SPEED-GRADIENT SCHEME AND ITS APPLICATION IN ADAPTIVE CONTROL PROBLEMS

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UDC 62-506:518.5

We consider a scheme for designing adaptive control algorithms that involves regulating the motion in the space of adjustable parameters in the direction of the gradient of the rate of change of the estimator functional. This scheme, which was first formulated in [1, 2] for the identification problem, includes a number of familiar adaptation and identification algorithms constructed by the direct Lyapunov method. General stability conditions for such systems are obtained. For regularizing algorithms we present ways that give a system robustness properties with respect to the effect on the object of uncontrollable perturbations and with respect to the discreteness of an adaptation algorithm. We give examples of the use of this scheme for designing adaptation algorithms in a number of problems of adaptive control of dynamic objects.

1. INTRODUCTION

During the last 10 or 15 years a large number of publications have appeared dealing with the development of algorithms for the adaptive control of dynamic objects (see, for example, the bibliographies in [3-6]). These algorithms relate to diverse particular problems, and diverse considerations, frequently heuristic, are used for designing them. Therefore, there is a natural interest in establishing general principles for the design of adaptive control algorithms and in determining the general properties of the systems designed. The first results in this direction are due to A. A. Krasovskii [1, 2], who obtained a general type of algorithm that is optimal by the "generalized operation" criterion for the problem of identification with an adaptive model. This algorithm is described by a functional series and is not realizable in pure form, but if we take the first approximation to it, then we get a family of well-known gradient algorithms that use sensitivity functions.

If we consider the second approximation, then (assuming a high frequency of the input signal and the quasi-stationarity of the tuning mode) we can determine a family of algorithms of which particular cases were constructed earlier with the help of Lyapunov functions. The present article deals with an investigation of the general properties of such algorithms. We obtain conditions for stability and for the attainment of the adaptation goal, and we present a method for regularizing algorithms that ensures the robustness of the system with respect to the effect of uncontrollable perturbations and the discreteness of the adaptation algorithm.

It turns out to be convenient in the beginning of the exposition (in §§ 2 and 3) not to refer to the identification problem or to any other adaptive control problem, but to speak of investigating the stability of a system of differential equations with a specified form, without indicating a concrete adaptation goal and class of adaptiveness. Concrete adaptive control problems are considered in § 4 as examples of the use of the general results.

The scheme considered in this paper for designing adaptation algorithms will be called the speed-gradient scheme. The origin of this name is explained below in § 2, where we give the general formulation of this principle and establish conditions for the stability and asymptotic optimality of systems constructed with its help. In § 3 we present a method for regularizing algorithms that ensures the robustness of the systems constructed with respect to the effect of uncontrollable perturbations and to the discrete realization of the algorithms. In § 4 we give examples of the application of the speed-gradient principle to the analysis and design of adaptive systems for controlling dynamic objects.

2. FORMULATION OF THE SPEED GRADIENT PRINCIPLE

Let us consider an adjustable object described by the differential equations

Leningrad. Translated from Avtomatika i Telemekhanika, No. 9, pp. 90-101, September, 1979. Original article submitted July 12, 1978.

0005-1179/79/4009-1333$07.50

$$\dot{x}=F(x,c,t), \tag{1}$$

where x is the state vector of the object, c is a vector of adjustable parameters, and F(x, c, t) is a smooth function.

The vector x can be made up of the state vectors of the controlled object itself, of the controlling and measuring devices, of the regulator, etc. For the formation of an algorithm for adjusting the vector c we are given a certain auxiliary (estimator) functional J_t that characterizes the quality of operation of the system. We consider two basic cases:

I. $J_t = Q(x(t), t)$ (local functional);

II. $J_t=\int\limits_0^t R(x(s),\ c(s),s)ds$ (integral functional).

Here Q(x, t) and R(x, c, t) are certain smooth functions.

In each of these cases we can compute the function $\dot{J}_t$, the rate of change of the functional J_t, by virtue of Eq. (1) (for a fixed vector c). For the first case $\dot{J}_t = F(x(t), c(t), t)^T \nabla_x Q(x(t), t) + \partial Q/\partial t(x(t), t)$, and for the second case $\dot{J}_t = R(x(t), c(t), t)$; i.e., in both cases $\dot{J}_T = \varphi(x(t), c(t), t)$, where $\varphi(x, c, t)$ is some function having continuous partial derivatives with respect to the components of the vector c.

We now define the speed-gradient algorithm to be the following law of change of the vector of adjustable parameters:

$$\dot{c}=-\Gamma\nabla_c\varphi(x,c,t). \tag{2}$$

Here $\Gamma = \Gamma^T > 0$ is a positive-definite matrix of the appropriate order (e.g., $\Gamma = \gamma I$, where $\gamma > 0$, and I is the identity matrix). According to the algorithm (2), for $\Gamma = \gamma I$ the vector c is varied in the direction of the gradient of the rate of change of the functional J_t, which justifies the name of the algorithm. As remarked in [1, 2], a number of algorithms proposed by various authors for solving concrete problems of adaptive control of dynamic objects reduce to the form (2) (see below, § 4).

The system (1), (2) of differential equations is closed, and we can pose the problem of investigating the qualitative properties of its trajectories in the phase space {x, c}. In particular, it is of interest to study the stability of the system (1), (2), which can be interpreted as stabilization of the object (1) with the help of the algorithm (2). Here it is natural to require the theoretical solvability of the stabilization problem, i.e., the existence of a "stabilizing" vector of parameters c_*. The following theorem shows that under the additional assumption that the function $\varphi(x, c, t)$ is convex in c the algorithm (2) stabilizes the object (1) in the sense that the trajectories of the system (1), (2) are bounded.

Theorem 1. Suppose that there is a vector c_* such that for any x and t

$$\varphi(x,c_*,t)\leqslant 0, \tag{3}$$

and, moreover, the function $\varphi(x, c, t)$ is convex in c; i.e., for any c', c", x, and t

$$\varphi(x,c',t)-\varphi(x,c'',t)\geqslant(c'-c'')^T\nabla_c\varphi(x,c'',t). \tag{4}$$

Then along any trajectory of the system (1), (2) the value of J_t is bounded above:

$$J_t\leqslant J_0+{}^1/_2[c(0)-c_*]^T\Gamma^{-1}[c(0)-c_*]. \tag{5}$$

If, moreover, the functional J_t is local and

$$\inf_{t\geqslant 0} Q(x,\ t)\to+\infty \quad \text{as} \quad \|x\|\to\infty, \tag{6}$$

then all the trajectories of the system (1), (2) are bounded.

Remark 1. The condition (4) is obviously satisfied if the function $\varphi(x, c, t)$ is linear in c.

Remark 2. It is easy to show that the theorem remains true if (3) is replaced by the weaker condition

$$\varphi(x(t),c_*,t)\leqslant\beta(t), \tag{3'}$$

where $\beta(t)\geqslant 0$, $\int\limits_0^\infty \beta(t)\,dt<\infty$.

A proof of Theorem 1 is given in the Appendix and is based on the use of the direct Lyapunov method. As the Lyapunov function we take the following functional V_t of the trajectories of the system (1), (2):

$$V_t = J_t + \tfrac{1}{2}[c(t) - c_*]^{\mathrm{T}}\Gamma^{-1}[c(t) - c_*]. \tag{7}$$

In addition to stability conditions we can also get estimates of the quality of the systems designed. The following theorem gives us the asymptotic optimality of speed-gradient algorithms in the sense of a minimum for the estimator function Q(x, t).

Theorem 2. Suppose that the right-hand sides of Eqs. (1), (2) are locally bounded uniformly with respect to $t \geq 0$; i.e.,

$$\sup_{t>0,\, \|z\|\leq\rho} (\|F(z, t)\| + \|\nabla_c\varphi(z, t)\|) < \infty, \tag{8}$$

for any $\rho > 0$, where $z = \{x, c\}$ is the state vector of the system (1), (2). Suppose that the functional J_t is uniformly continuous with respect to x, t in any region of the form $\{(z, t)\colon \|z\| \leq \rho,\ t \geq 0\}$, is local, is nonnegative, and satisfies the conditions (4), (6), and, instead of (3), the weaker inequality

$$\varphi(x, c_*, t) \leqslant -\alpha Q(x, t) \qquad (\alpha > 0). \tag{9}$$

Then $Q(x(t), t) \to 0$ as $t \to \infty$.

A proof of Theorem 2 is given in the Appendix.

Remark 1. The inequality (9), which can be rewritten in the form $\dot{J}_t \leq -\alpha J_t$, is the condition for exponential stability of the object (1) at $c = c_*$, while the condition (3) means that (1) is stable in the Lagrange sense [7].

It should also be mentioned that in the case of an integral functional J_t the asymptotic optimality in the sense of the criterion

$$\Phi_t = \frac{1}{t}\int_0^t R(x(s),\ c(s),\ s)\, ds$$

can be derived from Theorem 1. Indeed, for $R(x, c, t) \geq 0$ it follows from (5) that $\lim_{t\to\infty} \Phi_t = \min_{c,t} \Phi_t = 0$. Moreover, it follows from Lemma 1 (see the Appendix) that when the condition (8) holds, the relation $\lim_{t\to\infty} R(x(t), c(t), t) = 0$ is valid. In applying Theorem 2 the relations mentioned are interpreted as the adaptation goals, and the algorithm (2) is intended for attaining them.

3. REGULARIZATION OF SPEED-GRADIENT ALGORITHMS

For the practical application of an adaptation algorithm it is important to consider the behavior of the adaptive system when factors not taken into account in the initial design act on it. These factors usually include inaccuracies of the original mathematical model: determinate and random noises, additional nonlinearities and inertial properties, etc. Another important factor is the discreteness in time of the operation of the controlling device when it is realized on a digital computer. A practically efficient system must preserve its efficiency, at least under small actions of the indicated factors; i.e., it must be robust with respect to them.

Simple examples show that systems designed on the basis of the speed-gradient principle may not have robustness. In particular, the action on the object of arbitrarily small uncontrollable noises can destroy the stability of the system: the adjustable coefficients c(t) can grow unboundedly as $t \to \infty$ (see, e.g., [4, 8]). The speed-gradient scheme can fail to be robust because the system (1), (2) is in a certain sense at the boundary of stability: the derivative of the Lyapunov function V_t is not a negative-definite function.

In order to give the system (1), (2) robustness properties we can use the idea of regularization [3]. Instead of the functional J_t the following "regularized" functional can be used for designing an algorithm:

$$\mathcal{J}_t = J_t + \frac{\lambda}{2}\int_0^t \|c(s)\|^2\, ds, \tag{10}$$

where $\lambda > 0$ is the regularization parameter. Applying the speed-gradient principle, we get, instead of the algorithm (2), the regularized algorithm

$$\dot{c} = -\Gamma(\nabla_c\varphi(x, c, t) + \lambda c). \tag{11}$$

Accordingly, regularization has led to the introduction of a negative feedback in the circuit for tuning the parameters.

Passing to a study of the robustness of the system (1), (2) with respect to noises, we remark that by the stability of the system in the presence of uncontrollable noises of unknown intensity it is natural to mean its dissipativeness [7], i.e., the convergence as $t \to \infty$ of all its trajectories into some bounded region in phase space $\{x, c\}$ not depending on the initial conditions $x(0)$, $c(0)$.

Theorem 3. Suppose that the functional J_t is local and satisfies the conditions (4) and (6), the condition $\|\nabla_x Q(x, t)\|^2 \le \rho[1 + Q(x, t)]$, where $\rho > 0$, and, instead of (3), the inequality

$$\varphi(x, c, t) \leqslant -\alpha Q(x, t) + \beta, \tag{12}$$

where $\alpha > 0$, $\beta \ge 0$. Suppose, further, that the dynamics of the object is described by the following equation instead of (1):

$$\dot{x} = F(x, c, t) + \xi(t), \tag{13}$$

where $\|\xi(t)\| \le \varkappa$, $\varkappa > 0$.

Then the system (11), (13) is dissipative.

A proof of the theorem is given in the Appendix. Theorem 3 shows that the regularized algorithms ensure the efficiency of the system for an arbitrary level of uncontrollable noises, since no bounds are imposed on the quantity $\varkappa$. However, it is easy to see that the size of the limit set, which characterizes the quality of the system, increases with increasing $\varkappa$.

Remark 1. It is not hard to show (by modifying slightly the proof of Theorem 3) that an analogous result holds in the case when $\xi(t)$ is a random process that is bounded in the mean square ($M\|\xi(t)\|^2 \le C < \infty$) or has the form of a white noise with bounded intensity. Here the system (11), (13) is dissipative in the mean-square sense; i.e., we have the inequality

$$\overline{\lim_{t \to \infty}} M\{\|x(t)\|^2 + \|c(t)\|^2\} \leqslant D \tag{14}$$

for some $D > 0$.

Remark 2. The theorem remains true if we replace the integrand $\|c\|^2$ in the functional (10) by $\omega(c)$, and the λc in the algorithm (11) by $\lambda \nabla \omega(c)/2$. We can take the "penalty" function $\omega(c)$ to be any convex smooth function satisfying for some β_0, $\beta_1 > 0$ the quadratic growth condition $\omega(c) \ge \beta_1 \|c\|^2 - \beta_0$. In particular, $\omega(c)$ can be set equal to zero in a specified bounded "admissible" region.*

Let us now consider the question of the influence of the discreteness of a realization of an adaptation algorithm on the efficiency of the speed-gradient scheme. In this case the regularized algorithms (11) also have the property of robustness, although in a somewhat weakened sense. The fact of the matter is that the equations of the dynamics of adaptive control systems are usually nonlinear, and their right-hand sides do not satisfy a global Lipschitz condition. Such equations can lose stability (dissipativeness) properties upon discretization, even for an arbitrarily small discreteness step. However, it is frequently possible to show that the discretized system has the property of so-called limiting dissipativeness [13]. This property amounts to the existence in the phase space of the system of a ball D_∞ into which all the trajectories of the system with initial conditions in some ball D_0 converge in the course of time, where the radius of the ball D_0 grows unboundedly as the discreteness step decreases. From a practical point of view, limiting dissipativeness coincides in essence with the usual dissipativeness for a small discreteness step.

It turns out that the regularized speed-gradient algorithms (11) ensure the preservation of limiting dissipativeness of the system in a discrete realization. Let us formulate a precise statement for the stationary case.

Theorem 4. Suppose that the conditions of Theorem 3 hold, and that the functions $F(x, c, t)$ and $Q(x, t)$ do not depend on t. Suppose that the function $F(x, c)$ satisfies the conditions of local boundedness (8) and is locally Lipschitzian, $\|F(x', c) - F(x'', c)\| \le L(r, c)\|x' - x''\|$ for $\|x'\| \le r$, $\|x''\| \le r$, and that the function $Q(x)$ satisfies the

*A closely related result is given in [10]. A negative feedback in the adaptation circuit was used in a particular case in [11]. Other ways are also known for making the algorithms more robust, for example, the introduction of an insensitivity zone, which is widely used in the method of recursive goal inequalities [12].

inequality $\|\nabla^2 Q(x)\| \leq \varkappa_1$. Let the adaptation algorithm be described by the following difference equation instead of (11):

$$c(t_{k+1}) = c(t_k) - h\Gamma[\nabla_c \varphi(x(t_k), c(t_k), t_k) + \lambda c(t_k)], \tag{15}$$

where $t_k = kh$, $k = 0, 1, \ldots, h > 0$.

Then the system (13), (15) has the property of limiting dissipativeness as $h \to 0$.

A proof of the theorem is given in the Appendix.

In concluding the section we mention that regularization of the algorithms means, of course, giving up their optimality [for the algorithms (11), (15) Theorem 2 becomes false]; however, this is the price for increasing the robustness of a system (see, e.g., [14]).

4. EXAMPLES OF APPLICATION OF THE SPEED-GRADIENT SCHEME

By choosing various forms of the functional J_t, we can obtain various algorithms for adaptive control and identification described in the literature. Some examples are given in Table 1. In the majority of cases a direct check shows that the conditions of Theorems 1 and 2 hold, and the familiar results about stability of adaptive systems and about the attainment of adaptation goals follow from these theorems. Let us consider several examples in more detail.*

Example 1. "Direct" Adaptive Control with a Reference Model [3, 4, 15-20]

Suppose that an adjustable object is described by the equation

$$\dot{x} = (A + \Delta A)x + (B + \Delta B)r(t), \tag{16}$$

where $r(t)$ is a vector drive signal, A and B are matrices of unknown parameters of the object, and ΔA and ΔB are matrices of adjustable coefficients; i.e., $c = \{\Delta A, \Delta B\}$. For an estimate of the system's quality we choose the local functional $J_t = e^T(t)He(t)$, where $e(t) = x - x_M(t)$, $x_M(t)$ being the solution of the reference-model equation $\dot{x}_M = A_M x_M + B_M r(t)$. By the adaptation goal we understand the minimization of J_t, i.e., the satisfaction of the condition $e(t) \to 0$ as $t \to \infty$. Computing $\dot{J}_t = \varphi(x, c, t)$ according to the speed-gradient principle, we get $\varphi(x, c, t) = e^T H[(A + \Delta A)x + (B + \Delta B)r(t) - A_M x_M - B_M r(t)]$, from which $\nabla_{\Delta A}\dot{J}_t = Hex^T$, $\nabla_{\Delta B}\dot{J}_t = Her^T(t)$, and, consequently, the adaptation algorithm (2) for $\Gamma = \gamma I$ can be written in the form†

$$d\Delta A/dt = -\gamma Hex^T, \quad d\Delta B/dt = -\gamma Her^T(t). \tag{17}$$

The condition (4) holds here because $\dot{J}_t$ is linear in c. Moreover, the condition (9) holds because, setting $c_* = \{A_M - A, B_M - B\}$ and choosing the matrix H for the given Hurwitz matrix A_M from the condition $HA_M + A_M^T H < 0$, we get that

$$\varphi(x, c_*, t) = e^T H A_M e \leq -\alpha e^T He \ (\alpha > 0). \tag{18}$$

The condition (6) holds for a Hurwitz matrix A_M and bounded $r(t)$. Thus, the trajectories of the system (16), (17) are bounded and the adaptation goal formulated above is attained for any values of the unknown object parameters when the indicated conditions in Theorems 1 and 2 hold. An analogous result holds also in the case when not all the elements of the matrices ΔA and ΔB can be adjusted, but the matrices A, A_M, B, B_M have a special form (see Table 1).

Example 2. Adaptive Identification with a Predicting Model [4, 29]

Suppose that the object is described by the equations

$$A(p)y = B(p)u, \quad H(p)y_1 = y, \quad H(p)u_1 = u, \tag{19}$$

where $p = d/dt$, u, y, u_1, y_1 are scalars, $A(p) = p^n + a_{n-1}p^{n-1} + \ldots + a_0$, $B(p) = b_m p^m + \ldots + b_0$ are polynomials with unknown coefficients determining the dynamics of the object, and $H(p)$ is a known Hurwitz "filtering" polynomial

*The results obtained in Examples 1 and 2 are well known; the algorithm in Example 3 is apparently new.

†We mention that if in this example we regard the vector $x_M(t)$ as the state of the object and $x(t)$ as the state of the adjustable model, then we arrive at the identification algorithms considered in [20].

TABLE 1. Speed-Gradient Algorithms

No.	Equation of the object	Functional J_t	Adaptation algorithm for $\Gamma = \gamma I$	Convergence conditions; result	Authors
1	$\dot{x} = (A + \Delta A)x + (B + \Delta B)r(t)$, а) $c = \{\Delta A, \Delta B\}$	$J_t = e^T He$, $H = H^T > 0$, $e = x - x_M$, $\dot{x}_M = A_M x_M + B_M r(t)$	$\frac{d\Delta A}{dt} = -\gamma H e x^T$, $\frac{d\Delta B}{dt} = -\gamma H e r^T$,	A_M Hurwitz $HA_M + A_M^T H < 0$; $e(t) \to 0$	Landau [16], Lüdors, Narendra [17], Lindorff, Carroll [4] Ashimov, Syzdykov, Tokhtabaev [20]
	б) $\Delta A \begin{bmatrix} 0 \mid I_{n-1} \\ a^T \end{bmatrix}, \Delta B = \begin{bmatrix} 0 \\ \vdots \\ \beta \end{bmatrix}$ $B = 0$, $c = \{a, \beta\}$	$J_t = e^T He$, $A_M = \begin{bmatrix} 0 \mid I_{n-1} \\ a_M^T \end{bmatrix}$, $Hd = h$, $d = [0 \ldots\ldots 01]$	$\dot{a} = -\gamma h^T e x$, $\dot{\beta} = -\gamma h^T e r$	A_M Hurwitz Re $h^T (i\omega I_n - A_M)^{-1} d \geqslant$ $\geqslant \frac{\varepsilon}{1 + \omega^2}$; $e(t) \to 0$	Zemlyakov, Rutkovskii [15] Phillipson [18] Winsor, Roy [19]
2	$\dot{x} = Ax_0 + G(x - x_0) + Br(t)$ а) $c = \{A, B\}$	$J_t = e^T He$, $H = H^T > 0$, $e = x - x_0$, $\dot{x}_0 = A_0 x_0 + B_0 r(t)$	$\dot{A} = -\gamma(Hex_0^T + \lambda_1 A)$, $\dot{B} = -\gamma(Hex^T + \gamma_2 B)$, $\lambda_1, \lambda_2 > 0$	G Hurwitz $HG + G^T H < 0$; $\lVert e(t) \rVert \leqslant$ const	Kudva, Narendra [21]
	б) $A = \begin{bmatrix} a & I_{n-1} \\ & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \vdots \\ \beta \end{bmatrix}$ $c = \{a, \beta\}$	$J_t = e^T He$, $A_0 = \begin{bmatrix} a_0 & I_{n-1} \\ & 0 \end{bmatrix}$, $Hd = h$, $d = [0 \ldots 01]^T$	$\dot{a} = -\gamma h^T e x_0$, $\dot{\beta} = -\gamma h^T e r$	G Hurwitz Re $h^T (i\omega I_n - G)^{-1} d \geqslant$ $\geqslant \frac{\varepsilon}{1 + \omega^2}$; $e(t) \to 0$	Eremin, Nguyen Thuc Loan, Chkhartishvili [24]
3	$\dot{x} = f(x_0, c, t) + Ge$, $e = x - x_0$	$J_t = e^T He$, $\dot{x}_0 = f(x_0, c_*, t)$	$\dot{c} = -\gamma \left[\frac{\partial f}{\partial c}(x_0, c, t)\right]^T He$	G Hurwitz, $HG + G^T H < 0$; the point $\{e = 0, c = c_*\}$ is asymptotically stable	Seregin [22]

4	$\dot{x} = Gx + b\sum[c_j' y_j(t) + c_j'' u_j(t)]$, $c = \{c_j', c_j''\}$, $y_j = W_j(p)y$, $u_j = W_j(p)u \quad j = 1, \ldots, n$	$J_t = e^T H e$, $H = H^T > 0$, $e = x - x_0$, $\dot{x}_0 = Gx_0 + b\sum_j[\alpha_j y_j(t) + \beta_j u_j(t)]$	$\dot{c}_j' = -\gamma h^T e y_j$, $\dot{c}_j'' = -\gamma h^T e u_j$	G Hurwitz $\mathrm{Re}\, h^T(i\omega I_n - G)^{-1} b \geqslant$ $\geqslant \frac{\nu}{1+\omega^2}$; $e(t) \to 0$	Brusin [23] Narendra, Kudva [25]
5	$\dot{x} = Ax + bu$, $y = Lx$, $u = c^T y$, $W(\lambda) = L(\lambda I_n - A)^{-1} b$, $\delta(\lambda) = \det(\lambda I_n - A)$, $a(\lambda) = \delta(\lambda) W(\lambda)$	$J_t = x^T H x$, $H = H^T > 0$, $Hb = L^T g$	$\dot{c} = -\gamma(g^T y) y$	$g^T a(\lambda)$ a Hurwitz polynomial of degree $n-1$ $g^T a(0) > 0$; $x(t) \to 0$, $\Vert c(t)\Vert \to \mathrm{const}$	Fradkov [26]
6	$A(p) y = bu$, $p = d/dt$, $A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0$, $u = c_{n-1} p^{n-1} y + \ldots + c_0 y$, $c = [c_0 \ldots c_{n-1}]^T$	$J_t = \int_0^t \delta^2(s)\, ds$, $\delta(s) = G(p) y(s)$, $G(\lambda) = \lambda^n + g_{n-1}\lambda^{n-1} + \ldots + g_0$	$\dot{c}_j = -\gamma\,\vert\delta(t)\vert_\varepsilon\, p^j y$, $j = 0, 1, \ldots, n-1$, $\vert\delta\vert_\varepsilon = \begin{cases} \delta, & \vert\delta\vert > \varepsilon, \\ 0, & \vert\delta\vert < \varepsilon \end{cases}$	$G(\lambda)$ — a Hurwitz polynomial $\delta(t) \to 0$, $y(t) \to 0$	Aksenov, Fomin [27], Andrievskii, Fradkov [28]
7	$A(p) y = B(p) u$, $H(p) y_1 = y$, $H(p) u_1 = u$, $c = \{a_{M_j}, b_{M_j}\}$	$J_t = \int_0^t \delta^2(s)\, ds$, $\delta(s) = A_M(p) y_1(s) - B_M p(u_1) s$, $A_M(\lambda) = \lambda^n + a_{M_{n-1}}\lambda^{n-1} + \ldots + a_{M_0}$, $B_M(\lambda) = b_{M_m}\lambda^m + \ldots + b_{M_0}$	$\dot{a}_{M_j} = -\gamma\delta(t) p^j y_1$, $j = 0, 1, \ldots, n-1$, $\dot{b}_{M_k} = -\gamma\delta(t) p^k u_1$, $k = 0, 1, \ldots, m$	$M(\lambda)$ — a Hurwitz polynomial $\delta(t) \to 0$	Lion [29]

Notes. 1. The algorithm in example 2a) is regularized by introducing a negative feedback, and the algorithm in example 6 is regularized by introducing an insensitive zone. 2. If in example 3 the function $[f(x_0, c, t)]^T He$ is convex in c, then $e(t) \to 0$ for arbitrary initial conditions (this follows from Theorem 2).

[Eqs. (19) can easily be reduced to the form of the state equations (1)]. Let $J = 1/2\int_0^t \delta^2(x(s), c(s))\,ds$, where $\delta(x, c) = A_M(p)y_1 - B_M(p)u_1$, and the degrees of the polynomials $A_M(p)$ and $B_M(p)$ are equal to the degrees of $A(p)$ and $B(p)$, respectively. The coefficients of the polynomials $A_M(p)$ and $B_M(p)$ can be interpreted as parameters of a "predicting" model; then $\delta(x, c)$ has the meaning of the discrepancy between the model and the object. Therefore, we take the vector of adjustable parameters to be the vector of coefficients of the polynomials $A_M(p)$ and $B_M(p)$; i.e., $c^T = (a_{M_0},\ldots,a_{M_{n-1}}, b_{M_0},\ldots,b_{M_m})$. We understand the adaptation goal to be the minimization of the mean-square of the discrepancy: $\lim t^{-1}J_t = 0$. Computing $\dot{J}_t = 0.5\delta^2(x; c)$ and then $\nabla_c \dot{J}_t$ according to the speed-gradient principle, we arrive at the following algorithm for adjusting the model parameters:

$$\begin{aligned} \dot{a}_{M_i} &= -\gamma_i\delta(x(t), c(t))y_1^{(i)}, \quad i = 0, 1, \ldots, n-1, \\ \dot{b}_{M_i} &= \beta_i\delta(x(t), c(t))u_1^{(i)}, \quad i = 0, 1, \ldots, m. \end{aligned} \tag{20}$$

It is not hard to see that in this case the conditions of Theorem 1 hold with Remark 3 taken into account [for $c_*^T = (a_0,\ldots,a_{n-1}, b_0,\ldots,b_m)$ and $\beta(t) = \eta^2(t)$, where $\eta(t)$ is an arbitrary solution of the equation $H(p)\eta = 0$]. Therefore, the remarks following Theorem 2 imply that the adaptation goal is attained for any values of the unknown object parameters and, moreover, $\delta(x(t), c(t)) \to 0$. We mention that if the degree of the polynomial $H(p)$ is not less than the degrees of $A(p)$, $B(p)$, then the algorithm (20) can be realized without changing the derivatives of the signals $u(t)$, $y(t)$.

Example 3. Adaptive Stabilization of an Object with Amplification Coefficient Depending on the Phase Coordinates

Suppose that the object and the control law are described by the equations

$$\dot{x} = Ax + \mu(x)bu, \quad y = Lx, \quad u = c^T y, \tag{21}$$

where $x \in R^n$ is the state vector of the object, u is a scalar control, $y \in R^m$ is a vector of measurable coordinates of the object, $c \in R^m$ is a vector of adjustable coefficients of the controller, and A, B, and L are unknown object parameters. It is assumed that the amplification coefficient $\mu(x)$ is measurable, depends continuously on x, and satisfies the inequality $\mu(x) \geq \mu > 0$. Equations of the form (21) are encountered, e.g., in problems of controlling the power output in nuclear reactors [30]. Since we are considering the problem of stabilization [the object (21) can, generally speaking, be unstable], we choose $J_t = x^T(t)Hx(t)$, where $H > 0$ is some matrix of order $n \times n$. Proceeding according to the speed-gradient principle, we get

$$\dot{J}_t = \varphi(x, c) = x^T H(Ax + \mu(x)bc^T y), \quad \nabla_c\varphi(x, c) = x^T Hb\mu(x)y.$$

Since the quantity $x^T Hb$ appearing in the algorithm must depend only on the measurable coordinates, we impose the additional requirement that $Hb = L^T g$. Here the speed-gradient algorithm takes the form

$$\dot{c} = -\mu(x)g^T y\Gamma y. \tag{22}$$

Obviously, the conditions (4) and (6) hold in this case. We can show that the condition (9) holds if for some c_* we have the inequality $HA_* + A_*^T H < 0$, where $A_* = A + bc_*^T L$. Since the matrix H can also be varied, we get that the conditions of Theorem 2 hold and the system (21), (22) is efficient if for some $H > 0$, c_* we have the relations

$$Hb = L^T g, \quad HA_* + A_*^T H < 0, \quad A_* = A + bc_*^T L. \tag{23}$$

By Lemma 1 in [26], (23) holds if and only if the polynomial $\delta(p)g^T W(p)$, where $W(p) = L(pI - A)^{-1}b$ is the transfer function of the object and $\delta(p) = \det(pI - A)$, is Hurwitz of degree $n-1$ with positive coefficients. The last conditions determine a class of objects for which the adaptation goal $x(t) \to 0$ is attained, by Theorem 2.

CONCLUSION

It should be mentioned that the questions touched upon in this paper do not, of course, exhaust all the problematics of adaptive control theory. We have neglected stages in the design of an adaptive system such as the choice of the basic circuit structure, the choice of an estimator functional according to the specified control goal, and the choice of the adaptation circuit parameters according to the specified adaptiveness class.

The results above relate only to a unification of the stages of choosing the structure of the adaptation circuit and of analyzing the efficiency of the system.

The problem of comparing the various methods of designing adaptive control algorithms and of determining their limiting possibilities is as yet far from a definitive solution. The questions connected with the robustness of the algorithms and the possibility of their regularization are important aspects of this problem.

APPENDIX

Proof of Theorem 1. Computing the derivative of the functional (7) by virtue of the system (1), (2) and using first the inequality (4) for $c' = c_*$, $c'' = c(t)$ and then the inequality (3), we get $dV_t/dt = \varphi(x(t), c(t), t) + [c(t) - c_*]^T \Gamma^{-1} \dot{c} \leq \varphi(x(t), c_*, t) \leq 0$, from which $J_t \leq V_t \leq V_0 = J_0 + 0.5[c(0) - c_*]^T \Gamma^{-1} [c(0) - c_*]$; i.e., the inequality (5) holds. The second part of the statement of the theorem follows from the fact that, by what was proved, the state vector $\{x(t), c(t)\}$ of the system (1), (2) lies in the region $V(x, c, t) \leq V(x(0), c(0), 0)$, which, by the condition (6), is bounded uniformly with respect to $t \geq 0$. The theorem is proved.

Proof of Theorem 2. Computing (as in the proof of Theorem 1) the derivative of the functional (7) by virtue of the system (1), (2), we get from the condition (9) that

$$dV_t/dt \leqslant -\alpha Q(x(t), t) \leqslant 0,$$

from which $\alpha \int_0^t Q(x(s), s)\,ds \leqslant V_0 - V_t$ and, consequently, $\int_0^\infty Q\,ds \leqslant V_0/\alpha$. Since the trajectories of the system (1), (2) are bounded (by Theorem 1), the statement of Theorem 2 now follows from the following simple lemma.

Lemma 1. Let us consider a system $\dot{z} = F(z, t)$ whose right-hand side is locally bounded uniformly with respect to $t \geq 0$; i.e., $\|F(z, t)\| \leq \eta(r) < \infty$ for $z \in \Omega_r = \{z : \|z\| \leq r\}$, $t \geq 0$. Let $z(t)$ be a bounded solution of the system, and suppose that the function $Q(r, t) \geq 0$ is uniformly continuous with respect to z, t in any region Ω_r and is such that $\int_0^\infty Q(z(t), t)\,dt < \infty$. Then $\lim_{t\to\infty} Q(z(t), t) = 0$.

Proof of Theorem 3. We again consider the functional V_t of the form (7) and estimate the derivative of V_t by virtue of the system (11), (13). Writing $x(t) = x$, $c(t) = c$, we get

$$dV_t/dt \leqslant \varphi(x, c_*, t) + \nabla_x Q(x, t)^T \xi(t) - \lambda (c - c_*)^T c.$$

Using the conditions of the theorem and the easily checked inequalities

$$\nabla_x Q(x, t)^T \xi(t) \leqslant 0.5\varepsilon \|\nabla_x Q(x, t)\|^2 + (2\varepsilon)^{-1} \|\xi(t)\|^2,$$
$$-(c - c_*)^T c \leqslant 0.5\mu (c - c_*)^T \Gamma^{-1} (c - c_*) + 0.5\|c_*\|^2,$$

in which $\varepsilon = \alpha/\rho$ and μ is the greatest lower bound of the spectrum of the matrix Γ, we get that

$$dV_t/dt \leqslant -\frac{\alpha}{2} Q(x, t) - \frac{\lambda\mu}{2} (c - c_*)^T \Gamma^{-1} (c - c_*) + \beta_1 \leqslant -\alpha_1 V_t + \beta_1, \tag{A.1}$$

where $\alpha_1 = \min\{0.5\alpha, \lambda\mu\}$; $2\beta_1 = 2\beta + \alpha + \varkappa\rho/\alpha + \lambda\|c_*\|^2$.

It follows from the inequality (A.1) that $\lim_{t\to\infty} V_t \leq \beta_1/\alpha_1$. If we now recall that (6) holds, then we see that the theorem is proved.

Proof of Theorem 4. For a proof we use the results in [13, 31]. Along with the system (13), (15) let us consider the continuous model $\dot{z} = A(z)$ of it, where $z = \begin{pmatrix} x \\ c \end{pmatrix}$, $A(z) = \begin{pmatrix} F(x, c) \\ -\Gamma(\varphi(x, c) + \lambda c) \end{pmatrix}$. It follows from the proof of Theorem 3 that the system $\dot{z} = A(z)$ satisfies the exponential dissipativeness condition [13] with the function $V(x, c)$ of the form (7). The noise $\xi(t)$ in (13) is, by assumption, bounded. Therefore, the limiting (conditional) dissipativeness of the system (13), (15) follows directly from Theorem 1 in [13].

LITERATURE CITED

1. A. A. Krasovskii, "Optimal algorithms in problems of identification with an adaptive model," Avtom. Telemekh., No. 12, 75-82 (1976).

2. A. A. Krasovskii, V. N. Bukov, and V. S. Shendrik, Universal Algorithms for Optimal Control of Continuous Processes [in Russian], Nauka (1977).
3. B. N. Petrov, V. Yu. Rutkovskii, I. N. Krutova, and S. D. Zemlyakov, Principles of Construction and Design of Self-adjusting Control Systems [in Russian], Mashinostroenie (1972).
4. D. P. Lindorff and R. L. Carroll, "Survey of adaptive control using Lyapunov design," Int. J. Control, No. 5, 897–914 (1973).
5. J. D. Landau, "A survey of model reference adaptive techniques," Automatica, 10, 353–379 (1974).
6. R. B. Ésher, D. Andrisani, and P. P. Dorato, "The literature on the theory of adaptive systems," Tr. Inst. Inzh. Elektrotekh. Radioelektron., No. 8, 126–142 (1977).
7. B. P. Demidovich, Lectures on the Mathematical Theory of Stability [in Russian], Nauka (1967).
8. D. P. Lindorff, "Effects of incomplete adaptation and disturbance in adaptive control," in: Preprints of the 13th Joint Automatic Control Conf., AIAA, New York (1972), pp. 562–567.
9. A. N. Tikhonov and V. Ya. Arsenin, Methods for Solving Ill-Posed Problems [in Russian], Nauka (1974).
10. Yu. I. Neimark, Dynamic Systems and Control Processes [in Russian], Nauka (1978).
11. K. S. Narendra, S. S. Tripathi, G. Lüders, and P. Kudva, "Adaptive control using Lyapunov's direct method," Tech. Report No. CT-43, Yale Univ., New Haven (1971).
12. V. A. Yakubovich, "The method of recursive goal inequalities in the theory of adaptive systems," in: Problems in Cybernetics. Adaptive Systems [in Russian], Nauchn. Sov. Kompleks. Probl. Kibern., Akad. Nauk SSSR (1976), pp. 32–63.
13. D. P. Derevitskii and A. L. Fradkov, "An investigation of discrete adaptive systems for controlling dynamic objects with the help of continuous models," Tekh. Kibern., No. 5, 93–99 (1975).
14. Ya. Z. Tsypkin and B. T. Polyak, "A robust maximal likelihood method," in: Systems Dynamics [in Russian], No. 12, Gor'kovsk. Gos. Univ. (1977), pp. 22–46.
15. S. D. Zemlyakov and V. Yu. Rutkovskii, "Generalized adaptation algorithms for nonscanning self-adjusting systems with a model," Avtom. Telemekh., No. 6, 88–94 (1967).
16. J. D. Landau, "A hyperstability criterion for model-reference adaptive control systems," IEEE Trans., AC-14, No. 5, 552–555 (1969).
17. G. Lüders and K. S. Narendra, "Lyapunov functions for quadratic differential equations with application to adaptive control," IEEE Trans., AC-17, No. 6, 798–801 (1972).
18. P. H. Phillipson, "Design methods for model-reference adaptive systems," Proc. Inst. Mech. Eng., 183, No. 35, 695–700 (1968/1969).
19. C. A. Winsor and R. I. Roy, "Design of model-reference control systems by Lyapunov's second method," IEEE Trans., AC-13, No. 2, 204 (1968).
20. A. Ashimov, D. Zh. Syzdykov, and G. M. Tokhtabaev, "Nonscanning self-adjusting systems for identification," Avtom. Telemekh., No. 2, 184–188 (1973).
21. P. Kudva and K. S. Narendra, "An identification procedure for linear multivariable systems," Tech. Report No. CT-48, Yale Univ., New Haven (1972).
22. V. N. Seregin, "Design of an asymptotically stable identification algorithm for a nonlinear nonstationary system by the direct Lyapunov method," Avtom. Telemekh., No. 4, 28–32 (1978).
23. V. A. Brusin, "Design of a simulator for linear stationary objects," Avtom. Telemekh., No. 4, 55–59 (1977).
24. E. P. Eremin, Nguyen Thuc Loan, and G. S. Chkhartishvili, "A nonscanning identification system with a model, constructed by a hyperstability criterion," Avtom. Telemekh., No. 5, 54–65 (1973).
25. K. S. Narendra and P. Kudva, "Stable adaptive schemes for system identification and control," IEEE Trans. Syst. Man Cybern., SMC-4, No. 5, 542–560 (1974).
26. A. L. Fradkov, "Design of an adaptive system for stabilization of a linear dynamic object," Avtom. Telemekh., No. 12, 96–103 (1974).
27. G. S. Aksenov and V. N. Fomin, "On linear adaptive control systems," in: Computational Methods [in Russian], No. 8, Leningr. Gos. Univ. (1973), pp. 95–116.
28. B. P. Andrievskii and A. L. Fradkov, "Analysis of the dynamics of a certain algorithm for adaptive control of a linear dynamic object," in: Problems of Cybernetics. Adaptive Systems [in Russian], Nauchn. Sov. Kompleks. Probl. Kibern., Akad. Nauk SSSR (1976), pp. 99–103.
29. P. N. Lion, "Rapid identification of linear and nonlinear systems," AIAA J., 5, 1835–1842 (1967).
30. V. D. Goryachenko, Methods for Investigating the Stability of Nuclear Reactors [in Russian], Nauka (1976).
31. D. P. Derevitskii, "Design of a stochastic discrete adaptive stabilization system with the help of a continuous model," Avtom. Vychisl. Tekh., No. 6, 50–52 (1975).