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CONTROL THEORY

Adaptive Stabilization of Minimal-Phase Vector-Input Objects without Output Derivative Measurements

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We consider the problem of adaptive stabilization for multiple-input objects of control without measuring output derivatives. We propose a simple law of adaptive stabilization based on a compensating dynamic feedback introduced into the controller and on the implicit reference model method [1, 2]. We show that a compensator of the order $r-1$ exists for minimal-phase objects of the scalar order $r > 1$. Its inclusion into the object gives the latter the strict minimal-phase property ($r = 1$). This gives us the opportunity to apply the adaptive stabilization algorithm [2], as well as its generalization described below.

Consider a linear stationary object

$$\dot{x} = Ax + Bu, \quad y = Cx,$$
$$x \in \mathbf{R}^n, \quad y \in \mathbf{R}^l, \quad u \in \mathbf{R}^m, \tag{1}$$

with the matrix transfer function $W(\lambda) = C(\lambda I - A)^{-1}B$. Let $l = m$, $\delta(\lambda) = \det(\lambda I - A)$, and $\varphi(\lambda) = \delta(\lambda)\det W(\lambda)$.

Definition 1 [3]. We call the system (1) a minimal-phase system, if $\varphi(\lambda)$ is the Hurwitz polynomial, and a strictly minimal-phase system, if, in addition, CB is a symmetric positively definite matrix.

Definition 2 [4]. The system (1) is said to have the relative scalar order r, if

$$CA^iB = 0 \text{ for } i = 0, 1, \dots, r-2, \det CA^{r-1}B \neq 0. \tag{2}$$

Obviously, a strictly minimal-phase system has the relative scalar order $r = 1$. For $r = 1$, the polynomial $\varphi(\lambda)$ is of the degree $n - m$ and its leading coefficient is equal to $\det CB$.

We set up a problem of finding a linear adaptive controller

$$u = \theta^{\mathrm{T}} y, \tag{3}$$
$$\theta = \Theta(y, \theta), \tag{4}$$

where θ is the $(l \times m)$ matrix of adjustable parameters. The controller has to ensure the attainment of the stabilization, which is the objective of control:

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} \theta(t) = \text{const}. \tag{5}$$

Theorem 1. *Let the adaptation algorithm* (4) *have the form*

$$\frac{d\bar{\theta}}{dt} = -\Gamma \begin{bmatrix} g_1^{\mathrm{T}} y \cdot y \\ \dots \\ g_m^{\mathrm{T}} y \cdot y \end{bmatrix}, \tag{6}$$

where $\bar{\theta} = \mathrm{col}(\theta) \in \mathbf{R}^{lm}$ *is a column vector constructed from the columns of the matrix* θ, $\Gamma = \Gamma^{\mathrm{T}} > 0$ *is a* $(lm \times lm)$ *matrix;* g_i, $i = 1, \dots, m$, *are the l-dimensional vectors; and the matrix* $G^{\mathrm{T}}W(\lambda)$ *is a strictly minimal-phase one, with* $G = [g_1, \dots, g_m]$.

Then, the objective (5) *is attained in the system* (1), (3), (6).

In addition, the system has a quadratic Lyapunov function of the form

$$V(x, \theta) = x^{\mathrm{T}}Px + (\bar{\theta} - \bar{\theta}*)^{\mathrm{T}} P_1 (\bar{\theta} - \bar{\theta}*), \tag{7}$$

which has the property $V(x, \theta) > 0$ *for* $x \neq 0$, $\theta \neq \theta_*$, *and* $\dot{V}(x, \theta) < 0$ *for* $x \neq 0$.

The Theorem 1 is a generalization of the results obtained in [2]. It is proved by analogy with [2]. In [2], matrix Γ was assumed to have the block-diagonal form, and the adaptation algorithm (6) has the form

$$\dot{\theta}_i = \Gamma_i y y^{\mathrm{T}} G, \quad i = 1, \dots, m, \tag{8}$$

where θ_i are the columns of the matrix θ, $\Gamma_i = \Gamma_i^{\mathrm{T}} > 0$ are $(m \times m)$ matrices, and G is an $(l \times m)$ matrix. Following the lines of [2], it is possible to show that, if $\mathrm{rank}(B) = m$, and the structure of the main loop (3) is fixed, the conditions of the theorem are also necessary for the existence of a Lyapunov function (7) with the properties declared above. This means that the class of problems solved within the framework of the main loop structure (3) and of the Lyapunov function of the form (7) cannot be extended.

The conditions for the applicability of the algorithm (6) can be somewhat relaxed if the structure of the main

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loop is extended. Consider the generalized structure of the main loop, which contains an additional cross-link matrix

$$u = D\theta^{\tau} y, \tag{9}$$

where D is a constant $(m \times m)$ matrix.

Theorem 2. *For the system* (1), (6), (9) *with a certain matrix D, the objective* (4) *is attained if the system with transfer matrix $G^{\tau}W(\lambda)$ is a minimal-phase scalar one and has the relative order $r = 1$.*

The proof is obtained by reducing Theorem 2 to Theorem 1 by choice $D = (CB)^{-1}$ and substitution $u = D\tilde{u}$.

Note that the conditions of Theorems 1 and 2 are fulfilled only for $r = 1$, which excludes many important problems.

We consider below the general situation $r > 1$. Synthesizing and analyzing the stability of adaptive control systems is difficult in this case because the known solutions are quite cumbersome [5 - 7]. The adaptive stabilization algorithm that we propose for the case $r > 1$ contains a feedback dynamic compensator in the main loop. This is equivalent to modifying the control object. The analysis of stability of the adaptive system is based on the following statement.

Theorem 3. *Let the object with a transfer function G be a minimal-phase one with the relative scalar order $r > 1$ for a certain $(l \times m)$-matrix $G^{\tau}W(\lambda)$, for which the matrix $G^{\tau}CA^{r-1}$ has the Hurwitz form. Let $P(\lambda)$, $Q(\lambda)$ be the Hurwitz polynomials of degrees $r-2$, $r-1$, respectively, and the signs of their coefficients and those of $\varphi(\lambda) = \delta(\lambda)\det G^{\tau}W(\lambda)$ coincide. Finally, let*

$$W_{\kappa\varepsilon}(\lambda) = G^{\tau}W(\lambda) + \kappa\varepsilon P(\varepsilon\lambda)/Q(\lambda)I_m. \tag{10}$$

Then, there exist a number $\kappa_0 > 0$ and function $\varepsilon_0(\kappa) > 0$ such that the matrix $W_{\kappa\varepsilon}(\lambda)$ is a strictly minimal-phase one for $\kappa > \kappa_0$, $0 < \varepsilon < \varepsilon_0(\kappa)$.

The proof is based on the following lemma, analogous to Lemma 3 of [2].

Lemma 1. *Let $D(\lambda, \varepsilon)$, $E(\lambda, \varepsilon)$ be $(m \times m)$-matrix polynomials with coefficients that are continuous in ε in the point $\varepsilon = 0$:*

$$D(\lambda, \varepsilon) = D_r(\varepsilon)\lambda^r + \ldots + D_0(\varepsilon),$$

$$E(\lambda, \varepsilon) = E_{n-1}(\varepsilon)\lambda^{n-1} + \ldots + E_0(\varepsilon).$$

Let the polynomials $\det E(\lambda, 0)$, $\det[\lambda D(\lambda, 0) + E_{n-1}(0)]$ have the Hurwitz form.

Then, the polynomial $\pi(\lambda, \varepsilon) = \det[\varepsilon\lambda^n D(\varepsilon\lambda, \varepsilon) + E(\lambda, \varepsilon)]$ has the Hurwitz form for all sufficiently small $\varepsilon > 0$.

Proof. Let $\varepsilon \to 0$. Then, $\pi(\lambda, \varepsilon) \to \pi(\lambda, 0) = \det E(\lambda, 0)$. Consequently, $m(n-1)$ roots of the polynomial $\pi(\lambda, \varepsilon)$ tend to the roots of the polynomial $\det E(\lambda, 0)$, and the remaining $m(r+n) - m(n-1) = m(r+1)$ roots tend to infinity. Let us analyze the behavior of the roots, with a substitution $\varepsilon\lambda = \mu$ and assuming $\nu(\mu, \varepsilon) = \varepsilon^{m(n-1)}\pi(\mu/\varepsilon, \varepsilon)$. We obtain

$$\lim_{\varepsilon \to 0} \nu(\mu, \varepsilon) = \mu^{m(n-1)} \det[\mu D(\mu, 0) + E_{n-1}(0)].$$

Consequently, $m(r+1)$ roots of $\nu(\mu, \varepsilon)$ tend to μ_i, $i = 1, \ldots, m(r+1)$, which are the roots of $\det[\lambda D(\lambda, 0) + E^{n-1}(0)]$, and the remaining $m(n-1)$ roots tend to zero. Thus, $m(r+1)$ roots of the original polynomial $\pi(\lambda, \varepsilon)$ have the form $\mu_i/\varepsilon + O(1/\varepsilon)$ and, under the conditions of the lemma, for small $\varepsilon > 0$, these roots lie in the left half-plane. The lemma is proved.

Corollary. *Let the polynomials $\det E(\lambda, 0)$, $\det D(\lambda, 0)$ and $\det[\lambda D(0, 0) + E_{n-1}(0)]$ have the Hurwitz form. Then, there exist a number $\kappa_0 > 0$ and a function $\varepsilon_0(\kappa) > 0$, such that the polynomial $\pi(\lambda, \varepsilon) = \det[\varepsilon\lambda^n D(\varepsilon\lambda, \varepsilon) + E(\lambda, \varepsilon)]$ has the Hurwitz form for $\kappa > \kappa_0$, $0 < \varepsilon < \varepsilon_0(\kappa)$.*

This corollary is proved by analogy with the proof of Lemma 1.

To prove Theorem 3, let us represent $\det W_{\kappa\varepsilon}(\lambda)$ as

$$\det W_{\kappa\varepsilon}(\lambda) = \{\delta(\lambda) Q(\lambda)\}^{-m} \times \det\{R(\lambda) Q(\lambda) + \kappa\varepsilon\delta(\lambda) P(\varepsilon\lambda) I_m\},$$

where $R(\lambda) = G^{\tau}W(\lambda)\delta(\lambda)$. Under the conditions of the theorem, the matrix $R(\lambda)$ can be represented as $R(\lambda) = R_{n-r}\lambda^{n-r} + \ldots + R_0$, where $R_{n-r} = G^{\tau}CA^{n-r}B$ (see, e.g., [8]). The polynomial $\det R(\lambda)$ has the form $\det R(\lambda) = \delta(\lambda)^{m-1}\varphi(\lambda)$, where $\varphi(\lambda)$ is a Hurwitz polynomial. By lemma 1, the polynomial $\delta(\lambda)^{-m+1} \times Q(\lambda)^{-m}\det W_{\kappa\varepsilon}(\lambda)$ has the Hurwitz form for sufficiently small $\varepsilon > 0$, if the polynomial $\det\{\kappa\lambda P(\lambda)I_m + R_{n-r}Q_{r-1}\}$ has the Hurwitz form, which, in turn, is true for sufficiently large κ, if $-R_{n-r}$ is a Hurwitz matrix. To prove the theorem, it must be established that the matrix $\lim_{\lambda \to 0} W_{\kappa\varepsilon}(\lambda)$ is symmetric and positively definite. This is true, because $\lim_{\lambda \to 0} W_{\kappa\varepsilon}(\lambda) = \kappa\varepsilon P_{r-2} I_m$. Theorem 3 is proved.

Note. If $r = 2$, the statement of Theorem 3 is true for $\kappa_0 = 0$, i.e., we can put $\kappa = 1$ in the filter (10).

Now we can describe the structure of the adaptive controller proposed and conditions of its applicability.

Theorem 4. *Let an object have the transfer function $G^{\tau}W(\lambda)$, the minimal-phase property for a certain $(l \times m)$-matrix G, and the relative scalar order $r \geq 1$. Let the matrix $-G^{\tau}CA^{r-1}B$ have the Hurwitz form. Let $P(\lambda)$, $Q(\lambda)$ be Hurwitz polynomials of degrees $r-2$, $r-1$, respectively, and the signs of their coefficients and those of $\varphi(\lambda) = \delta(\lambda)\det G^{\tau}W(\lambda)$ be the same. Let the control algorithm have the form*

$$u_i = \theta_{1i}^{\tau} y + \theta_{2i} v_i, \quad i = 1, \ldots, m, \tag{11}$$

$$\frac{d\tilde{\theta}_1}{dt} = -\Gamma_1 \begin{bmatrix} (g_1^{\tau} y + v_1) y \\ \ldots \\ (g_m^{\tau} y + v_m) y \end{bmatrix}, \tag{12}$$

$$\frac{d\bar{\theta}_2}{dt} = -\Gamma_2 \begin{bmatrix} (g_1^{\mathrm{T}} y + v_1)\, v_1 \\ \dots \\ (g_m^{\mathrm{T}} y + v_m)\, v_m \end{bmatrix}, \quad (13)$$

where $\bar{\theta}_1 = \mathrm{col}(\theta_{1i}) \in \mathbf{R}^{lm}$ *is a column vector constructed from the columns* θ_{1i}, *and* $\bar{\theta}_2 = \mathrm{col}(\theta_{2i}) \in \mathbf{R}^m$ *is a column vector constructed from the numbers* θ_{2i}; $\Gamma_1 = \Gamma_1^{\mathrm{T}} > 0$ *is an* $(lm \times lm)$ *matrix*; $\Gamma_2 = \Gamma_2^{\mathrm{T}} > 0$ *is a* $(m \times m)$ *matrix*, $G = [g_1, \dots, g_m]$, g_i, $i = 1, \dots, m$, *are the l-dimensional vectors*; v_i, $i = 1, \dots, m$, *are the outputs of the subsidiary systems* (*of the filters*):

$$Q(p) v_i = \kappa\varepsilon P(\varepsilon p) u_i, \quad p = d/dt. \quad (14)$$

Then objective (4) *is attained in the system* (1), (12), (13) *for all* $\kappa > \kappa_0$, $0 < \varepsilon < \varepsilon_0(\kappa)$. *The objectives*

$$\lim_{t\to\infty} v_i(t) = 0, \ \lim_{t\to\infty} \theta_i(t) = \mathrm{const}, \quad i = 1, \dots, m, \quad (15)$$

are attained as well.

P r o o f. Let us introduce the extended object

$$\frac{d\tilde{x}}{dt} = \tilde{A}\tilde{x} + \tilde{B}u, \quad \tilde{y} = \tilde{C}\tilde{x} \quad (16)$$

with the state vector $\tilde{x} = \mathrm{col}(x, x_1, \dots, x_m) \in R^{n+m(r-1)}$, where x_i is the state vector of the ith filter (14); with the output vector $\tilde{y} = \mathrm{col}(y, v_1, \dots, v_m)$; and with transfer function $\tilde{W}(\lambda) = \mathrm{col}(W(\lambda), W_F(\lambda))$, where $W_F(\lambda) = \kappa\varepsilon P(\varepsilon\lambda)/Q(\lambda) I_m$. Evidently, the algorithm (12) - (14) is a specific case (up to the choice of notation) of the algorithm (3), (6), namely, that of the extended object with block-diagonal matrices $\Gamma = \mathrm{diag}[\Gamma_1, \Gamma_2]$ and $\tilde{G} = \mathrm{diag}[G, I_m]$. In applying Theorem 3, we choose $\kappa > 0$, $\varepsilon > 0$ in such a way that the function $G^{\mathrm{T}} W(\lambda) = G^{\mathrm{T}} W(\lambda) + W_F(\lambda)$ is a strictly minimal-phase one. Now the statement of the theorem follows directly from Theorem 1.

C o r o l l a r y. *Under the conditions of Theorem* 4, *it can be easily shown that the problem of adaptive stabilization can be solved by algorithms structured in a simpler way, e.g., by an algorithm with* 2*m adjustable parameters*

$$u_i = \theta_{1i} g_i^{\mathrm{T}} y + \theta_{2i} v_i, \quad i = 1, \dots, m, \quad (17)$$

$$\dot{\theta}_{1i} = -\gamma_{1i} (g_i^{\mathrm{T}} y + v_i)\, g_i^{\mathrm{T}} y, \quad (18)$$

$$\dot{\theta}_{2i} = -\gamma_{2i} (g_i^{\mathrm{T}} y + v_i)\, v_i, \quad (19)$$

where θ_{1i}, θ_{2i} *are scalars.*

To conclude, we note that the existing adaptive stabilization algorithms [5 - 7] applicable to the case $r > 1$ contain several additional filters with the total order three to five times higher than the order of the controlled object. The algorithms we proposed have the dynamic compensator of the order $m(r - 1)$. Such structures were, for the first time, studied in [9], in nonadaptive situations. Let us also note that when the parameter κ of the compensator increases and ε decreases, the stability domain of the adaptive system in the space of object parameters extends to infinity and covers any compact subset of the set of minimal-phase objects. In practice, the compensator parameters κ and ε should be chosen according to *a priori* information on the parameters of controlled object.

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