Integrodifferentiating velocity gradient algorithms

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We describe a class of control and adaptation algorithms for nonlinear nonstationary systems which extend the class of velocity gradient algorithms.[1-3] Applicability conditions for the algorithms are established and the time to attain the objective is estimated.

1. Consider the controlled system

$$\frac{dx}{dt} = F(x, u, t), \quad t > 0, \tag{1}$$

where $x \in R^n$ is the state, $u \in R^m$ the control, $F(\cdot)$ is continuously differentiable. Let the control objective be specified.

$$Q(x(t), t) \leq \Delta \quad \text{for} \quad t \geq t_*, \tag{2}$$

where $Q(x, t) \geq 0$ is a continuously differentiable objective function, $\Delta > 0$. It is required to find a control algorithm

$$u(t) = \mathcal{U}_t\{x(s), u(s), 0 \leq s \leq t\}, \tag{3}$$

which ensures that for all initial conditions $x_0 = x(0)$, $u_0 = u(0)$ the system (1)–(3) attains the control objective (2) for some $t_* > 0$.

The term "control" is understood in a broad sense: the vector u(t) does not necessarily represent physical action on the controlled system coordinates. Equation (1) may represent a closed-loop plant–controller system (a generalized tunable plant), and u(t) may be the vector of tunable controller parameters, the parametric control vector, etc.; (3) may correspond to the system adaptation algorithm. The function Q(x, t) is generally a measure of deviation of the controlled-system path from the desired path; the control objective (2) requires minimizing Q(x, t) with specified accuracy.

2. Consider control algorithms of the form

$$\frac{d(u + \psi(x, u, t))}{dt} = -\Gamma \nabla_u \varphi(x, u, t), \tag{4}$$

where $\Gamma = \Gamma^T > 0$ is a $m \times m$, Δ is the gradient symbol, $\varphi(x, u, t) = \partial Q/\partial t + [\nabla_x Q]^T F(x, u, t)$ is the derivative function of Q(x, t) on (1) for u = const. $\psi(x, u, t)$ is some vector function which satisfies the pseudogradient inequality[4]

$$\psi(x, u, t)^T \nabla_u \varphi(x, u, t) \geq 0. \tag{5}$$

For instance, we may assume

$$\psi(x, u, t) = \Gamma_1 \nabla_u \varphi(x, u, t); \tag{6}$$

$$\psi(x, u, t) = \gamma \operatorname{sign} \nabla_u \varphi(x, u, t), \tag{6a}$$

where $\Gamma_1 = \Gamma_1^T > 0$ is a $m \times m$ matrix, $\gamma > 0$ a number, φ a vector whose components are the signs of the components of the vector φ.

For $\psi(x, u, t) \equiv 0$ the algorithms (4) reduce to velocity-gradient algorithms.[1-3]

Theorem 1. Suppose that for every $v \in R^m$ there exists a unique solution $u = \chi(x, v, t)$ of the equation $u + \psi(x, u, t) = v$, the functions $F(x, u, t)$, $\nabla_x Q(x, t)$, $\nabla_u Q(x, u, t)$, $\chi(x, u, t)$ are bounded in every region $\{(x, u, t) : \|x\| + \|u\| \leq \beta\}$, and the growth condition $\inf_{t>0} Q(x, t) \to \infty$ for $\|x\| \to \infty$ holds. Moreover, let the function $\varphi(x, u, t)$ be convex in u and let there exist a vector $u_* \in R^m$ and a number $\alpha > 0$ such that for all x, t

$$\varphi(x, u_*, t) \leq -\alpha Q(x, t). \tag{7}$$

All the paths of the system (1), (4) will then be bounded and $uQ(x(t), t) \to 0$ for $t \to \infty$, i.e., the control objective (2) is attained for every $\Delta > 0$.

3. Consider a controlled system with perturbation

$$\frac{dx}{dt} = F(x, u, t) + f(x, t) \tag{8}$$

and, as in Refs. 2 and 3, the robustified algorithm

$$\frac{d(u + \psi(x, u, t))}{dt} = -\Gamma \nabla_u [\varphi(x, u, t) + \mu\omega(u + \psi(x, u, t))], \tag{9}$$

where $\omega(u)$ is a convex regularizing function, $\Gamma = \Gamma^T > 0$, $\mu > 0$.

Theorem 2. Let the conditions of Theorem 1 hold with (7) replaced by

$$\varphi(x, u_*, t) + [\nabla_x Q(x, t)]^T f(x, t) \leq -\alpha Q(x, t) + \beta \tag{10}$$

Let $\omega(u) \geq \rho_1 \|u\|^2 - \rho_2$ (α, β, ρ_1, ρ_2 are positive numbers). Then (8), (9) is a dissipative system and for every $\Delta > \beta/\alpha$ there exist a matrix $\Gamma_0 > 0$ and a number $\mu_0 > 0$ such that for $\Gamma \geq \Gamma_0$, $0 < \mu \leq \mu_0$ the control objective (2) is attained. Here Q(x(t), t) approaches the objective with an exponential velocity, and for the time t_* we have the estimate $t_* \leq \alpha^{-1} \ln [2(V(x_0, u_0, 0) - \beta/\alpha)(\Delta - \beta/\alpha)^{-1}]$.

The proof of Theroems 1 and 2 requires taking the derivative of the function

$$V(x, u, t) = Q(x, t) + 0.5 \|u - u_* + \psi(x, u, t)\|^2_{\Gamma^{-1}} \tag{11}$$

in the system (1), (4), or (8), (9).

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Remark. a) The inequality (1) implies that for some (generally unknown) $u = u_*$ the control objective (2) is attained in the system (3) for $\Delta = \beta/\alpha$. This inequality holds, for instance, in controlled systems which are exponentially stable for $u = u_*$, $f(x, t) \equiv 0$ if $Q(x, t)$ is quadratic and $f(x, t)$ is bounded. Theorem 2 implies that in this case the algorithm (9) ensures that for every $u(0)$ the system attains a control objective which is arbitrarily close to the best attainable objective.

b) In practice, the regularizing function $\omega(u)$ is often chosen in the form $\omega(u) = 0.5\|u\|^2$, i.e., $\nabla\omega = u$; if we also take $\psi(x, u, t)$ in the form (6), then the algorithm (9) is described by a matrix integrodifferentiating element with the transfer function $W(\lambda) = (\Gamma + \Gamma_1\lambda)(\alpha + \lambda)^{-1}$, whose input is the velocity gradient $\nabla_u\varphi(x, u, t)$. This suggests the term integrodifferentiating velocity gradient algorithms for this class of algorithms.

4. For $\Gamma = 0$, the algorithm (4) is best written in finite, rather than differential, form

$$u = u_0 - \gamma_1 \psi(x, u, t), \tag{12}$$

which explicitly includes the stepping factor $\gamma_1 > 0$. Let $\psi(x, u, t)$ satisfy the strong pseudogradient condition

$$\psi(x, u, t)^T \nabla_u \varphi(x, u, t) \geqslant \rho \|\nabla_u \varphi(x, u, t)\|^\delta \tag{13}$$

for some $\rho > 0$, $\delta \geq 1$ and all x, u, t. [The function (6), for instance satisfies (13) for $\delta = 2$, $\Gamma_1 \geq \rho I$, and the function (6a) satisfies (13) for $\delta = 1, \rho = \gamma/\sqrt{m}$.)

Theorem 3. Let (12) be uniquely solvable for u, let $\varphi(x, u, t)$ be convex in u, and let (7), (13) hold. If $\delta > 1$, then for every $\Delta > 0$ the control objective (2) is attained in the system (1), (12) for

$$\gamma_1 > \gamma_0(\Delta) = \frac{\|u_0 - u_*\|}{\rho\delta} \left(\frac{\|u_0 - u_*\|(\delta - 1)}{\Delta\alpha\delta}\right)^{\delta - 1}$$

If $\delta = 1$, then $\delta = 1$, $\gamma_1 \geqslant \|u_0 - u_*\|/\rho$, το $Q(x(t), t) \leqslant Q(x_0, 0)e^{-\alpha t}$ The time to attain the objective within ε has tne exact estimate $t_* \leqslant \alpha^{-1} \ln [Q(x_0, 0)/\varepsilon]$.

Remarks. a) The theorem remains valid if $u_* = u_*(x, t)$ but $\|u_0 - u_*(x, t)\|$ is bounded.

b) The algorithm (12), (6a) coincides with the optimal control algorithm for damping the function[5] $Q(x, t)$.

5. As an example let us consider the design of model reference adaptive systems. Let the controlled system (1) have the form $\frac{dx}{dt} = A(x, t) + B(x, t)u$, $u \in R^m$, and let $Q(x, t) = e^T He$, where $e = x - x_M(t)$, $\frac{dx_M}{dt} = A_M x_M + B_M r(t)$ is the model equation, $H = H^T > 0$ is a $m \times m$ matrix, $r(t) \in R^m$ is the setting signal. Then $\varphi(x, u, t) = e^T H[A_M e + (A - A_M)x + Bu - B_M r]$ and the velocity gradient has the form $\nabla_u \varphi = B^T He$. Using $\psi(x, u, t)$ in the form (6) and (6a), we obtain the control algorithms

$$\frac{du}{dt} = -\Gamma B^\tau He - \frac{d(\Gamma_1 B^\tau He)}{dt}; \tag{14}$$

$$\frac{du}{dt} = -\Gamma B^\tau He - \frac{d(\gamma \operatorname{sign} B^\tau He)}{dt}. \tag{15}$$

For the controlled system by $\frac{dx}{dt} = A(x, t) + B(x, t)(u^\tau x)$, which corresponds to the coefficient adjusting problem for a linear controller, using the same $Q(x, t)$, we obtain $\nabla_u\varphi = (B^T He)\, x$ and different algorithms:

$$\frac{du}{dt} = -\Gamma B^\tau Hex - \frac{d(\Gamma_1 B^\tau Hex)}{dt}; \tag{16}$$

$$\frac{du}{dt} = -\Gamma B^\tau Hex - \frac{d(\gamma \operatorname{sign} B^\tau Hex)}{dt} \tag{17}$$

For linear controlled system, the algorithms (14)-(17) reduce to well-known algorithms from Ref. 6. The structure of (14) corresponds to the classical PI-controller; the algorithm (15) for $\Gamma = 0$ is the signal adaptation algorithm which has been systematically used, say, in Ref. 7. Algorithms of the form (16) have been previously designed by the hyperstability method.[3] and the "signal-parametric" algorithms (17) with integral and relay components have been studied in detail in Ref. 9. The unknown applicability conditions for the algorithms (14)-(17) and their regularized forms follows from Theorems 1-3.

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