SPEED-GRADIENT LAWS OF CONTROL AND EVOLUTION

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Abstract. A class of evolution laws of manmade and natural dynamic systems is described. The essence of these laws is the motion along the gradient of the speed of appropriate goal functional change. It is shown that such systems (SG-systems) cover many different control and adaptation algorithms as well as a set of well known physical systems evolution laws. Theorems about SG-systems stability and robustness as well as some examples are given.

Keywords. Adaptation, speed-gradient algorithms, evolution, Lyapunov function, stability, robustness.

1. INTRODUCTION

In automatics as well as in the other fields f science an interest grows to general principles nifying the solutions of different kinds of roblems. Conventionally such principles are ariational ones and postulate the solution sought o realize the extremal value of appropriate unctional. For instance, some optimal control ethods [1-2] give the equations, the solutions of hich define control action as a function of time nd/or controlled plant state variables. Analogous rinciples based on calculus of variations are sed in physics (minimal action principle [3]).

Perhaps it is more convenient to obtain the olution moving step by step along time axis and sing only current information provided by the ocal goal functional. We can mention Gauss inimal forcing principle as an example of this local" approach. Local principles do not require he knowledge neither of the future of the system r of the final time of its evolution. So they do t require the anticipation actions. On the other nd the local approach to system design requires e further analysis of global system behavior: ability, convergence, etc. Speaking about local inciples advantages M.Planck regretted that they e applicable only to the problems of mechanics).

In the control science the local principles e used rather often, e.g. Zubov principle of imal transient processes damping [5], locally imal adaptive control algorithms [6,7], etc.

In the 2nd section of this paper one more al principle named speed-gradient (SG) ciple is described. This principle was ulated in [8] and then generalized in [9-11]. covers many different laws of adaptation and control as well as many evolution laws of different natural systems. The further sections contain the stability and robustness theorems for SG-systems as well as some examples of such systems: proportional-integral regulator, model reference adaptive systems designed by Lyapunov function and hyperstability methods, identification systems and some physical systems (Newton's law, diffusion and heat conduction equations, viscous fluid equations).

2. SPEED-GRADIENT PRINCIPLE

Consider process equations in the form

$$dx/dt=F(x,\theta,t) \quad , \quad t\geq 0 \qquad (1)$$

where $x\in R^n$ is a process state vector, $\theta\in R^m$ is an input vector, $F(\cdot):R^n\to R^m$ is continuously differentiable vector-function in x,θ. Input variables may be of arbitrary nature : real control action for the plant, adjustable parameters or smth. else. The problem is to choose the evolution law[1)]

$$\theta(t)=\Theta[x_0^t,\theta_0^t,t] \qquad (2)$$

according to some criterion of "good" functioning of the system.

Suppose this criterion requires to provide low values of some goal functional $Q_t=Q[x_0^t,\theta_0^t,t]$. Typically Q_t may be of the local form

[1]Notation x_0^t means the set of values $\{x(s), 0\leq s\leq t\}$, $\nabla_x Q$ denotes the gradient of Q in x, sign "T" denotes transposition and $x=\mathrm{col}(x_1,\ldots,x_n)$ means that x is the column vector which components include all the components of vectors or matrices $x_1,\ldots,x_n$.

$Q_t=Q(x(t),t)$, where $Q(x,t)\geq 0$ is a scalar smooth goal function or of the integral form : $Q_t=\int_0^t q(x(s),\theta(s),s)ds$.

In either case one can determine a function $\omega(x,\theta,t)$ - the velocity of Q_t change along trajectories of (1). For example, $\omega(x,\theta,t)=(\nabla_x Q)^T F(x,\theta,t)$ for a local case and $\omega(x,\theta,t)=q(x,\theta,t)$ for anintegral case. Obviously $\dot{Q}_t=\omega(x(t),\theta(t),t)$.

In [8] the following law has been introduced

$$d\theta/dt=-\Gamma\nabla_\theta\omega(x,\theta,t) \qquad (3)$$

called speed-gradient (SG) algorithm in differential form. Later the finite form was suggested [10]:

$$\theta-\theta_0=-\Gamma\nabla_\theta\omega(x,\theta,t). \qquad (4)$$

In (3), (4) $\Gamma=\Gamma^T>0$ is a positive definite matrix. The most general form is a combined form of SG-law:

$$\frac{d}{dt}\left[\theta+\psi(x,\theta,t)\right]=-\Gamma\nabla_\theta\omega(\dot{x},\theta,t), \qquad (5)$$

where $\psi(\cdot)$ satisfies pseudogradientity condition $\psi^T\nabla_\theta\omega\geq 0$. We may rewrite (5) in an integral form

$$\theta=-\psi(x,\theta,t)-\Gamma\int_0^t \nabla_\theta\dot{Q}_s ds. \qquad (6)$$

The formulation of speed-gradient principle is as follows.

Of all possible motions those motions are realized for which input variables change proportionally to the speed-gradient of appropriate goal functional.

Now let us give some examples illustrating SG-laws derived on the basis of this principle.

2.CONTROL AND ADAPTATION LAWS

Example 1. Proportional-integral regulation law.

Let the process equation be

$$\dot{x}=Ax+Bu, \qquad y=Cx, \qquad (7)$$

where $x\in R^n$ is a process state vector, $u\in R^1$ is a scalar control action, $y\in R^1$ is controlled variable, A,B,C are matrices of appropriate size. Let control goal be $y(t)\to y_*$ as $t\to\infty$, where y_* is required value of y. Thus, the goal function may be taken as

$$Q_t=\frac{1}{2}\,(y(t)-y_*)^2=\frac{1}{2}e^2(t), \qquad (8)$$

where $e=y-y_*$ is an error function.

If we choose u(t) as the plant input, that is $\theta=u$, we can calculate the speed of Q_t change along trajectories of (7):

$$\dot{Q}_t=\omega(x,\theta,t)=eC(Ax+B\theta)$$

and then determine the speed-gradient $\nabla_\theta\omega(x,\theta,t)=eCB$. Assuming that sign(CB) is known (CB>0 for simplicity) we define $\psi=\gamma_p\nabla_\theta Q_t/CB=\gamma_p e$, $\gamma_1=\Gamma CB$, where $\Gamma>0$, $\gamma_p>0$, $\gamma_1>0$. The combined SG-law (5) is then given by

$$\frac{d}{dt}\,(\theta+\gamma_p e)=-\gamma_1 e. \qquad (9)$$

After integrating and substituting $\theta\to u$ we get the usual PI-regulation law:

$$u(t)=-\gamma_p e(t)-\gamma_1\int_0^t e(s)ds. \qquad (10)$$

Example 2. Model reference adaptive control via hyperstability criterion.

Let controlled plant be described by equation

$$\dot{x}=(A+\Delta A)x + (B+\Delta B)r(t), \qquad (11)$$

where $x\in R^n$ is a state vector, $r(t)\in R^m$ is command action vector, A,B, - unknown parameter matrices, ΔA, ΔB, - adjustable parameter matrices of appropriate size. Let control goal be $e(t)\to 0$ as $t\to\infty$ where $e=x-x_m$, $x_m\in R^n$ is a state vector of reference model

$$\dot{x}_m(t)=A_m x + B_m r(t). \qquad (12)$$

Choosing $Q(x,t)=e^T He$ where $H=H^T>0$ as the goal function and $\theta=col(\Delta A,\Delta B)$ as an input vector, we may calculate the speed and the speed-gradient as follows

$$\dot{Q}_t=e^T H[(A+\Delta A)x+(B+\Delta B)r(t)-A_m x_m-B_m r(t)],$$

$$\nabla_{\Delta A}\dot{Q}_t=Hex^T, \quad \nabla_{\Delta B}\dot{Q}_t=Her^T.$$

The combined SG-law (6) for this case has a form

$$\Delta A=-\Psi_A-\Gamma_B\int_0^t He(s)x^T(s)ds$$
$$\Delta B=-\Psi_B-\Gamma_B\int_0^t He(s)r^T(s)ds \qquad (13)$$

where $Sp(\Psi_A Hex^T)\geq 0, Sp(\Psi_B Her^T)\geq 0$. (For instance $\Psi_A=sign(Hex^T)$, $\Psi_B=sign(Her^T)$). The laws (13) coincide with adaptation algorithms synthesized via hyperstability criterion [12,13]. So SG-algorithms may be regarded as the generalization of conventional MRAS algorithms for nonlinear case.

Example 3. Nonlinear plant identification with explicit and implicit models.

Consider a nonlinear system (e.g. a pendulum) described by second-order equation

$$\ddot{y}=a*\sin(y)+b*f(t), \qquad (14)$$

ere y is generalized coordinate, $f(t)$ is asurable external force, a,b are unknown rameters to be determined. Form adjustable del of system (14) as follows

$$\ddot{y}_m=d_1(y-y_m)+d_2(\dot{y}-\dot{y}_m)+a_m\sin(y)+b_m f(t), \qquad (15)$$

ere $d_1>0$, $d_2>0$ are introduced to ensure ability. Choosing $Q_t=e^T He$, $\theta=col(a_m,b_m)$, where $x-x_m=col(e_1,e_2)$, $e_1=y-y_m$, $e_2=\dot{y}-\dot{y}_m$ and lculating speed-gradient as it was done above we n write differential SG-law (3) in the form

$$\dot{a}_m=-\gamma_a(h_{12}e_1+h_{22}e_2)\sin(y),$$
$$\dot{b}_m=-\gamma_b(h_{12}e_1+h_{22}e_2)f(t). \qquad (16)$$

entification laws of type (16) were suggested many authors (see [8] for bibliography). To rive another type of identification law let us andon the explicit adjustable model (15) and use tegral goal functional $Q_t=0.5\int_0^t\delta^2(s)ds$ where $\delta(t)=\ddot{y}(t)-a_m\sin(y)-b_m f(t)$ is the measure of model viation from the system (14) (equation error). en straightforward calculation gives

$$\dot{Q}_t=\frac{1}{2}\delta^2(t), \quad \frac{\partial\dot{Q}_t}{\partial a_m}=\delta\sin(y), \quad \frac{\partial\dot{Q}_t}{\partial b_m}=\delta f(t)$$

ith the following SG-law

$$\dot{a}_m=-\gamma_a\delta\sin(y), \quad \dot{b}_m=-\gamma_b\delta f(t). \qquad (17)$$

4. PHYSICAL SYSTEM EVOLUTION LAWS

In this section we apply the SG-principle for scription of natural systems. The goal nctional for such systems must be chosen in such way that real system behavior corresponds to all or decreasing values of the goal functional.

Example 4. Newton's evolution law.

Consider the motion of a material point in a tential force field. Let the state vector x be e vector of point coordinates $x=col(x_1,x_2,x_3)$ d the input vector be $\dot{x}$. Then the process uation (1) is

$$\dot{x}=\theta. \qquad (18)$$

know that the point motion is characterized by creasing of potential energy $Q(x)$.So we try to ke $Q(x)$ as the goal function. It is clear that $\dot{Q}_t=[\nabla_x Q(x)]^T\theta$ and speed-gradient of Q_t due to (18) s $\nabla_\theta\dot{Q}_t=\nabla_x Q(x)$. Choosing the SG-law in fferential form (3) with scalar gain matrix $m^{-1}I_3$ we obtain the law of motion

$$\ddot{x}=-m^{-1}\nabla_x Q(x) \qquad (19)$$

incIding (naturally !) with Newton's law when m the point mass.

Example 5. Wave, diffusion ,heat conduction and viscous fluid transfer equations.

Note that the SG-principle may be extended to distributed parameter processes. For instance state vector x may be an element of Gilbert space X and $F(\cdot)$ - nonlinear operator (possibly unbounded) defined at the dense set $D_F\subset X$. The solution of (1)-(5) may be defined as a generalized one.

Suppose $x=x(r)$ is the temperature or the concentration of a substance in some region $\Omega\subset R^3$, $r=col(r_1,r_2,r_3)\in\Omega$ and process equation is specified in the form (18).Choose the goal functional as the measure of the field ununiformity :

$$Q(x)=\frac{1}{2}\int_\Omega|\nabla_r x(r)|^2dr. \qquad (20)$$

Under zero boundary conditions we have

$$\dot{Q}_t=-\int_\Omega\Delta x(r)\theta(r)dr, \quad \nabla_\theta\dot{Q}_t=-\Delta x(r),$$

where $\Delta=\sum_{i=1}^{3}\frac{\partial^2}{\partial r_i^2}$ - Laplace operator. It is easy to see that the differential form of SG-law (3) corresponds to D'Alamber wave equation, while the finite form (4) with $\theta_0=0$ gives the heat conduction (diffusion) equation.

Now let us choose

$$Q_t=\int_\Omega p(r,t)dr + \eta\int_0^t\int_\Omega|\nabla_r v(r,s)|^2drds, \qquad (21)$$

where $v(r,t)$ - field of fluid particles velocities, $p(r,t)$ - pressure, $\eta>0$ - viscosity coefficient. For this case the speed-gradient is $\nabla_r p-\eta\Delta v$. Combining (3) with (18) we get Navier-Stocks equation for viscous fluid transfer:

$$\dot{v}=-[\nabla_r p(r,t)-\eta\Delta v(r,t)]. \qquad (22)$$

Similarly choosing the goal functional as appropriate energy or entropy function one can obtain equations for different electric, thermodynamic and other systems. It is worth mentioning that the differential form (3) of SG-laws corresponds to reversible processes while the finite form (4) generates irreversible ones. Some other examples of SG-principle application in control science and physics are given in [11,14].

5. STABILITY AND ROBUSTNESS OF SG-SYSTEMS

It is convenient to study dynamic properties of SG-systems by means of Lyapunov functions. For example the stability theorems for combined SG-law are formulated as follows.

Theorem 1 [10] (local goal functional). Let system (1),(5) have unique solution for any initial conditions $x(0),\theta(0)$, functions $F(x,\theta,t)$, $\nabla_x Q(x,t)$, $\psi(x,t)$, $\nabla_x\omega(x,\theta,t)$ be locally bounded in t (bounded in any region $\{(x,\theta,t): |x|+|\theta|\leq\beta, t\geq 0\}$) and following conditions are held:

1. *Growth condition:* $\inf_t Q(x,t)\to\infty$ as $|x|\to\infty$.

2. *Convexity condition*: function $\omega(x,\theta,t)$ is convex in θ.

3. *Attainability condition:* vector $\theta_*\in R^m$ and function $\rho(Q)$ exist such that $\rho(Q)>0$ when $Q>0$ and

$$\omega(x,\theta_*,t)\leq-\rho(Q). \qquad (23)$$

Then all solutions of system (1),(5) are bounded and $Q_t\to 0$ as $t\to\infty$.

Theorem 2 [14] (integral goal functional). Let conditions of theorem 1 are fulfilled with $\rho(Q)\equiv 0$ in (23).

Then all solutions of system (1),(5) are bounded and $q(x(t),\theta(t),t)\to 0$ as $t\to\infty$.

Consider local case and finite SG-laws which we can write as

$$\theta=\theta_0(x,t)+\gamma(x,t)\psi(x,t) \qquad (24)$$

where $\gamma(x,t)$ is a scalar.

Theorem 3 [14]. Let all conditions of the theorem 1 are fulfilled as well as strong pseudogradientity condition

$$\psi(x,t)^T\nabla_\theta\omega(x,\theta,t)\geq\rho|\nabla_\theta\omega(x,\theta,t)|^\delta \qquad (25)$$

for some $\rho>0$, $\delta\geq 1$ and inequality

$$\rho\gamma(x,t)|\nabla_\theta\omega(x,\theta,t)|^{\delta-1}\geq|\theta_0-\theta_*| \qquad (26)$$

(vector θ_* may depend on x,t).

Then all solutions of system (1), (24) are bounded and $Q_t\to 0$ as $t\to\infty$.

The proof of theorems 1,2 is based on Lyapunov functional

$$V_t=Q_t+\tfrac{1}{2}[\theta-\theta_*+\psi(x,t)]^T\Gamma^{-1}[\theta-\theta_*+\psi(x,t)], \qquad (27)$$

while for theorem 3 one can use $V_t=Q_t$. The theorems can be generalized for the case when process state vector x belongs to an infinite dimensional Gilbert space. The precise formulations use appropriate existance and uniqness theorems (see [15]). More detailed investigation of SG-system stability and robustness can be find in [11].

Let us illustrate the usage of the theorems for example 2. In this case regularity and convexity conditions are carried out owing to linearity of plant and reference model equations both in the coordinates and in the adjustable parameters. The growth condition is held when H>0. For the validity of the attainability condition the existance of $\theta_*=col(\Delta A_*,\Delta B_*)$ it is nesessary such that $A+\Delta A_*=A_m$, $B+\Delta B_*=B_m$. Thus for $\theta=\theta_*$ we have $\dot Q_t=e^THA_me$. Obviously if A_m is stable and matrix $H=H^T>0$ is chosen as the solution of Lyapunov equation

$$HA_m+A_m^TH=-R \qquad (28)$$

for some $R=R^T>0$, then condition (23) will be held with $\rho(Q)=\rho Q$, $\rho=\min\lambda_i(R)$, $\lambda_i(R)$ - eigenvalues of R. Hence $e(t)\to 0$ as $t\to\infty$. Thus we have obtained some well known results (see e.g. [13]) by means of the theorem 1.

The process model (1) is often not known precisely so that the true model has a form

$$\dot x=F(x,\theta,t)+\phi(x,\theta,t), \qquad (29)$$

where ϕ is an unknown bounded disturbance. Adaptive SG-systems are known to become unstable even for arbitrary small disrurbance level [16]. To provide the robustness of the system the modified laws are usually used, e.g.

$$\dot\theta=-\Gamma\nabla_\theta\dot Q_t-\nabla\omega(\theta), \qquad (30)$$

where $\omega(\theta)$ is convex penalty function (see [16,17] for linear plant). System (29),(30) trajectories tend not to the point but to the region. The bounds for final set can be found in [11]. Note that the modified SG-system (29),(30) keeps its properties in presence of unmodelled dynamics (singular perturbations, additional filters etc.,see [11]).

6. CONCLUSION

The unified approach to system evolution description was given. This approach has different applications, the control science being the first of them. The general methodology for control and adaptation algorithms analysis and synthesis appears to be fruitful for their comparison and rational choice.

The second application field is physics where some well known problems may be considered from new point of view. For instance a simple proof was suggested in [11] for Onzagger symmetry principle validity for SG-systems.

But it seems that the most interesting and intrinsic applications are generated by physical-cybernetic analogy. An example of such result is as follows. The particle in electromagnetic field is known to belong to the class of SG-systems only when magnetic part of the field is absent. The reason of it is that magnetic field action generates rotational motions which doesn't meet the requirements for SG-systems.

Immediately the question arises: can't we generate analytically such rotational motions in control SG-systems? Occasionally it is the case and such systems may possess some new properties. For instance one can introduce the rotational motions into model reference adaptive systems (see example 2) when adding the antisymmetric component to H matrix. Matrices of such kind may be generated as Lyapunov equation (28) solutions for non-symmetric right-hand side. So we receive some new algorithms which one can name "rotational" ones. These algorithms are not of speed-gradient type because both goal functional and Lyapunov function are invariant to asymmetry of H.

Simulation results for MRAS with implicit reference model showed that the rotational algorithms provide them additional possibility of transient processes oscillability regulation. See Fig.1, where transient processes of system

$$\dot{x}=Ax+bu, \quad u=\theta^T x, \quad \dot{\theta}=-\gamma(x^T Hb)x \qquad (31)$$

for $x\in R^2$, $\gamma=3.5$ are presented). Plant transfer function is $W(p)=1/(16p^2+4*1.41p+1)$. Matrix H is found as solution of equation (28) for $A_m=A+\Delta A$,

$$R=\begin{bmatrix}1 & \varepsilon\\ -\varepsilon & 1\end{bmatrix}.$$

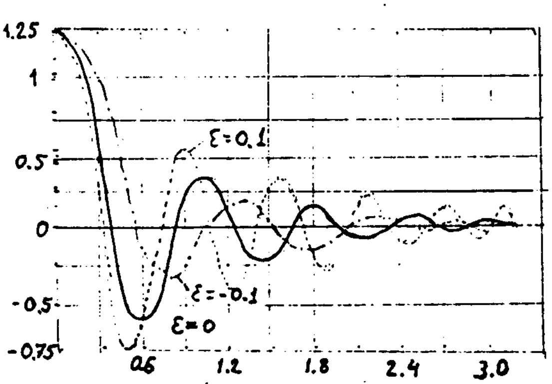

Fig. 1

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