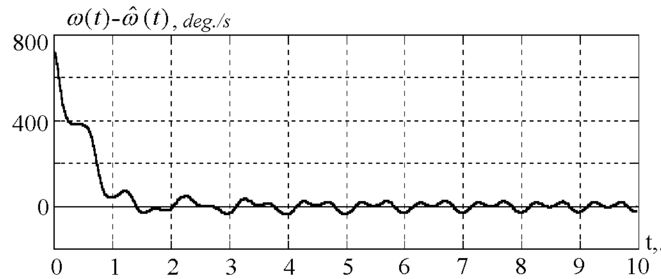

Figure 6: Estimation error time history for the perturbed system.

$\rho = 0.24$ kg·m^{2}s^{-1} and $M(t)$ as the harmonic process with the angular frequency 12 s^{-1} and the magnitude $2 \cdot 10^{-3}$ kg·m. The estimation error, obtained for considered case by simulation, is shown in the Fig. 6.

5.2 State vector estimation, based on the measurements of the time intervals

The identification of the system state vector based on the measurements of the time intervals $d_k = t_k - t_{k-1}$ (where $\varphi(t_k) = 0$, $k = 0, 1, 2, ...$) is a challenging task. In order to investigate its feasibility the state observer design was performed.

Again neglecting the motion of the second link and the friction the simplified pendulum-like system model (8) is used to obtain the state estimation algorithm.

The observer equation in the interior of the interval $[t_{k-1}, t_k]$ coincides with (8). By means of relation (5) one can calculate the state variables of the observer at time t_k as follows: $\tilde{\varphi}(t_k) = 0$, $\tilde{\omega}(t_k) = \Omega(\varphi_0)$. The initial (amplitude) value φ_0, in its turn, can be found as $\varphi_0 = \Phi(T)$, where $\Phi(T)$ is the inverse function (5) resolved for A_2 ($A_1 = 0$). It can be determined numerically *a priori*. For example, figure 7 presents the plot of $T(\varphi_0)$ for the considered system (3), the dependence $\Omega(\varphi_0)$ is shown in the Fig. 8. Two kinds of estimation algorithms are considered below.

The first one consists in re-counting the state estimates in the lower points by means of function $\omega_k = F(\varphi_0)$, where ω_k denotes the values of the angular velocity at the

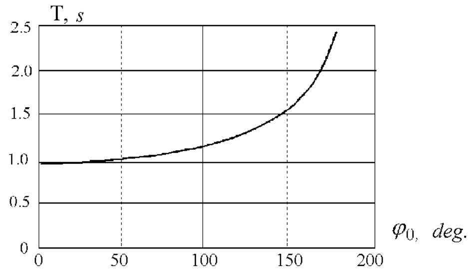

Figure 7: $T = T(\varphi_0)$ plot.

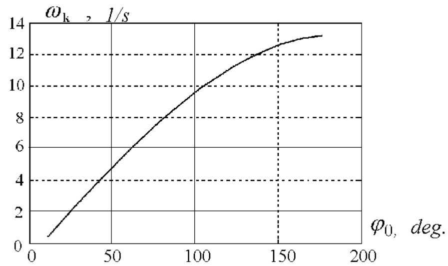

Figure 8: $\omega_k = \Omega(\varphi_0)$ plot.

instances t_k, $\varphi(t_k) = 0$. This algorithm is described by the following equations

$$\begin{array}{l} \hat{\varphi}(t_k) = 0, \quad \tilde{\omega}_k = \Omega(\varphi_0(T_k)), \\ \hat{\omega}(t_k) = d\hat{\omega}(t_{k-1}) + (1-d)\tilde{\omega}_k. \end{array} \tag{11}$$

Renewal coefficient d, $|d| < 1$, is used to provide the algorithm with the disturbance filtering properties. If $d = 0$, only current measurements are taken into account. The simulation example for this case is given in the Fig. 9, 10.

Another *gradient energy based correction* algorithm is proposed to reduce the influence of the disturbances. This algorithm has the form

$$\begin{array}{l} \hat{\varphi}_{k+1} = \Big(\hat{\varphi}_k + \\ \alpha \dfrac{\Delta H_k mgl \sin(\hat{\varphi}_k - \varphi_0)}{\big(mgl_e \sin(\hat{\varphi}_k - \varphi_0)\big)^2 + (J_e\hat{\omega}_k)^2}\Big) \bmod 2\pi, \\ \hat{\omega}_{k+1} = \hat{\omega}_k + \alpha \dfrac{\Delta H_k J_e \hat{\omega}_k}{(mgl_e \sin(\hat{\varphi}_k - \varphi_0))^2 + (J_e\hat{\omega}_k)^2}, \end{array} \tag{12}$$

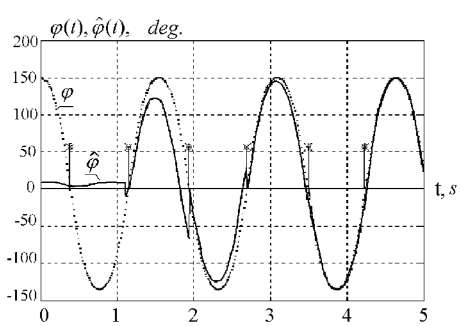

Figure 9: $\varphi(t)$ estimation by means the algorithm (11).

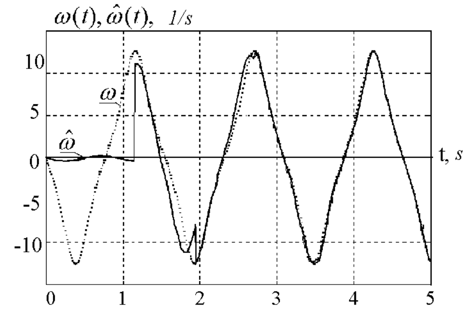

Figure 10: $\omega(t)$ estimation by means the algorithm (11).

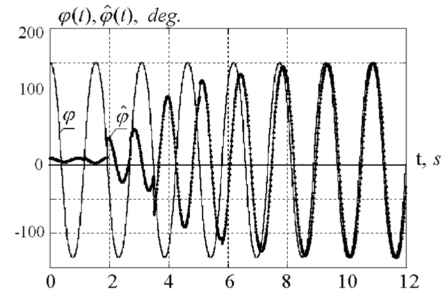

Figure 11: $\varphi(t)$ estimation by means the algorithm (12).

where $\alpha > 0$ is a gain coefficient, $\Delta H_k = \hat{H}_k - \tilde{H}_k$, $\tilde{H}_k = J_e\tilde{\omega}(t_k)^2/2$, $\hat{H}_k = J\hat{\omega}_k^2/2 + mgl_e\big(1 - \cos(\hat{\varphi}_k - \varphi_0)\big)$. The simulation results are shown in the Fig. 11, 12. The following parameter values were taken: $J_e = 0.63 \cdot 10^{-3}$ kg·ms^2, $mgl_e = 0.028$ kg·m^{2}s^{-1}, $\varphi_0 = 0.13$ rad ≈ 7.3 deg., $\alpha = 0.5$. As it is seen, the observation error becomes sufficiently small (below 5% level) after 4 – 5 periods of oscillation.

6 Parameter identification

Solutions of the control and observation problems, given above, are based on the knowledge of the plant parameters. For parameters identification the linearized autonomus plant model $J\ddot{\varphi}(t) + mgl_e(\varphi(t) - \varphi_0) = 0$ was transformed to the discrete form

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = 0, \quad n = 0, 1, 2, \ldots, \tag{13}$$

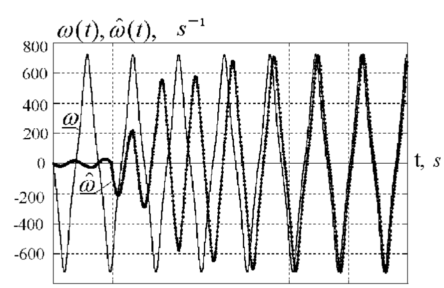

Figure 12: $\omega(t)$ estimation by means the algorithm (12).

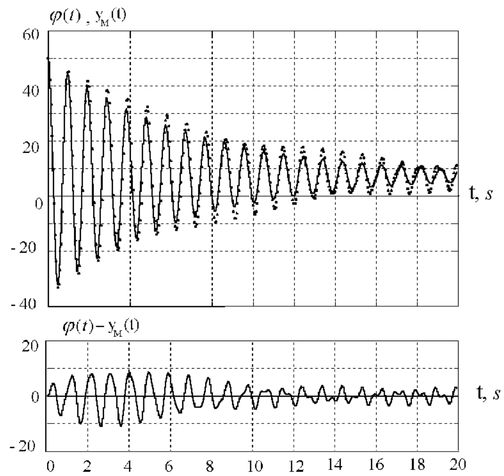

Figure 13: The plant and the tunable model outputs.

where $y[n] \stackrel{\text{def}}{=} \varphi(t_n)$, $t_n = nT_0$, T_0 =const is *the sample time*, a_1, a_2 are the *a priori* undetermined plant parameters.

Applying the standard LS-estimation procedure, one gets the estimates of a_1, a_2. Coming back to the continuous equation gives the parameters of the original plant model. This approach is based on the linear plant description, but can be used also to obtain the nonlinear model parameters if the experiment is conducted in the linear area of oscillations (when $\max|\varphi(t)| \approx 30$ deg.)

The sensitivity of LS-estimates to the disturbances is illustrated the Fig. 13, where the comparative plots of the plant and *the tunable model* outputs $\varphi(t), y_m(nT_0)$ are given. The model is described by the equation

$$y_m[n] + \hat{a}_1 y_m[n-1] + \hat{a}_2 y_m[n-2] = 0, \quad n = 0, 1, 2, \ldots,$$

where $\hat{a}_1, \hat{a}_2$ denote the LS-estimates of the plant (13) parameters a_1, a_2. The viscous friction and the disturbance torque are taken as those in the Sec. 5.1.

7 Conclusion

The mathematical model of oscillatory mechanical system based on aggregative method is given. The laboratory equipment for experiments with the controlled double links pendulum is described. The simple pulsing control law is used to provide the given total energy (or, in other words, given amplitude of oscillations) of the pendulum. Results of the numerical experiments are close to the laboratory ones. The special form of the observer is developted to obtain the state estimates via conditions when only period of oscillations can be measured.

The work was supported by the RFBR (grant 99-01-00672) and the Federal Program Integration (project N^o 589).

References

[1] Konjukhov A.P., Protopopov E. G. "Oscillations Control of Coupled Pendula Systems", *5th International Student Olympiad on Automatic Control. Abstracts.* St.Petersburg, Russia, October 2 – 4, pp. 75 – 79, (1996).

[2] Konjukhov, A.P., Tomchina O.P, Nagibina O.V. "Energy Based Double Pendulum Control in Periodic and Chaotic Mode", *Prepr. MOVIC'96*, (1996).

[3] Konoplev V.A., Konjukhov A.P. "Modelling , Control , and Laboratory Experiments with Oscillatory Mechanical System", *Proc. Intern. Conf. on Inform. and Control,* St. Petersburg, Russia, pp. 990 – 996, (1997).

[4] Andrievsky B.R., Fradkov A.L., Konoplev V.A., Konjukhov A.P. "Control, State Estimation and Laboratory Experiments with Oscillatory Mechanical System", *Prepr. Nonlin. Contr. Design Symp. NOLOCOS'98,* Twente, Holland, pp. 761 – 764, (1998).

[5] Fradkov A. L., Guzenko, P. Yu., Hill D. J., Pogromsky A. Yu. "Speed Gradient Control and Passivity of Nonlinear Oscillators", *IFAC Symposium on Nonlinear Control Systems (NOLCOS'95),* USA, 28 – 30 June 1995, pp. 655 – 659, (1995).

[6] Fradkov A.L. "Swinging control of nonlinear oscillations" *International Journal of Control*, vol. **6**, pp.1189-1202, (1996).

[7] Fradkov A. L., Egorenkov D. L., Kharlamov V. Yu. "Basics of mathematical modeling with samples on MATLAB^R language. " (A.L. Fradkov, Ed.) 192. p., BSTU St.Petersburg, (in Russian) (1996).

[8] Konoplev V. A. "Aggregative Models of Mechanic Solid Body Systems with Tree-Like Structure", *Mechanics of Solid*, vol. 24, **6**, pp.45 – 52, USA, New-York, (1989).

[9] Konoplev V.A. "The aggregative mechanic of solid systems", Science. St. Petersburg, (1996)

[10] Migulin V.V., Medvedev V.I., Mustel E.R., Parygin V.N. "Basics of oscillation theory" Science, Moscow, (in Russian) (1978).