European Journal of Control (2003)6:573–582

European Journal of Control

Passification of Non-square Linear Systems and Feedback Yakubovich–Kalman–Popov Lemma*

Alexander Fradkov

Institute for Problems of Mechanical Engineering, Russian Academy of Sciences, 61 Bolshoy ave V.O., St. Petersburg, 199178, Russia

The concepts of G-passivity and G-passifiability (feedback G-passivity) are introduced extending the concepts of passivity and passifiability to nonsquare systems (systems with different numbers of inputs and outputs). Necessary and sufficient conditions for strict G-passifiability of nonsquare linear systems by output feedback are given. Simple description of a broad subclass of passifying feedbacks is proposed. The proofs are based on a version of the celebrated Yakubovich–Kalman–Popov lemma.

Keywords: Linear Systems; Passification; Passivity; Yakubovich–Kalman–Popov Lemma

1. Introduction

During the last decade there was a growing interest in passivity-based or passification design methods. It attracted an attention to feasibility conditions for passification and to *passifiable* or *feedback passive* systems – ones that can be made passive by means of state or output feedback [7,15,25,27,29,31,32]. Such passifiability conditions are of interest even for linear systems because they are used not only for linear feedback design, but also for passivity-based design of cascade nonlinear systems [29,31]. Since passivity of a linear system is equivalent to positive realness of its transfer function, the linear passifiable (strictly passifiable) systems were also called "almost positive real" ("almost strictly positive real") [23,33]. The conditions for passifiability by output feedback are of most importance.

Necessary and sufficient conditions for passifiability of linear systems by linear output feedback were suggested in [34,41] (for SISO systems) and in [1,18,21] for MIMO systems. Note that feedback structures in [1,18] and in [21] are different (see also [37]). State feedback case was considered in [24,29]. In [13], it was shown that strict passifiability conditions for state feedback and output feedback case coincide. In a number of works the problem of positive real synthesis for systems with feedthrough (relative degree zero case) was considered, see [35] and references therein. The obtained results have applications in robust control [1,18,34,41], adaptive control [2,21,23], stabilization of partially linear cascaded systems [24,29,31].

In the Russian literature the problem of feedback design ensuring strict positive realness was studied still in the 1970s, see [9] for SIMO systems and [10] for MIMO systems. The results were systematically applied to adaptive control [8,11,36], to VSS design [4], to synchronization systems design [14,17,16]. Although the papers [9,10] (translations from Russian) were published in the West and some results of [9,10] were recalled in [2,3,13], these results were somewhat overlooked in the Western literature.

*The work was supported in part by the Russian Foundation for Basic Research (project 02-01-00765) and by the Program 19 of the Presidium of RAS (project 1.4).
Fax: +7 (812) 321-4771. E-mail: alf@control.ipme.ru

Received 25 October 2001; Accepted in revised form 4 June 2003. Recommended by V. Rasvan and S. Morse.

Particularly, the necessary and sufficient conditions of output strict passifiability in [1,18,21,34,41] are straightforward consequences of the corresponding results of [9,10] and the adaptive output feedback control algorithms proposed in [21] (and recalled in [25]) coincide with those of [10].

The existing passification and passifiability results are developed for the case of systems with equal number of inputs and outputs (square systems). However, nonsquare systems appear in many applications. Using the squaring down procedure (multiplying the input or output vector by a rectangular matrix of appropriate size, see [30]) for output passification design leads to unnecessary reduction of the designer's possibilities. Indeed, the passifying feedback law of the squared down system may contain less gain coefficients that may reduce flexibility of the design and ease of interpreting the controller parameters. For adaptive control systems, the nonsquaring down designs lead to better parameter convergence: it was shown by computer simulation that parameters may converge faster and their limit values are smaller [8,11]. Therefore, it is interesting to extend the passification framework to nonsquare systems.

In this paper, the concepts of G-passivity and G-passifiability (feedback G-passivity) are introduced extending the concepts of passivity and passifiability to nonsquare systems. Necessary and sufficient conditions for strict G-passifiability of nonsquare linear systems by output feedback are given in Theorems 2 and 3. Besides, simple parametrization of a broad subclass of passifying feedbacks is proposed (see Corollary 3). In the proofs some auxillary matrix inequalities results are used (Theorem 4 and Lemma 5) are used which, in turn, are based on a "semi-singular" version of the celebrated Yakubovich–Kalman–Popov lemma established by V.A. Yakubovich in 1966 [40] (Theorem 1).

The problem is formulated in Section 2 where the G-passivity definitions for nonsquare systems are also introduced. The main results are presented in Section 3, while the proofs are given in Section 4.

2. G-passivity and G-passification

Consider an affine in control system

$$\dot{x} = f(x) + g(x)u, \quad y = h(x), \tag{1}$$

where $x = x(t) \in \mathbb{R}^n, u = u(t) \in \mathbb{R}^m, y = y(t) \in \mathbb{R}^l$ are state, input and output vectors, respectively, f, h are smooth vector-valued functions of x and g is smooth matrix-valued function of x. In some applications (e.g., in control of quantum-mechanical systems), the case with complex-valued variables and parameters ($x = x(t) \in \mathbb{C}^n, u = u(t) \in \mathbb{C}^m, y = y(t) \in \mathbb{C}^l$) is also important. Such a case will be called *complex case* while the case of real-valued variables and parameters will be called *real case*.

Let G be a prespecified $m \times l$-matrix.

Definition 1. System (1) is called *G-passive* if there exists a nonnegative scalar function $V(x)$ (*storage function*) such that:

$$V(x) \leq V(x_0) + \int_0^t u(t)^* G y(t)\, \mathrm{d}t \tag{2}$$

for any solution of system (1) satisfying $x(0) = x_0, x(t) = x$. In (2) and below, the asterisk denotes transposition of the matrix and complex conjugation of its elements which is just transposition in real case.

Definition 2. System (1) is called *strictly G-passive*, if there exist a nonnegative scalar function $V(x)$ and a scalar function $\mu(x)$, where $\mu(x) > 0$ for $x \neq 0$, such that

$$V(x) \leq V(x_0) + \int_0^t [u(t)^* G y(t) - \mu(x(t))]\, \mathrm{d}t \tag{3}$$

for any solution of system (1) satisfying $x(0) = x_0$, $x(t) = x$.

In this paper, we will be dealing with strict version of G-passivity property for linear systems

$$\dot{x} = Ax + Bu, \quad y = Cx, \tag{4}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, A, B, C are matrices of appropriate size. For linear systems, the storage function $V(x)$ is quadratic form $V(x) = 0.5x^* Hx$ (or Hermitian form in complex case), while function $\mu(x)$ is just Euclidean norm of the vector: $\mu(x) = \mu|x|^2$, $\mu > 0$.

Obviously, if $l = m$ and $G = I_m$ is identity matrix, then G-passivity coincides with conventional passivity property. In turn, passivity is closely related to *hyperstability*, introduced by V.M. Popov [28]. In fact V.M. Popov was the first who studied passivity in detail for linear control systems and gave its characterization in terms of frequency-domain inequality meaning positive realness of the system.

Note that if the storage function $V(x)$ is smooth, the integral dissipation inequalities (2), (3) are equivalent to their differential forms. For a nonlinear system (1), the integral inequality (3) is equivalent to fulfillment of the differential dissipation inequality

$$\frac{\partial V}{\partial x}(f(x) + g(x)u) \leq u^* Gy - \mu(x). \qquad (5)$$

For linear system (4) and for quadratic storage function $V(x) = 0.5x^* Hx$, the integral inequality (3) is equivalent to

$$x^* H(Ax + Bu) \leq u^* Gy - \varrho |x|^2 \qquad (6)$$

for some $\varrho > 0$ and all $x \in \mathbb{R}^n, u \in \mathbb{R}^m$.

In its turn, the dissipation inequality (5) is equivalent to the relations

$$\frac{\partial V}{\partial x} f(x) \leq -\varrho |x|^2, \quad \frac{\partial V}{\partial x} g(x) = (Gh(x))^*, \qquad (7)$$

while the inequality (6) is equivalent to the following matrix relations

$$HA + A^* H < 0, \quad HB = (GC)^*. \qquad (8)$$

The solvability conditions for (8) and related versions of the dissipation inequalities for linear systems are given by the seminal Yakubovich–Kalman–Popov or Kalman–Yakubovich lemma (for this special case also called positive real lemma). In what follows we need the "semi-singular" version of the Yakubovich–Kalman–Popov lemma established by V.A. Yakubovich [40] in 1966. Introduce the following notations[1]:

$$G = HA + A^* H + R, \quad g = -Ha - b,$$
$$Q(H) = \begin{bmatrix} -G & g \\ g^* & \varrho \end{bmatrix},$$
$$\pi(s) = \varrho - 2\mathrm{Re}\, b^* (sI_n - A)^{-1} a - a^* (s^* I_n - A^*)^{-1} R (sI_n - A)^{-1} a,$$

where $H = H^*$ is $n \times n$-matrix, $R = R^*$ is $n \times n$-matrix, $\varrho = \varrho^*$ is $m \times m$-matrix, a, b are $n \times m$-matrices. Let $m = m_1 + m_2$, where m_1, m_2 are integer numbers and let matrices ϱ, π, a be split into the corresponding blocks as follows:

$$\varrho = \begin{bmatrix} \varrho_{11} & \varrho_{12} \\ \varrho_{21} & \varrho_{22} \end{bmatrix}, \quad \pi = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}, \quad a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

and $\varrho_{12} = \varrho_{21}^* = 0, \varrho_{22} = 0$.

[1]Notation Re K stands for the Hermitian part of the matrix: $\mathrm{Re}\, K = (K + K^*)/2$.

Theorem 1 (*Yakubovich, 1966*)**.** Let A be Hurwitz matrix, $\varrho_{11} \geq 0$ and rank $a_2 = m_2$. Necessary and sufficient conditions for existence of matrix $H = H^*$ such that $Q(H) \geq 0$ and rank $Q(H) = n + m_1$ are

(1) $\pi(i\omega) > 0$ for all real ω; $(i = \sqrt{-1})$.
(2) $\lim_{\omega \to \infty} \omega^2 (\pi_{22}(i\omega) - \pi_{21}(i\omega)\pi_{11}^{-1}(i\omega)\pi_{12}(i\omega)) > 0$.

Note that for $m_2 = 0$ Theorem 1 turns into "non-singular" Yakubovich–Kalman–Popov lemma, while for $m_1 = 0$, $R = 0$ it provides solvability conditions for matrix inequalities $HA + A^* H < 0, Ha = -b$ corresponding to SPR property of the matrix function $b^*(sI_m - A)^{-1} a$.

For the purposes of control systems design the following *G-passification problems* are important.

Problem A. Find m-vector function $\alpha(y)$ and $m \times m$-matrix function $\beta(y)$ such that the system (1) with the output feedback

$$u = \alpha(y) + \beta(y)v, \qquad (9)$$

where $v \in \mathbb{R}^m$ is new input, is strictly G-passive.

Problem B. Find m-vector function $\alpha(y)$ such that the system (1) with the output feedback (9) is strictly G-passive with fixed $m \times m$-matrix function $\beta(y)$.

For linear systems, a linear passifying feedback is considered instead of (9) and the passification problems are formulated as follows.

Problem AL. Find $m \times l$-matrix K and $m \times m$-matrix L such that the system (4) with the output feedback

$$u = Ky + Lv, \qquad (10)$$

(where $v \in \mathbb{R}^m$ is new input, det $L \neq 0$, see Fig. 1) is strictly G-passive.

Problem BL. Find $l \times m$-matrix K such that the system (4) with the output feedback (10) is strictly G-passive with fixed matrix L.

For complex case all the variables and functions in (9), (10) are complex valued. Important for G-passification are *G-passifiability* problems: checking solvability of the Problems A, B, AL and BL.

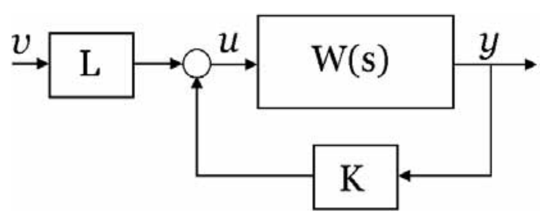

Fig. 1

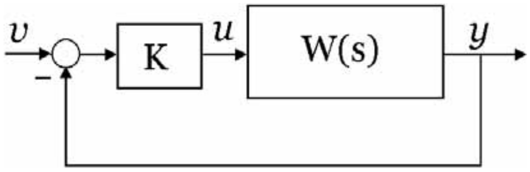

Fig. 2

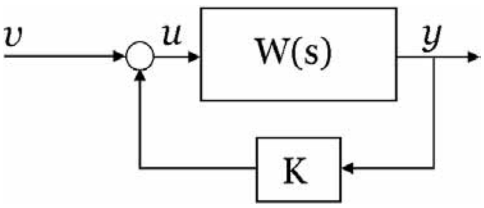

Fig. 3

In this paper, the passification and passifiability problems AL, BL for linear nonsquare systems ($l \neq m$) are studied. Related problems for square MIMO linear systems were considered in [1,18,21,37]. In [1,18] the special case $L = K$ (configuration Fig. 2) was considered, while configuration in the papers [21,37] corresponds to the case $L = I$ (Fig. 3).

Note that the problems cannot be reduced to the "square" case by the "squaring down" procedure (see, e.g. [30]). Indeed, initially the unknown matrix K is rectangular and the squaring down will reduce the number of constructive parameters. It follows from our main results, however, that squaring down by introducing new output $\overline{y} = Gy \in \mathbb{R}^m$ does not change the passifiability conditions, while the passifying feedback can always be found in the form $u = \bar{K}\bar{y} + Lv$ (see Section 3). In other words, the initial problems related to finding a *rectangular* $m \times l$-matrix K have same solvability conditions as the corresponding reduced problems related to finding a *square* $m \times m$-matrix $\overline{K}$. However, the problems are *not equivalent* since they have different number of unknowns. Preserving initial number of parameters is important, e.g. for adaptive control, where reducing the number of adjustable parameters may decrease transient performance of adaptive systems, see [8,11].

3. Main Results

In order to formulate the solutions to the above problems, introduce the following notations:

$$\delta(s) = \det(sI_n - A), \quad W(s) = C(sI_n - A)^{-1}B,$$
$$A(K) = A + BKC,$$

$$\delta(s, K) = \det[sI_n - A(K)],$$
$$W(s, K) = C[sI_n - A(K)]^{-1}B,$$

where K is $m \times l$-matrix. Obviously, $\delta(s, K)$ and $W(s, K)$ are characteristic polynomial and transfer matrix, respectively, of the system (4) closed with the feedback

$$u = Ky + v. \tag{11}$$

It is easy to show that the following identities are valid:

$$\delta(s, K) = \delta(s)\det[I_m - KW(s)], \tag{12}$$
$$W(s, K) = W(s)[I_m - KW(s)]^{-1}. \tag{13}$$

Let G be $m \times l$-matrix. Define $\varphi(s) = \delta(s)\det GW(s)$, $\Gamma = \lim_{s\to\infty} sGW(s)$. It can be shown (see Lemma 1 below) that $\varphi(s)$ is a polynomial of degree not exceeding $n - m$, invariant with respect to feedback transformation (11). Since $\Gamma = GCB$, the $m \times m$-matrix Γ is also invariant with respect to the feedback transformation (11).

Definition 3. The system (4) is called *G-minimum phase* if the polynomial $\varphi(s)$ is Hurwitz (its zeros belong to the open left half-plane). It is called *strictly G-minimum phase* if it is minimum phase and $\det\Gamma \neq 0$, and *hyper minimum phase* if it is minimum phase and $\Gamma = \Gamma^* > 0$.

Since the above terms are correctly defined using the transfer matrix of the system (4), it is no abuse to use same terms as related to the transfer function itself.

It is worth noticing that for square systems the introduced terms are related to the frequently used notion of minimum phaseness defined via *transmission zeros* of the system (4) and to the notion of *strong* detectability* introduced by M. Hautus [19]. Namely, for $m = l$ and $G = I_m$ the roots of the polynomial $\varphi(s)$ coincide with transmission zeros since $\varphi(s) = \det\Sigma$, where Σ is the Rosenbrock matrix:

$$\Sigma = \begin{bmatrix} sI_n - A & -B \\ C & 0 \end{bmatrix}.$$

For nonsquare case the notions are different. Indeed, let $m = 1$. Then, the set of transmission zeros is the set of common zeros of all $\nu_j(s)$, $j = 1, \ldots, l$ where $\nu_j(s)$ is the numerator of the transfer function for j-th output. Even if $\nu_j(s)$ have only left common zeros, their linear combination is not necessary Hurwitz because the set of all Hurwitz polynomials of degree k is neither linear nor convex for $k > 2$. In fact, if $m \leq l$ and the number q of common roots of the polynomials $\nu_j(s)$ is less than

m, then the remaining $m - q$ zeros of $\det GW(s)$ can be assigned more or less arbitrarily, as illustrated by the following example.

Example 1. Consider a second order system with one input, two outputs and transfer function

$$W(s) = \frac{1}{s^2 + a_1 s + a_2} \begin{bmatrix} s + b_1 \\ s + b_2 \end{bmatrix}. \tag{14}$$

Let G be a 1×2-matrix, $G = [g_1, g_2]$. Then the root of the binomial $GW(s) = (g_1 + g_2)s + (g_1 b_1 + g_2 b_2)$ can be assigned everywhere in the plane, if $b_1 \neq b_2$, i.e. if the system does not have transmission zeros (because in that case the equations $g_1 + g_2 = a$, $g_1 b_1 + g_2 b_2 = b$ are always solvable for g_1, g_2).

Now we formulate the main results of the paper. Assume that rank $B = m$, i.e. there is no redundancy in inputs. The following two theorems give solvability conditions for the Problems AL and BL.

Theorem 2. The system (4) is strictly G-passifiable by output feedback (10) if and only if it is strictly G-minimum-phase.

Theorem 3. The system (4) is strictly G-passifiable by output feedback (10) with fixed matrix L if and only if the system with the transfer matrix $W(s)L$ is hyper G-minimum-phase.

Corollary 1. In view of results of [19,20,26] existence of the matrix G, rendering the transfer function $GW(s)$ strictly minimum-phase is necessary and sufficient for existence of the *unknown input observer*.

Proofs of the Theorems 2 and 3 are based on the solution to the following algebraic problem, posed and solved in [10].

Given complex-valued matrices A, B, C, G, R of the dimensions $n \times n$, $n \times m$, $l \times n$, $m \times l$ and $n \times n$, respectively ($m \leq n$, $l \leq n$). Let $R = R^* \geq 0$.

Find existence conditions for a Hermitian $n \times n$ matrix $H = H^* > 0$ and a complex valued $m \times l$ matrix K such that

$$HA(K) + A(K)^* H + R < 0, \tag{15}$$

$$HB = (GC)^*, \tag{16}$$

where

$$A(K) = A + BKC. \tag{17}$$

The case when all the matrices A, B, C, G, R are real valued will be called the *real case*.

Theorem 4. (*Fradkov, 1976*). For the existence of the matrices $H = H^* > 0$, K satisfying relations (15)–(17) and being real valued in the real case, it is sufficient (and, when $\text{rank}(B) = m$, it is also necessary) that the matrix $GW(s)$ be hyper minimum phase.

Obviously, relations (15), (16), (17) for fixed K coincide with the linear matrix inequalities (LMI) (6) arising in a version of Yakubovich–Kalman–Popov lemma. Therefore, Theorem 4 deals with the existence of a feedback rendering the system satisfy conditions of Yakubovich–Kalman–Popov lemma. In other words, Theorem 4 can be called the *Feedback Yakubovich–Kalman–Popov lemma*. Note also that the inequalities (15) are *bilinear* matrix inequalities and the problem of their solvability is in general *NP*-hard. However, for the above special case the solvability conditions for (15), (16), (17) are simple and constructive that is seen from Theorem 4 formulation.

4. Proof of Main Results

In order to make the paper self-contained we start with the proof of the Theorem 4[2]. Then we will derive Theorems 2 and 3 from Theorem 4. First of all, we need to establish several auxiliary lemmas.

Lemma 1. Let p, q be $n \times m$-matrices, $\alpha(s) = (sI_n - A)^{-1}\delta(s)$, and $\Sigma(s) = p^*\alpha(s)q$. Then $\det \Sigma(s) = \delta^{m-1}(s)\sigma(s)$, where $\sigma(s)$ is a polynomial of degree no higher then $n - m$ with the leading coefficient $s^{n-m} \det p^* q$. Besides, $\sigma(s)$ does not change with substitution $A + qr^*$ instead of A, where r is any $n \times m$ matrix.

Proof of Lemma 1. It is seen that $\sigma(s) = \det \Phi(s)$, where

$$\Phi(s) = \begin{bmatrix} sI_n - A + qp^* & -q \\ p^* & 0 \end{bmatrix}.$$

Let $\sigma(s) = \sigma_n s^n + \cdots + \sigma_1 s + \sigma_0$. Then, σ_{n-k} is equal to a sum of three principal minors of order $m + k$ of the matrix $\Phi(0)$, expansion of which contains exactly k elements from the upper left block. Therefore $\sigma_{n-k} = 0$ for $k < m$. Furthermore

$$\begin{aligned} \sigma_{n-m} &= \lim_{s\to\infty} \frac{\sigma(s)}{s^{n-m}} = \lim_{s\to\infty} \det \begin{bmatrix} sI_n - A + qp^* & -q \\ p^* & 0 \end{bmatrix} \\ &= \det \begin{bmatrix} I_n & -q \\ p^* & 0 \end{bmatrix} \\ &= \det p^* q. \end{aligned}$$

[2]The proof is slightly simplified compared with [10,15]. Some misprints in [10,15] are also fixed.

The second statement of the Lemma follows from the fact that substitution $A \to A + qr^*$ gives the same result as if we added the last m columns of the matrix $\Phi(s)$, multiplied by corresponding elements of the matrix r, to the first n columns of the same matrix $\Phi(s)$. Obviously, such a manipulation does not change the polynomial $\det \Phi(s)$. □

Corollary 2. The polynomial $\varphi(s) = \delta(s)GW(s) = GC[(sI_n - A)^{-1}\delta(s)]B$, defined above, has the leading coefficient $s^{n-m} \det \Gamma$ and is invariant under "feedback transformation" of the form $A \to A(K) = A + BKC$.

Lemma 2. Any k-th order minor of the matrix $\Sigma(s)$, defined in Lemma 1, is divisible by $\delta(s)^{k-1}$, with the fraction being a polynomial of degree not exceeding $n-k$. Its leading coefficient is equal to the corresponding minor of the matrix p^*q.

Proof of Lemma 2. The matrix $\Sigma'(s)$ of the mentioned minor has the form $\Sigma'(s) = e_1^*\Sigma(s)e_2$, where e_1, e_2 are constant matrices of dimensions $m \times k$. Therefore $\Sigma'(s) = a_1^*\alpha(s)a_2$, where a_1, a_2 are matrices of dimensions $n \times k$. Application of Lemma 1 to the matrix $\Sigma'(s)$ proves Lemma 2. □

Lemma 3. Let the polynomial $P_\epsilon(s)$ have the form $P_\epsilon(s) = s^{n-m}Q_\epsilon(s) + R_\epsilon(s)$, where

$$Q_\epsilon(s) = \sum_{k=0}^{m}(q_k + q_k'(\epsilon))\epsilon^k s^k,$$
$$R_\epsilon(s) = \sum_{k=0}^{n-m-1}(r_k + r_k'(\epsilon))s^k,$$

and $q_k'(\epsilon) = O(\epsilon)$, $r_k'(\epsilon) = O(\epsilon)$ for $\epsilon \to 0$. Let, additionally, the polynomials

$$Q(s) = \sum_{k=0}^{m} q_k s^k, \quad R(s) = q_0 s^{n-m} + \sum_{k=0}^{n-m-1} r_k s^k$$

be Hurwitz. Then the polynomial $P_\epsilon(s)$ is Hurwitz for all sufficiently small $\epsilon > 0$.

Proof of Lemma 3. It is easy to see that $n-m$ roots of the polynomial $P_\epsilon(s)$ tend to the roots of polynomial $R(s) = \lim_{\epsilon\to 0} P_\epsilon(s)$ as $\epsilon \to 0$, while remaining m roots tend to infinity. Let us substitute $\epsilon s = \mu$ and denote $S_\epsilon(\mu) = \epsilon^{n-m}P_\epsilon(\mu/\epsilon)$. Then,

$$S_\epsilon(\mu) = \mu^{n-m}\sum_{k=0}^{m}(q_k + q_k'(\epsilon))\mu^k + \sum_{k=0}^{n-m-1}(r_k + r_k'(\epsilon))\mu^k\epsilon^{n-m-k}.$$

From the latter expression it is obvious that $n-m$ roots of the polynomial $S_\epsilon(\mu)$ tend to zero as $\epsilon \to 0$, while remaining m roots tend to the roots of $Q(\mu)$ with the rate $O(\epsilon)$. Therefore, m roots $s_1, \ldots, s_m$ of the polynomial $P_\epsilon(s)$ as $\epsilon \to 0$ take the form

$$s_i = \frac{\mu_i}{\epsilon} + O(1), \quad i = 1, \ldots, m,$$

where $\mu_i, \ldots, \mu_m$ are the roots of $Q(\mu)$. Thus, the roots of $P_\epsilon(s)$ will belong to the left half-plane, if the roots of $Q(s)$ and $R(s)$ lie in the same half plane. □

Lemma 4. Let $\Phi(\omega), \omega \in \mathbb{R}^1$ be complex valued $m \times m$-matrix function satisfying for all $x \neq 0$ and all $\omega \in \mathbb{R}^1$ the inequality: $\mathrm{Re}\, x^*\Phi(\omega)x > 0$. Then

$$|\Delta \arg \det \Phi(\omega)| \leq m\pi,$$

where

$$\Delta \arg \Psi(\omega) = \lim_{\omega\to+\infty}[\arg \Psi(\omega) - \arg \Psi(-\omega)].$$

Proof of Lemma 4. Let $s_i(\omega), x_i(\omega)$, $i = 1, \ldots, m$, be the eigenvalues and eigenvectors of the matrix $\Psi(\omega)$. Then,

$$\mathrm{Re}\, s_i(\omega) = \frac{\mathrm{Re}\, x_i^*(\omega)\Phi(\omega)x_i(\omega)}{\|x_i(\omega)\|^2} > 0,$$
$$i = 1, \ldots, m, \quad \omega \in \mathbb{R}^1.$$

Therefore $s_i(\omega) \neq 0$ and

$$\det \Phi(\omega) = \sum_{i=1}^{m} s_i(\omega) \neq 0$$

for any $\omega \in \mathbb{R}^1$. Besides $|\Delta \arg s_i(\omega)| \leq \pi$, and hence

$$|\Delta \arg \det \Phi(\omega)| \leq \sum_{i=1}^{m}|\Delta \arg s_i(\omega)| \leq m\pi. \quad \square$$

Finally, we will need another version of the Yakubovich–Kalman–Popov lemma which deals with positive definite solutions of the matrix relations (8).

Lemma 5. Let A_0, R, B, C_0 be matrices of dimensions $n \times n$, $n \times n$, $n \times m$, $m \times n$ respectively, and $R = R^* \geq 0$, rank $B = m$. Let

$$\Pi(s) = 2\,\mathrm{Re}\, C_0(sI_n - A_0)^{-1}B - B^*(sI_n - A_0^*)^{-1}R(sI_n - A_0)^{-1}B. \quad (18)$$

For the existence of $n \times n$-matrix $H = H^* > 0$ such that

$$HA_0 + A_0{}^* H + R < 0, \quad HB = C_0^*, \tag{19}$$

which is real valued in the real case, the following conditions are necessary and sufficient:

(i) $\det(sI_n - A_0)$ is a Hurwitz polynomial;
(ii) $\Pi(i\omega) > 0$ for all $\omega \in \mathbb{R}^1$;
(iii) $\lim_{\omega \to \infty} \omega^2 \Pi(i\omega) > 0$.

Proof of Lemma 5. The sufficiency of conditions (i), (ii) and (iii) follows from Theorem 1 for $m_1 = 0$. Let us prove their necessity. The necessity of condition (i) follows from the fact that fulfillment of (19) implies asymptotic stability of the system $\dot{x} = A_0 x$ (it follows from the Lyapunov theorem about solution of the matrix inequality (19)). Finally, validity of (ii) and (iii) when (19) and (i) hold also follows from Theorem 1. □

Note that the requirement $H = H^* > 0$ makes the solvability problem for linear matrix inequalities more complicated. Although the case of inequalities (8), corresponding to passivity is trivial, constructive solvability conditions for the case of general quadratic forms, corresponding to dissipativity property are still unknown. Of course, numerical solution of LMI is always possible, but for our purposes analytical conditions are needed. The case considered in Lemma 5 is an intermediate one. The inequalities with $R > 0$ are important for treating stochastic stabilization problems, see [11].

Proof of Theorem 4. First, let us prove the sufficiency. To this end, choose an $m \times l$-matrix K_0 such that matrices $A_0 = A + BK_0C$, $C_0 = GC$ satisfy conditions (i), (ii) and (iii) of Lemma 5. We will show that K_0 can be chosen as $K_0 = -kG$, where $k > 0$ is sufficiently large (this guarantees that K_0 is real valued in the real case). For brevity we denote

$$\delta_k(s) = \delta(s, -kG), \quad A_k = A - kBGC,$$
$$W_k(s) = W(s, -kG).$$

To check condition (i) we use the equality (12) which means that

$$\delta_k(s) = \frac{k^m}{\delta(s)^{m-1}} \det\left[\frac{\delta(s)}{k} I_m + Ga(s)\right], \tag{20}$$

where $a(s) = W(s)\delta(s)$. Expanding the determinant in the right-hand side of expression (20), we obtain

$$\delta_k(s) = \frac{k^m}{\delta(s)^{m-1}}\left[\frac{\delta(s)^m}{k^m} + \varphi_1(s)\frac{\delta(s)^{m-1}}{k^{m-1}} + \cdots + \varphi_{m-1}(s)\frac{\delta(s)}{k} + \varphi_m(s)\right],$$

where $\varphi_1(s), \ldots, \varphi_{m-1}(s), \varphi_m(s) = \det Ga(s)$ are the coefficients of the characteristic polynomial of the matrix $Ga(s)$. By Lemma 2 $\varphi_k(s) = \delta(s)^{k-1}\psi_k(s)$, where $\psi_k(s)$ is a polynomial of degree $n - k$, the leading coefficient of which ψ_k is equal to the sum of principal minors of order k for the matrix $GCB\,(k = 1, \ldots, m)$.

Therefore,

$$\begin{aligned}\frac{1}{k^m}\delta_k(s) &= \frac{\delta(s)}{k^m} + \frac{\psi_1(s)}{k^{m-1}} + \cdots + \frac{\psi_{m-1}(s)}{k} + \psi_m(s)\\ &= s^{n-m}\left[\left(\frac{s}{k}\right)^m + \left(\frac{s}{k}\right)^{m-1}\left(\psi_1 + O\left(\frac{1}{k}\right)\right)\right.\\ &\quad \left. + \cdots + \left(\frac{s}{k}\right)\left(\psi_{m-1} + O\left(\frac{1}{k}\right)\right)\right]\\ &\quad + \psi_m(s) + \psi(s,k),\end{aligned}$$

where $\psi(s,k)$ is some polynomial of degree not higher then $n - m$ with coefficients of order $O(1/k)$ as $k \to \infty$. Applying Lemma 3 for $\epsilon = 1/k$, we obtain that the polynomial $\delta_k(s)$ is Hurwitz for sufficiently large k, if the following polynomials are also Hurwitz

$$Q(s) = s^m + \sum_{k=0}^{m-1}\psi_k s^k, \quad R(s) = \psi_m(s).$$

However, $Q(s)$ and $R(s)$ are Hurwitz by assumptions of the theorem since $Q(s) = \det(sI_m + GCB) = \det(sI_m + \Gamma)$, while $R(s) = \varphi(s)$. Thus the condition (i) is valid for $k > k_1$ and some positive number k_1.

To prove condition (ii), let us rewrite it in the following form:

$$2\,\mathrm{Re} GW_k(i\omega) > B^*(-i\omega I_n - A_k^*)^{-1} R(i\omega I_n - A_k)^{-1} B. \tag{21}$$

By virtue of invariance of the polynomial $\varphi(s)$ for any k, we conclude that $\det GW_k(s) = \varphi(s)/\delta_k(s)$. In accordance with the conditions of the theorem, the polynomial $\varphi(s)$ is Hurwitz, and, hence, $\det GW_k(i\omega) \neq 0$ for all $\omega \in \mathbb{R}^1$. Therefore for any $\omega \in \mathbb{R}^1$ we obtain

$$\mathrm{Re}\, GW_k(i\omega) = [GW_k(i\omega)]^* \times \mathrm{Re}\,[GW_k(i\omega)]^{-1} GW_k(i\omega).$$

In view of the latter equation, inequality (21) is equivalent to the following inequality

$$2\,\mathrm{Re}\,[GW_k(i\omega)]^{-1} > [W_k^*(i\omega)G^*]^{-1}B^*(-i\omega I_n - A_k)^{-1} \times B[GW_k(i\omega)]^{-1}. \quad (22)$$

However,

$$[GW_k(s)]^{-1} = kI_m + [GW(s)]^{-1}.$$

Therefore, it is sufficient to show that $\mathrm{Re}\,[GW(i\omega)]^{-1}$ and the right-hand side of (22) is bounded for $\omega \in \mathbb{R}^1$. Since $\det GW(i\omega) = \varphi(i\omega)/\delta(i\omega)$ for $\omega \in \mathbb{R}^1$, the matrix $[GW(i\omega)]^{-1}$ is bounded in any bounded set of ω.

Now let us show that $\mathrm{Re}\,[GW(i\omega)]^{-1}$ is bounded for $\omega \to \pm\infty$. By virtue of the conditions of the theorem, $\Gamma = \Gamma^* > 0$, where $\Gamma = GCB$. Therefore

$$\begin{aligned}\mathrm{Re}\,[GW(i\omega)]^{-1} &= \mathrm{Re}\,i\omega[i\omega GW(i\omega)]^{-1}\\ &= \mathrm{Re}\,i\omega\left[\Gamma^{-1} + O\left(\frac{1}{\epsilon}\right)\right]\\ &= i\omega[\Gamma^{-1} - (\Gamma^{-1})^*] + O(1) = O(1)\\ &\quad \text{for}\quad \omega \to \pm\infty.\end{aligned}$$

To end the proof we need to show the boundedness of the right-hand side of (22) as $\Psi_k(\omega)$). For $k > k_1$, the matrix function $\Psi_k(\cdot)$ is continuous, and, hence, bounded on any bounded set. Let us show that for any k there exists a finite limit $\lim_{\omega\to\pm\infty}\Psi_k(\omega)$. Denote $B_s = (sI_n - A_k)^{-1}B$. Then $B_s = B/s + O(1/|s|^2)$ for $s \to \infty$, and

$$\lim_{\omega\to\pm\infty} B_{i\omega}[GCB_{i\omega}]^{-1} = B[GCB]^{-1}.$$

Therefore, there exists a finite limit

$$\begin{aligned}\lim_{\omega\to\pm\infty}\Psi_k(\omega) &= \lim_{\omega\to\pm\infty}[B_{i\omega}^*C^*G^*]^{-1}B_{i\omega}^*RB_{i\omega}[GCB_{i\omega}]^{-1}\\ &= [B^*C^*G^*]^{-1}B^*RB[GCB]^{-1}\\ &= \Gamma^{-1}B^*RB\Gamma.\end{aligned}$$

Thus, condition (ii) of Lemma 5 is valid for $k > k_2$ with some positive number $k_2 > k_1 > 0$.

Finally, to verify the condition (iii) for sufficiently large $k > 0$ we use the following apparent relationships

$$\begin{aligned}\lim_{\omega\to\infty}\omega^2\Pi(i\omega) &= -2\,\mathrm{Re}GCA_kB - B^*RB\\ &= -2\,\mathrm{Re}GCAB - B^*RB + 2k[GCB]^2.\end{aligned}$$

Thus the sufficiency part of the theorem is proved.

To prove necessity, assume that the relationships (15), (16), (17) are fulfilled for some $H_0 = H_0^* > 0$ and K_0. By Lemma 5, the polynomial $\delta_0(s) = \det(sI_n - A(K_0))$ is Hurwitz and $\mathrm{Re}\,GW(i\omega, K_0) > 0$ for any $\omega \in \mathbb{R}^1$, where $W(s, K_0) = C[sI_n - A(K_0)]^{-1}B$. Due to the invariance of the polynomial $\varphi(s)$ we obtain

$$\varphi(s) = \delta_0(s)\det GW(s, K_0).$$

Calculating the increment of the argument for $s = i\omega$, where ω varies from $-\infty$ to $+\infty$ for both sides of this equality, we have

$$\Delta\arg\varphi(i\omega) = n\pi + \Delta\arg\,\det GW(i\omega, K_0).$$

By Lemma 4 we obtain $\Delta\arg\varphi(i\omega) \geq (n-m)\pi$. However, $\varphi(s)$ is a polynomial of degree $n-m$ with the leading coefficient $s^{n-m}\det\Gamma$ (see Lemma 1). Therefore $|\Delta\arg\varphi(i\omega)| \leq (n-m)\pi$, and $\Delta\arg\varphi(i\omega) = (n-m)\pi$. Hence $\varphi(s)$ is Hurwitz, and $\det\Gamma > 0$.

It remains to show that $\Gamma = \Gamma^* \geq 0$. By Lemma 5

$$\begin{aligned}&\lim_{\omega\to\infty}\omega^2\,\mathrm{Re}\,GW(i\omega, K_0)\\ &\quad= \lim_{\omega\to\infty}\mathrm{Re}\,[i\omega GCB - GCA(K_0)B + O(1/|\omega|)]\\ &\quad= \lim_{\omega\to\infty} i\omega[\Gamma - \Gamma^*] - \mathrm{Re}\,GCA(K_0)B > 0,\end{aligned}$$

and, hence, $\Gamma = \Gamma^*$. It is worth noting that the relationships (15), (16), (17) are, obviously, valid if we substitute $A(K_0) - kI_n (k > 0)$ instead of $A(K_0)$. Applying Lemma 5 again, we conclude that $-GCA(K_0)B + k\Gamma > 0$ for any $k > 0$. Therefore $\Gamma \geq 0$. To complete the proof of the theorem note that in the real case the matrix $K = -kG$ is real by its choice and the matrix H can be chosen real by Lemma 5. □

Proof of Theorems 2 and 3. Proof of the main results is now straightforward. If the matrix L is fixed, then the strict output passifiability is equivalent to existence of matrices K, H, satisfying relations (15)–(17) where B is replaced by BL. By Theorem 4, it is equivalent to the minimum phaseness and fulfillment of $D = D^T > 0$, where $D = GCBL$, i.e. to the hyper G-minimum-phaseness of the transfer matrix $W(s)L$. If matrix L is subject to choice, we can ensure the relation $D = D^T > 0$ as soon as system is strictly minimum phase, i.e. if $\det(GCB) \neq 0$. In this case the simplest choice is $L = (GCB)^{-1}$. □

Corollary 3. The set of matrices K satisfying (15), (16), (17) i.e. G-passifying the initial system contains the subset of matrices of form $K = -k\,G$, where

$$k > k_0 = \sup_{\omega \in \mathbb{R}^1} \sigma_{\max}([GW(i\omega)]^{-1}) \tag{23}$$

($\sigma_{\max}$ is the maximum singular value of the matrix).

Proof of the Corollary 3. It is clear from the proof of Theorem 4 that the most restrictive condition among conditions (i), (ii) and (iii) of Lemma 5 for ensuring strict G-positive realness by choice of $K = -kG$ is the condition (ii): $\mathrm{Re} GW_k(i\omega) > 0$ for all $\omega \in \mathbb{R}^1$, where matrix inequality is understood in sense of Hermitian forms (quadratic forms in the real case). Indeed, the condition (ii) is equivalent to $\mathrm{Re}\,[GW_k(i\omega)]^{-1} > 0$. Multiplying the identity $[GW_k(s)]^{-1} = kI_m + [GW(s)]^{-1}$, by x^*, x and choosing $s = i\omega$, $\omega \in \mathbb{R}^1$ we obtain

$$\begin{aligned} x^*\mathrm{Re}\,[GW_k(i\omega)]^{-1}x &= kx^*x + x^*[GW(i\omega)]^{-1}x \\ &= kx^*x[1 + x^*[GW(i\omega)]^{-1}x/x^*x/k] \\ &\geq kx^*x[1 - \sigma_{\max}([GW(i\omega)]^{-1})/k], \end{aligned}$$

where $\sigma_{\max}$ stands for the maximum singular value of the matrix. Therefore if the condition (23) holds then condition (ii) holds. Since $\det(I_m - |[GW(i\omega)]^{-1}|) > 0$ for $k > k_0$, the conditions (i), (iii) hold too. □

5. Conclusions

The paper clarifies some interrelations between passification of nonsquare linear systems and solvability of a class of bilinear matrix inequalities. The presented passification results can be useful for design purposes in various situations. For example, they justify design of SPR system based on providing some minimum phase property and applying high gain output feedback (this approach has numerous applications in adaptive control, see [9,10,22,23,25,33,38]).

Note that Theorem 4 provides solvability conditions for the inequalities (15), (16), (17) rather than complete description of the set of solutions. A complete description of matrices H, K satisfying to (15), (16), (17) can be obtained similarly to [21] based on solving some LMI.

An extension of G-passification approach to infinite-dimensional linear systems is presented in [5,6]. Different G-passifiability conditions for affine nonlinear systems (1) can be obtained by extension of the results of [7,13] and will be presented elsewhere.

Acknowledgement

Author is grateful to the anonymous reviewers for their useful comments.

References

1. Abdallah C, Dorato P, Karni S. SPR design using feedback. In: Proceedings of the American Control Conference, Boston, MA. 1991, pp 1742–1743
2. Andrievsky BR, Churilov AN, Fradkov AL. Feedback Kalman–Yakubovich Lemma and its applications to adaptive control. In: Proceedings of the 35th IEEE Conference on Decision and Control. December 1996, pp 4537–4542
3. Andrievsky BR, Fradkov AL, Kaufman H. Necessary and sufficient condition for almost strict positive realness and their application to direct implicit adaptive control. In: Proceedings of the American Control Conference, Baltimore. 1994, pp 1265–1266
4. Andrievsky BR, Stotsky AA, Fradkov AL. Velocity gradient algorithms in control and adaptation problems. A survey Autom Rem Control 1988; 49(12): 1533–1563
5. Bondarko VA, Fradkov AL. G-passification of infinite-dimensional linear systems. In: proceedings of the 41st IEEE Conference on Decision and Control, Las Vegas. 2002, pp 3396–3401
6. Bondarko VA, Fradkov AL. Necessary and sufficient conditions for the passifiability of linear distributed systems. Autom Rem Control 2003; 64(4): 517–530
7. Byrnes CI, Isidori A, Willems JC. Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems. IEEE Trans Autom Control 1991; 36(11): 1228–1240
8. Fomin VN, Fradkov AL, Yakubovich VA. Adaptive control of dynamic objects. Nauka, Moscow, 1981 (in Russian)
9. Fradkov AL. Synthesis of an adaptive system for linear plant stabilization. Autom Rem Control 1974; 35(12): 1960–1966
10. Fradkov AL. Quadratic Lyapunov functions in the adaptive stabilization problem of a linear dynamic plant. Siberian Math J 1976; (2): 341–348
11. Fradkov AL. Adaptive control in complex systems. Nauka, Moscow, 1990 (in Russian)
12. Fradkov AL. Passification of nonsquare linear systems. In: Proceedings of the 5th European Control Conference, Porto. 2001, pp 3338–3343
13. Fradkov AL, Hill DJ. Exponential feedback passivity and stabilizability of nonlinear systems. Automatica 1998; 34(6): 697–703
14. Fradkov AL, Markov AYu. Adaptive synchronization of chaotic systems based on speed gradient method and passification. IEEE Trans Circ Syst 1997; 44(10): 905–912
15. Fradkov AL, Miroshnik IV, Nikiforov VO. Nonlinear and adaptive control of complex systems. Kluwer Academic Publishers, Dordrecht, 1999
16. Fradkov AL, Nijmeijer H, Markov AYu. Adaptive observer-based synchronization for communication. Int J Bifurcations Chaos 2000; 10(12): 2807–2814

17. Fradkov AL, Pogromsky AYu. Introduction to control of oscillations and chaos. World Scientific, Singapore, 1998
18. Gu G. Stabilizability conditions of multivariable uncertain systems via output feedback control. IEEE Trans Autom Control 1990; 35(8): 925–927
19. Hautus MLJ. Strong detectability and observers. Linear Algebra Appl. 1983; 50: 353–368
20. Hou M, Muller PC. Disturbance decoupled observer design: a unified viewpoint. IEEE Trans Autom Control 1994; 39(6): 1338–1341
21. Huang CH, Ioannou PI, Maroulas J, Safonov MG. Design of strictly positive real systems using constant output feedback. IEEE Trans Autom Control 1999; 44(3): 569–573
22. Ilchmann A. Non-identifier-based adaptive control of dynamic systems. Lecture Notes in Control and Information Science, Springer-Verlag, New York, 1993
23. Kaufman H, Bar-Kana I, Sobel K. Direct adaptive control algorithms. Springer, New York, 1994
24. Kokotović PV, Sussmann HJ. A positive real condition for global stabilization of nonlinear systems. Syst Contr Lett 1989; 13: 125–133
25. Lozano R, Brogliato B, Egeland O, Maschke B. Dissipative systems analysis and control. Springer-Verlag, London, 2000
26. Moreno J. Existence of unknown input observers and feedback passivity for linear systems. In: Proceedings of the 40th IEEE Conference on Decision and Control. 2001, pp 3366–3371
27. Picci G, Pinzoni S. On feedback dissipative systems. J Math Syst. Contr 1992; 2(1): 1–30
28. Popov VM. On the problem in the theory of absolute stability of automatic control. Autom Rem Control 1964; 25(9): 1129–1134
29. Saberi A, Kokotović PV, Sussmann HJ. Global stabilization of partially linear composite systems. SIAM J Contr Optimiz 1990; 28(6): 1491–1503
30. Saberi A, Sannuti P. Squaring down by static and dynamic compensators. IEEE Trans Autom Control 1988; 33(4): 358–365
31. Sepulchre R, Janković M, Kokotović PV. Constructive nonlinear control. Springer-Verlag, New York, 1996
32. Seron MM, Hill DJ, Fradkov AL. Adaptive passification of nonlinear systems. In: Proceedings of the 33rd Conference on Decision and Control. 1994, pp190–195
33. Sobel K, Kaufman H, Mabius L. Implicit adaptive control for a class of MIMO systems. IEEE Trans Aerosp Electr Syst 1982; 18: 576–590
34. Steinberg A. A sufficient condition for output feedback stabilization on uncertain dynamical systems. IEEE Trans Autom Control 1988; 33(7): 676–677
35. Sun W, Khargonekar P, Shim D. Solution to the positive real control problem for linear time-invariant systems. IEEE Trans Autom Control 1994; 39(10): 2034–2046
36. Tsykunov AM. Adaptive control of systems with delay. Nauka, Moscow, 1984 (in Russian)
37. Weiss H, Wang Q, Speyer JL. System characterization of positive real conditions. IEEE Trans Autom Control 1994; 39(3): 540–544
38. Willems IC, Byrnes CI. Adaptive stabilization of multivariable linear system. In: Proceedings of the 23rd IEEE Conference on Decision and Control. 1984, pp 1574–1577
39. Yakubovich VA. The solution of certain matrix inequalities in automatic control theory. Soviet Math Dokl 1962; 3: 620–623
40. Yakubovich VA. Periodic and almost-periodic limit modes of controled systems with several, in general discontinuous, nonlinearities. Soviet Math Dokl 1966; 7(6): 1517–1521
41. Zeheb A. A sufficient condition for output feedback stabilization on uncertain systems. IEEE Trans Autom Control 1986; 31(11): 1055–1057