Integrodifferentiating velocity gradient algorithms

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We describe a class of control and adaptation algorithms for nonlinear nonstationary systems which extend the class of velocity gradient algorithms.\(^{1,3}\) Applicability conditions for the algorithms are established and the time to attain the objective is estimated.

1. Consider the controlled system

\[
\frac{dx}{dt} = F(x, u, t), \quad t > 0,
\]

(1)

where \(x \in \mathbb{R}^n\) is the state, \(u \in \mathbb{R}^m\) the control. \(F(\cdot)\) is continuously differentiable. Let the control objective be specified.

\[
Q(x(t), t) < \Delta \quad \text{for} \quad t > t_0,
\]

(2)

where \(Q(x, t) \geq 0\) is a continuously differentiable objective function, \(\Delta > 0\). It is required to find a control algorithm

\[
u(t) = \nabla_{x} Q(x(t), u(t)), \quad 0 < s < t
\]

(3)

which ensures that for all initial conditions \(x_0 = x(0), u_0 = u(0)\) the system (1)-(3) attains the control objective (2) for some \(t_0 > 0\).

The term "control" is understood in a broad sense: the vector \(u(t)\) does not necessarily represent physical action on the controlled system coordinates. Equation (1) may represent a closed-loop plant-controller system (a generalized tunable plant), and \(u(t)\) may be the vector of tunable controller parameters, the parametric control vector, etc.; (3) may correspond to the system adaptation algorithm. The function \(Q(x, t)\) is generally a measure of deviation of the controlled-system path from the desired path; the control objective (2) requires minimizing \(Q(x, t)\) with specified accuracy.

2. Consider control algorithms of the form

\[
\frac{dx}{dt} = F(x, u, t) + \phi(x, u, t),
\]

(4)

where \(\Gamma = \Gamma^T \geq 0\) is a \(m \times m\) matrix, \(\Delta\) is the gradient symbol, \(\phi(x, u, t) = \partial Q / \partial t + [\nabla_{x} Q] \Gamma F(x, u, t)\) is the derivative function of \(Q(x, t)\) on (1) for \(u = \text{const}\); \(\psi(x, u, t)\) is some vector function which satisfies the pseudogradient inequality

\[
\psi(x, u, t)^T \nabla_{u} Q(x, u, t) > 0.
\]

(5)

For instance, we may assume

\[
\psi(x, u, t) = \Gamma \nabla_{x} \phi(x, u, t);
\]

(6)

\[
\psi(x, u, t) = \gamma \nabla_{x} \psi(x, u, t),
\]

(6a)

where \(\Gamma \geq 0\) is a \(m \times m\) matrix, \(\gamma > 0\) a number, \(\phi\) a vector whose components are the signs of the components of the vector \(\psi\).

For \(\epsilon(x, u, t) = 0\) the algorithms (4) reduce to velocity-gradient algorithms.\(^{1,3}\)

**Theorem 1.** Suppose that for every \(v \in \mathbb{R}^m\) there exists a unique solution \(u = \chi(x, v, t)\) to the equation \(u + \psi(x, u, t) = v\), the functions \(F(x, u, t), \nabla_{x} Q(x, t), \nabla_{u} Q(x, u, t)\) are bounded in every region \(\{(x, u, t) : \|x\| + \|u\| \leq \beta\}\), and the growth condition \(\inf_{0 \leq \sigma < \infty} Q(x(t), t) = \infty\) for \(\|x\| = \infty\) holds. Moreover, let the function \(\phi(x, u, t)\) be convex in \(u\) and let there exist a vector \(u^* \in \mathbb{R}^m\) and a number \(\alpha > 0\) such that for all \(x, t\)

\[
\phi(x, u^*, t) < -\alpha Q(x, t),
\]

(7)

All the paths of the system (1), (4) will then be bounded and \(u^* = 0\) for \(t \rightarrow \infty\), i.e., the control objective (2) is attained for every \(\Delta > 0\).

3. Consider a controlled system with perturbation

\[
\frac{dx}{dt} = F(x, u, t) + f(x, t)
\]

(8)

and, as in Refs. 2 and 3, the robustified algorithm

\[
\frac{d(\psi(x, u, t))}{dt} = -\Gamma \nabla_{u} \phi(x, u, t) + \mu \omega(u + \psi(x, u, t)),
\]

(9)

where \(\omega(u)\) is a convex regularizing function, \(\Gamma = \Gamma^T > 0, \mu > 0\).

**Theorem 2.** Let the conditions of Theorem 1 hold with (7) replaced by

\[
\phi(x, u^*, t) + \inf_{v \in \mathbb{R}^m} f(x, t) < -\alpha Q(x, t) + \Delta
\]

(10)

Let \(\omega(u) = \rho_{0} \|u\|^{2} \leq \rho_{1}(\alpha, \beta, \rho_{1}, \rho_{2}\) are positive numbers). Then (8), (9) is a dissipative system and for every \(\Delta > \Delta_0\) there exist a matrix \(\Gamma > 0\) and a number \(\mu > 0\) such that for \(\Gamma \geq \Gamma, \mu \leq \mu_{0}\) the control objective (2) is attained. Here \(Q(x(t), t)\) approaches the objective with an exponential velocity, and for the time \(t_0\) we have the estimate \(t_0 < c^{-1} \ln \left(2\|f(x(0), 0) - \beta o(\Delta - \beta o)\right)^{-1}\).

The proof of Theorems 1 and 2 requires taking the derivative of the function

\[
V(x, u, t) = Q(x(t)) + 0.5 \|u - u^* + \psi(x, u, t)\|^{2}_{1},
\]

(11)

in the system (1), (4), or (8), (9).
Remark. a) The inequality (1) implies that for some (generally unknown) $u = u_0$, the control objective (2) is attained in the system (3) for $\Delta = \beta/\delta$. This inequality holds, for instance, in controlled systems which are exponentially stable for $u = u_0$, $x(t) = 0$ if $Q(x)$ is quadratic and $x(t)$ is bounded. Theorem 2 implies that in this case the algorithm (9) ensures that for every $u(0)$ the system attains a control objective which is arbitrarily close to the best attainable objective.

b) In practice, the regularized function $\omega(u)$ is often chosen in the form $\omega(u) = 0.5u^TJ^2$, i.e., $V_u = u$ if we also take $V(x, u, t)$ in the form (6), then the algorithm (9) is described by a matrix integro-differential element with the transfer function $W(a) = (I + \Gamma I)(\alpha + \lambda)^{-1}$, whose input is the velocity gradient $V_u(x, u, t)$. This suggests the term integro-differential velocity gradient algorithms for this class of algorithms.

4. For $\Gamma = 0$, the algorithm (1) is best written in finite, rather than differential, form

$$u = u_0 - \gamma_1 V(x, u, t),$$

which explicitly includes the stepping factor $\gamma_1 > 0$. Let $V(x, u, t)$ satisfy the strong pseudogradient condition

$$V(x, u, t) = \rho \nabla V(x, u, t)$$

for some $\rho > 0, \delta > 0$, and all $x, u, t$. (The function $V$, for instance satisfies (13) for $\delta = 2, \Gamma = \rho I$, and the function (6a) satisfies (13) for $\delta = 1, \rho = \gamma/\sqrt{2}$.)

Theorem 3. Let (12) be uniquely solvable for $u, \lambda(x, u, t)$ be convex in $u$, and let (7), (13) hold. Then if $\delta > 1$, then for every $\Delta > 0$ the control objective (2) is attained in the system (1), (13) for

$$\gamma_1 > \gamma_0(\Delta) = \frac{\parallel u_0 - u_0 \parallel}{\rho \delta} \parallel (u_0 - u_0, \parallel (\delta - 1) \delta^{-1}.$$  

If $\delta = 1$, then $\delta = 1, \gamma_1 = \parallel u_0 - u_0 \parallel / m, m Q(x(t), t) < Q(x_0, 0)$. The time to attain the objective within $\varepsilon$ is the exact estimator $\varepsilon = \ln(\gamma_0)(x_0, 0, \varepsilon).$

Remarks. a) The theorem remains valid if $u_0 = u_0(x, t)$ but $\parallel u_0 - u_0(x, t)\parallel$ is bounded.

b) The algorithm (12), (6a) coincides with the optimal control algorithm for damping the function $Q(x, t)$.

5. As an example let us consider the design of model reference adaptive systems. Let the controlled system (1) have the form

$$\frac{dx}{dt} = A(x, t) + B(x, t)u, u \in \mathbb{R}^m,$$

and let $Q(x, t) = e^THe$, where $e = x - x_0(t)$, $\frac{dx_0}{dt} = A_0x_0$.

For the controlled system $Q(x, t) = A(x, t) + B(x, t)(u x)$, which corresponds to the coefficient adjusting problem for a linear controller, using the same $Q(x, t)$, we obtain $\frac{dx}{dt} = A(x, t) + B(x, t)(u x)$ and different algorithms:

$$\frac{dx}{dt} = -\Gamma B^T He - \frac{d(\Gamma B^T He)}{dt};$$

$$\frac{dx}{dt} = -\Gamma B^T Hex - \frac{d(\gamma \text{sign } B^T He)}{dt};$$

For linear controlled systems, the algorithms (14)-(17) reduce to well-known algorithms from Ref. 6. The structure of (14) corresponds to the classical PI-controller; the algorithm (15) for $\Gamma = 0$ is the signal adaptation algorithm which has been systematically used, say, in Ref. 7. Algorithms of the form (16) have been previously designed by hyperstability method, and the "signal-parametric" algorithms (17) with integral and relay components have been studied in detail in Ref. 9. The unknown applicability conditions for the algorithms (14)-(17) and their regularized forms follow from Theorems 1-3.

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