QUADRATIC LYAPUNOV FUNCTIONS IN THE ADAPTIVE STABILITY PROBLEM OF A LINEAR DYNAMIC TARGET

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INTRODUCTION

We consider the system of differential equations†

$$dx/dt = Ax + Bu + f(t),\ y = L^*x,\ 0 \leqslant t < \infty, \tag{0.1}$$

$$u = C^*y, \tag{0.2}$$

$$dC/dt = F(y), \tag{0.3}$$

where x, u, y, f, A, B, L, C are real vectors and matrices of orders n, m, l, n, n × n, n × m, n × l, l × m, respectively, and F(y) is a continuous matrix function.

Equations (0.1)-(0.3) arise in the theory of automatic equations in the mathematical description of a system of adaptive stability. In this connection, Eqs. (0.1) describe the dynamics of a controlled target in a linear approximation [u = u(t) is the vector of inputs or "controls," y = y(t) is the vector of observable inputs, x = x(t) is the "state" vector of the target, f(t) is the "perturbation" vector], and Eqs. (0.2)-(0.3) assign the "adaptive" linear control. We will assume that the matrices A, B, L (target parameters) do not change with time (the target is stationary). The existence of the "contour of adaptation" (0.3) is stipulated so that in a series of practical problems (cf., e.g., [1, 2]) the target parameters and the perturbations f(t) are unknowns of the designer of the system; under this, the usual stability system with "rigid" feedback, described by Eqs. (0.1)-(0.2) under C(t) = const, cannot be stable under all possible values of the parameters. We will give a precise definition of adaptivity of a system of the form (0.1)-(0.3), following the common formulation of the problem of the design of an adaptive system [3].

Let $A = A(\xi)$, $B = B(\xi)$, $L = L(\xi)$, $f(t) = f_\xi(t)$, $\xi \in \Xi$, where ξ is the abstract vector of all unknown target parameters and the characteristics of the perturbations, and Ξ is the known set of possible values of ξ.

Definition 1. The stability system (0.1)-(0.3) is said to be adaptive in the set Ξ if for every x(0), C(0) and, for all $\xi \in \Xi$, the solution (x(t), C(t)) of the system of differential equations (0.1)-(0.3) is defined for all $t \geq 0$ and satisfies the conditions:

(I) $\lim_{t\to\infty} x(t) = 0$;

(II) there exists a finite limit $\lim_{t\to\infty} C(t)$.

The problem of the design of an adaptive stability system consists of defining the function F(y), independent of $\xi \in \Xi$, so that conditions (I) and (II) are satisfied.

In the present paper, a solution is proposed for the stated problem.

Let G be a matrix of order l × m, and λ a complex variable. Let us introduce the notation: $\chi_\xi(\lambda) = L^*(\xi)[\lambda I_n - A(\xi)]^{-1}B(\xi)$,

†Below, finite-dimensional vectors will be identified with column matrices of the appropriate order. The asterisk is the symbol of the Hermitian conjugate (for real matrices this is the conjugate transpose). We denote by $\|\cdot\|$ the equivalent vector or matrix norm, and by I_n the n × n identity matrix. The diagonal matrix with elements $\tau_1, \ldots, \tau_m$ on the main diagonal is denoted by $\mathrm{diag}\{\tau_1, \ldots, \tau_m\}$.

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$$\delta_\xi(\lambda)=\det[\lambda I_n-A(\xi)],\ \varphi_\xi(\lambda)=\delta(\lambda)\det G^*\chi_\xi(\lambda),\ D_\xi=\lim_{\lambda\to\infty}\lambda G^*\chi_\xi(\lambda).$$

The principal result of the paper is contained in the following theorem.

THEOREM 1. Let the adaptive contour (0.3) be described by the equation

$$dc_j/dt=-\left[g_j^*y\right]P_jy,\quad j=1,\ldots,m, \tag{0.4}$$

in which $P_j = P_j^* > 0$ is an arbitrary positive definite $l \times l$ matrix, and g_j is an l-dimensional vector. Let G be an $l \times m$ matrix with columns g_j, $j = 1, \ldots, m$. The system (0.1), (0.2), (0.4) is adaptive in the set Ξ if for every $\xi \in \Xi$, the following conditions are satisfied:

1) $\varphi_\xi(\lambda)$ is a Hurwitzian polynomial;

2) the matrix $\tau_\xi D_\xi$ is symmetric and positive definite under some $\tau_\xi = \operatorname{diag}\{\tau_1, \ldots, \tau_m\}$, $\tau_j = \tau_j(\xi) > 0$, $j = 1, \ldots, m$;

3) $\int_0^\infty \|f_\xi(t)\|^2dt < \infty$.

In the proof of Theorem 1 the problem of the existence of a Lyapunov function of the form†

$$V_\xi(x,C)=x^*H_0x+\sum_{j=1}^m\left(c_j-c_j^0\right)^*H_j\left(c_j-c_j^0\right) \tag{0.5}$$

with properties:

(A) $V_\xi(x, C) > 0$ for $x \neq 0$, $C \neq C^0$;

(B) $\dot V_\xi(x, C) < 0$ for $x \neq 0$, $f_\xi(t) \equiv 0$

for the system (0.1)-(0.3) for every $\xi \in \Xi$. Here $H_0 = H_0^* = H_0(\xi)$ is a matrix of order $n \times n$; $H_j = H_j^* = H_j(\xi)$ is a matrix of order $l \times l$, c_j, $c_j^0(\xi)$, $j = 1, \ldots, m$ are the columns of the $l \times m$ matrices C, $C^0(\xi)$, respectively; $\dot V_\xi(x, C)$ is the derivative of the function $V_\xi(x, C)$ by virtue of the system (0.1)-(0.3).

THEOREM 2. For the existence of a function of the form (0.5) with properties (A), (B) for the system (0.1)-(0.3) and for every $\xi \in \Xi$ it is sufficient, and if $\chi_\xi(\lambda) \not\equiv 0$ and the rank of $B(\xi)$ is equal to m for every $\xi \in \Xi$ then it is also necessary, that the adaptive contour be described by the relations (0.4), in which $P_j = P_j^* > 0$ is an arbitrary positive definite $l \times l$ matrix, and the $l \times m$ matrix G with columns $g_1, \ldots, g_m$ is such that for every $\xi \in \Xi$ conditions 1, 2 of Theorem 1 are satisfied.

The proof of Theorems 1, 2 is deduced in §2. The central moment of the proof is the application of a theorem on the determination of the conditions of the matrix inequalities (1.1)-(1.3) (Theorem 3), the proof of which is dealt with in §1. In turn, the proof of Theorem 3 rests on the frequency conditions for the existence of solutions of other matrix inequalities, appearing first in the analyses of the stability of nonlinear control systems [4, 5] and are well known under the name of the Yakubovich – Kalman lemma (cf., e.g., [6]).

It should be noted that similar problems are considered (without rigorous mathematical formulation) in the theory of nonrandom, self-adjusting systems (BSNS) with a standard pattern [2, 7-10]. The idea of the application for the design of the adaptive contour of the Lyapunov function of the form (0.5) was taken by the author from well known and very significant works [2, 7]. However, the results of [2, 7-10], and also a series of other works on BSNS theory relate only to the case m = 1 (scalar control) and $L = I_n$ (target state accessible to observation). In addition, the methods of the design of adaptive algorithms, following from the results of [2, 7-10], are based on the determination of the matrix H_0 in (0.5) from the relation $H_0A_0 + A_0^*H_0 = -Q$, where $A_0 = A + B[C^0]^*L^*$, and $Q = Q^* > 0$ is a given matrix. In addition, the matrix A_0 must be known. The method, proposed here, does not require the determination of the matrix A_0, including at the same time all the algorithms which can be obtained by means of a Lyapunov function of the form (0.5). Finally, the proposed algorithms, in contrast to those of [2, 7-10], do not require the integration of the differential equation of the "standard pattern," which simplifies the model.

A problem similar to that stated is considered in [11], where, however, u(t) is a step function and the algorithms of control and adaptivity work in discrete time and have a form different from (0.2), (0.3).

†Below we will omit the index ξ everywhere, where this does not cause confusion.

§ 1. CONDITIONS FOR THE EXISTENCE OF SOLUTIONS OF THE FUNDAMENTAL MATRIX INEQUALITIES

We consider the following algebraic problem. The complex matrices A, B, L, G, R of orders $n \times n$, $n \times m$, $n \times l$, $l \times m$, $n \times n$, respectively ($m \leq n$, $l \leq n$), where† $R = R^* \geq 0$. It is necessary to find conditions for the existence of an Hermitian $n \times n$ matrix $H = H^* > 0$ and a complex $l \times m$ matrix C such that

$$HA(C)+A^*(C)H+R<0, \tag{1.1}$$
$$HB=LG \tag{1.2}$$

where

$$A(C)=A+BC^*L^*. \tag{1.3}$$

The case when all the matrices A, B, L, G, R are real will be called the real case. Let us introduce the notation $\delta(\lambda) = \det(\lambda I_n - A)$, $\chi(\lambda) = L^*(\lambda I_n - A)^{-1}B$, $\delta(\lambda, C) = \det[\lambda I_n - A(C)]$, $\chi(\lambda, C) = L^*(\lambda I_n - A(C))^{-1}B$, $\varphi(\lambda) = \delta(\lambda)\det G^*\chi(\lambda)$, $D = \lim_{\lambda\to\infty} \lambda G^*\chi(\lambda)$. It is possible to show that (cf. Lemma 1) $\varphi(\lambda)$ is a polynomial of degree $n - m$, invariant relative to the substitution of $\delta(\lambda, C)$ and $\chi(\lambda, C)$ for $\delta(\lambda)$ and $\chi(\lambda)$. It is obvious that $D = G^* L^* B$, and therefore the $m \times m$ matrix D is also invariant relative to this substitution. It is easy to verify the following identities:

$$\delta(\lambda, C)=\delta(\lambda) \det [I_m-C^*\chi(\lambda)], \tag{1.4}$$
$$\chi(\lambda, C)=\chi(\lambda)[I_m-C^*\chi(\lambda)]^{-1}. \tag{1.5}$$

As usual, we will say that a polynomial is Hurwitzian if all its zero lie in the open left half-plane. The solution of the stated problem is given by the following theorem.

THEOREM 1. For the existence of the matrices $H = H^* > 0$ and C, satisfying (1.1)-(1.3) and real in the real case, it is sufficient, and if the rank of B is equal to m then it is necessary, that the polynomial $\varphi(\lambda)$ be Hurwitzian and the matrix D be Hermitian and positive definite.

Before turning to the proof of Theorem 1, we prove an auxiliary assertion.

LEMMA 1. Let $\alpha(\lambda) = (\lambda I_n - A)^{-1}\delta(\lambda)$. Let p, q be arbitrary $n \times m$ matrices and $\Sigma(\lambda) = p^*\alpha(\lambda)q$. Then $\det \Sigma(\lambda) = \delta(\lambda)^{m-1} \cdot \sigma(\lambda)$, where $\sigma(\lambda)$ is a polynomial of degree not greater than $n - m$ with highest term $\lambda^{n-m} \det p^*q$. In addition, $\sigma(\lambda)$ does not change under the substitution of $A + qr^*$ for A, where r is an arbitrary $n \times m$ matrix.

Proof. From the proof of Lemma 4 [12], it follows that $\sigma(\lambda) = \det \Phi(\lambda)$, where

$$\Phi(\lambda)=\begin{Vmatrix}\lambda I_n - A + qp^* & -q \\ p^* & 0\end{Vmatrix}.$$

Let $\sigma(\lambda) = \sigma_n\lambda^n + \ldots + \sigma_1\lambda + \sigma_0$. Then σ_{n-k} is equal to the sum of those principal minors of order $m + k$ of the matrix $\Phi(0)$, whose expansion consists of exactly k elements of the left upper block. It follows that $\sigma_{n-k} = 0$ for $k < m$. Further,

$$\sigma_{n-m}=\lim_{\lambda\to\infty}\sigma(\lambda)/\lambda^{n-m}=\lim_{\lambda\to\infty}\det\begin{Vmatrix}I_n-\lambda^{-1}(A-qp^*) & -q \\ p^* & 0\end{Vmatrix}=\det\begin{Vmatrix}I_n & -q \\ p^* & 0\end{Vmatrix}=\det p^*q.$$

The second assertion of the lemma follows from the substitution $A \to A + gr^*$, which is equivalent to the addition to the first n columns of the matrix $\Phi(\lambda)$ its last m columns, multiplied by the corresponding elements of the matrix r. Obviously, the polynomial $\det \Phi(\lambda)$ does not change in this connection.

COROLLARY. The polynomial $\varphi(\lambda) = \delta(\lambda)G^*\chi(\lambda) = G^*L^*[(\lambda I_n - A)^{-1}\delta(\lambda)]B$, defined above, has as its highest term $\lambda^{n-m}\det G^*L^*B$ and does not change under the feedback transformation $A \to A(C) = A + BC^*L^*$. [The last assertion can be deduced directly from (1.4), (1.5).]

LEMMA 2. Any k-th order minor of the matrix $\Sigma(\lambda)$, introduced in Lemma 1, divides into $\delta(\lambda)^{k-1}$, and the quotient is a polynomial of degree not greater than $n - k$ with coefficients in λ^{n-k} equal to the corresponding minor of the matrix p^*q.

†Here and later, the notation $R \geq 0$ ($R > 0$) means that the matrix R is nonnegative (positive definite), i.e., $x^*Rx \geq 0$ ($x^*Rx > 0$ for $x \neq 0$).

Proof. The matrix $\Sigma'(\lambda)$ of the indicated minor has the form $\Sigma'(\lambda) = e_1^*\Sigma(\lambda)e_2$, where e_1, e_2 are constant matrices of order m × k. Therefore, $\Sigma'(\lambda) = a_1^*\alpha(\lambda)a_2$, where a_1, a_2 are matrices of order n × k. If we apply Lemma 1 to the matrix $\Sigma'(\lambda)$, we obtain the required statement.

LEMMA 3. Let the polynomial $P_\varepsilon(\lambda)$ have the form $P_\varepsilon(\lambda) = \lambda^{n-m}Q\ (\lambda) + R_\varepsilon(\lambda)$, where $Q_3(\lambda) = \sum_{k=0}^{m}(q_k + q_k'(\varepsilon))\varepsilon^k\lambda^k$, $R_\varepsilon(\lambda) = \sum_{k=0}^{n-m-1}(r_k + r_k'(\varepsilon))\lambda^k$, and $q_k'(\varepsilon) = O(\varepsilon)$, $r_k'(\varepsilon) = O(\varepsilon)$ for $\varepsilon \to 0$.

In addition, let the polynomials $Q(\lambda) = \sum_{k=0}^{m} q_k\lambda^k$, $R(\lambda) = q_0\lambda^{n-m} + \sum_{k=0}^{n-m-1} r_k\lambda^k$ be Hurwitzian. Then the polynomial $P_\varepsilon(\lambda)$ is Hurwitzian for all sufficiently small $\varepsilon > 0$.

Proof. It is easy to verify that n − m zeros of the polynomial $P_\varepsilon(\lambda)$ converge to the m zeros of the polynomial $\varepsilon \to 0$, and the remaining $R(\lambda) = \lim_{\varepsilon\to 0} P_\varepsilon(\lambda)$ zeros diverge to infinity. We make the substitution $\varepsilon\lambda = \mu$ and set $S_\varepsilon(\mu) = \varepsilon^{n-m}P_\varepsilon(\mu/\varepsilon)$. Then $S_\varepsilon(\mu) = \mu^{n-m}\sum_{k=0}^{m}(q_k + q_k'(\varepsilon))\mu^k + \sum_{k=0}^{n-m-1}(r_k + r_k'(\varepsilon))\mu^k\varepsilon^{n-m-k}$. Therefore n − m zeros of the polynomial $S_\varepsilon(\mu)$ converge to zero for $\varepsilon \to 0$, and the remaining m zeros converge to the zeros of $Q(\mu)$ with rate of convergence of order $O(\varepsilon)$. Consequently, the m zeros $\lambda_1, \ldots, \lambda_m$ of the polynomial $P_\varepsilon(\lambda)$ at $\varepsilon \to 0$ bearing in mind $\lambda_i = \mu_i/\varepsilon + 0(1)$, $i = 1, \ldots, \mu_m$ are the zeros of $Q(\mu)$. Thus, the zeros of the polynomial $P_\varepsilon(\lambda)$ will lie in the left half-plane if the zeros of $Q(\lambda)$ and $R(\lambda)$ lie there. This proves Lemma 3.

Remark 1. An assertion similar to Lemma 3 appears in [13].

Remark 2. The condition of Hurwitzian for the polynomials $Q(\lambda)$ and $R(\lambda)$ is "almost" necessary. Namely, in order that $P_\varepsilon(\lambda)$ be Hurwitzian for all sufficiently small $\varepsilon > 0$, it is necessary that $Q(\lambda)$ and $R(\lambda)$ not have zeros in the right half-plane (this follows from the proof of the lemma).

LEMMA 4. Let $\Phi(\omega)$, $\omega \in R^1$ be a complex m × m matrix satisfying for every $x \neq 0$, $\omega \in R^1$ the inequality $\mathrm{Re}\, x^*\Phi(\omega) > 0$. Then $|\Delta \arg\det\Phi(\omega)| \leq m\pi$, where $\Delta\arg\psi(\omega)$ is regarded as the increment of the argument of the complex-valued function $\psi(\omega)$ $(\psi(\omega) \neq 0)$ under ω, changing from $-\infty$ to $+\infty$, i.e., the quantity $\lim_{\omega\to+\infty}[\arg\psi(\omega) - \arg\psi(-\omega)]$.

Proof. Let $\lambda_i(\omega)$, $x_i(\omega)$, $i = 1, \ldots, m$ be the eigenvalues and eigenvectors of the matrix $\Phi(\omega)$ (it is possible to have coincidence among these). Then $\mathrm{Re}\,\lambda_i(\omega) = \mathrm{Re}\, x_i^*(\omega)\Phi(\omega)x_i(\omega)/\|x_i(\omega)\|^2 > 0$ for all $i = 1, \ldots, m$, $\omega \in R^1$. Hence $\lambda_i(\omega) \neq 0$ and consequently $\det\Phi(\omega) = \prod_{i=1}^{m}\lambda_i(\omega) \neq 0$ for every $\omega \in R^1$. Thus, $|\Delta\arg\lambda_i(\omega)| \leq \pi$, so that $|\Delta\arg\det\Phi(\omega) \leq \sum_{i=1}^{m}|\Delta\arg\lambda_i(\omega)| \leq m\pi$, completing the proof.

The following lemma is a rephrasing of one of the versions of the Yakubovich – Kalman lemma [14].

LEMMA 5. Let A_0, R, B, Q be matrices of orders n × n, n × n, n × m, n × m, respectively, where $R = R^* \geq 0$ and the rank of B equals m. We set†

$$\Pi(\lambda) = 2\mathrm{Re}\, Q^*(\lambda I_n - A_0)^{-1}B - B^*[\lambda I_n - A_0]^{*-1}R[\lambda I_n - A_0]^{-1}B. \tag{1.6}$$

For the existence of an n × n matrix $H = H^* > 0$, satisfying the relation

$$HA_0 + A_0^*H + R < 0, \quad HB = Q \tag{1.7}$$

and real in the real case, it is necessary and sufficient that the conditions:

a) $\det(\lambda I_n - A_0)$ is a Hurwitzian polynomial;

b) $\Pi(i\omega) > 0$, $\forall\omega \in R^1$;

c) $\lim_{\omega\to\infty}\omega^2\Pi(i\omega) > 0$

be fulfilled.

Proof. The sufficiency of the conditions a), b), c) is demonstrated in Theorem 4 [14]. The necessity of condition a) follows from the fact that when (1.7) is fulfilled $HA_0 + A_0^*H < 0$, i.e., the system $dx/dt = A_0x$ is

†By Re Π, where Π is an arbitrary square complex matrix, we denote the Hermitian matrix $(\Pi + \Pi^*)/2$.

asymptotically stable. Finally, the validity of b) and c) when (1.7) and a) are satisfied is also proved in Theorem 4 [14].

Proof of Theorem 1. We first prove the sufficiency of the conditions of the theorem. We note that, from the condition $G^*L^*B = D > 0$, the rank of the matrix B is equal to m (since the rank of a product of matrices is not less than the rank of each of the components). It is therefore possible to use Lemma 5, by virtue of which it is sufficient to find an $l \times m$ matrix C_0 such that the conditions a), b), c) of the lemma will be fulfilled for $A_0 = A + BC_0^*L^*$, $Q = LG$. We show that as such a matrix it is possible to take $C_0 = -\varkappa G$, where the number $\varkappa > 0$ is sufficiently large. (It is obvious that in the real case C_0 will be real.) For brevity, we introduce the notation $\delta_\varkappa(\lambda) = \delta(\lambda, -\varkappa G)$, $\chi_\varkappa(\lambda) = \chi(\lambda, -\varkappa G)$, $A_\varkappa = A - \varkappa BC^*L^*$. To verify condition a) we use the identity (1.4), from which it follows that

$$\delta_\varkappa(\lambda) = \frac{\varkappa^m}{\delta(\lambda)^{m-1}} \det\left[\frac{\delta(\lambda)}{\varkappa} I_m + G^*\alpha(\lambda)\right], \quad (1.8)$$

where $\alpha(\lambda) = \chi(\lambda)\delta(\lambda)$. By expanding the determinant on the right side of (1.8), we obtain

$$\delta_\varkappa(\lambda) = \frac{\varkappa^m}{\delta(\lambda)^{m-1}} \left[\frac{\delta(\lambda)^m}{\varkappa^m} + \varphi_1(\lambda)\frac{\delta(\lambda)^{m-1}}{\varkappa^{m-1}} + \ldots + \varphi_{m-1}(\lambda)\frac{\delta(\lambda)}{\varkappa} + \varphi_m(\lambda)\right],$$

where $\varphi_1(\lambda), \ldots, \varphi_{m-1}(\lambda)$, $\varphi_m(\lambda) = \det G^*\alpha(\lambda)$ are the coefficients of the characteristic polynomial of the matrix $G^*\alpha(\lambda)$. By Lemma 2, $\varphi_k(\lambda) = \delta(\lambda)^{k-1}\psi_k(\lambda)$, where $\psi_k(\lambda)$ is a polynomial of degree $n - k$ whose leading coefficient ψ_k is equal to the sum of the principal minors of order k of the matrix G^*L^*B ($k = 1, \ldots, m$). Thus

$$\frac{1}{\varkappa^m}\delta_\varkappa(\lambda) = \frac{\delta(\lambda)}{\varkappa^m} + \frac{\psi_1(\lambda)}{\varkappa^{m-1}} + \ldots + \frac{\psi_{m-1}(\lambda)}{\varkappa} + \psi_m(\lambda)$$

$$= \lambda^{n-m}\left[\left(\frac{\lambda}{\varkappa}\right)^m + \left(\frac{\lambda}{\varkappa}\right)^{m-1}\left(\psi_1 + O\left(\frac{1}{\varkappa}\right)\right) + \ldots + \left(\frac{\lambda}{\varkappa}\right)\left(\psi_{m-1} + O\left(\frac{1}{\varkappa}\right)\right)\right] + \psi_m(\lambda) + \psi(\lambda, \varkappa), \quad (1.9)$$

where $\psi(\lambda, \varkappa)$ is a polynomial of degree not greater than $n - m$ with coefficients of order $O(1/\varkappa)$ as $\varkappa \to \infty$. Applying Lemma 3 with $\varepsilon = 1/\varkappa$, we find that the polynomial $\delta_\varkappa(\lambda)$ is Hurwitzian for sufficiently large $\varkappa$ if the polynomials $Q(\lambda) = \lambda^m + \sum_{k=0}^{m-1} \psi_k\lambda^k$, $R(\lambda) = \psi_m(\lambda)$ are Hurwitzian. But $Q(\lambda)$ and $R(\lambda)$ are Hurwitzian since $Q(\lambda) = \det(\lambda I_m + G^*L^*B) = \det(\lambda I_m + D)$, and $R(\lambda) = \varphi(\lambda)$. Consequently, condition a) is fulfilled for $\varkappa > \varkappa_1$ for any $\varkappa_1 > 0$.

Proceeding to the verification of condition b), we rewrite this in the form

$$2\,\mathrm{Re}\,G^*\chi_\varkappa(i\omega) > B^*(-i\omega I_n - A_\varkappa^*)^{-1}(i\omega I_n - A_\varkappa)^{-1}B. \quad (1.10)$$

Since the polynomial $\varphi(\lambda)$ is invariant, for every $\varkappa$ the equality $\det G^*\chi_\varkappa(\lambda) = \varphi(\lambda)/\delta_\varkappa(\lambda)$ is valid. Under the conditions of the theorem, the polynomial $\varphi(\lambda)$ is Hurwitzian, from which it follows that $\det G^*\chi_\varkappa(i\omega) \neq 0$, $\forall\omega \in R^1$. Thus, for every $\omega \in R^1$ the identity

$$\mathrm{Re}\,G^*\chi_\varkappa(i\omega) = [G^*\chi_\varkappa(i\omega)]^*\,\mathrm{Re}\,[G^*\chi_\varkappa(i\omega)]^{-1}G^*\chi_\varkappa(i\omega),$$

is valid, so that (1.10) is equivalent to the inequality

$$2\,\mathrm{Re}\,[G^*\chi_\varkappa(i\omega)]^{-1} > [\chi_\varkappa^*(i\omega)G]^{-1}B^*(i\omega I_n - A_\varkappa^*)^{-1}R(i\omega I_n - A_\varkappa)^{-1}B[G^*\chi_\varkappa(i\omega)]^{-1}. \quad (1.11)$$

But $[G^*\chi_\varkappa(\lambda)]^{-1} = \varkappa I_m + [G^*\chi(i\omega)]^{-1}$, and therefore it is sufficient to prove that $\mathrm{Re}\,[G^*\chi(i\omega)]^{-1}$ and the right side of (1.11) are bounded for $\omega \in R^1$. Since $\det G^*\chi(i\omega) = \varphi(i\omega)/\delta(i\omega) \neq 0$ for $\omega \in R^1$, the matrix $[G^*\chi(i\omega)]^{-1}$ is bounded under variations in ω on any finite interval. We show that for every $\mathrm{Re}\,[G^*\chi(i\omega)]^{-1}$ there exists a finite limit $\omega \to \pm\infty$. By the condition of the Theorem $D = D^* > 0$, where $D = G^*L^*B$, so that

$$\mathrm{Re}\,[G^*\chi(i\omega)]^{-1} = \mathrm{Re}\,i\omega\,[i\omega G^*\chi(i\omega)]^{-1} = \mathrm{Re}\,i\omega\left[D^{-1} + O\left(\frac{1}{\omega}\right)\right]$$

$$= i\omega\left[D^{-1} - (D^{-1})^*\right] + O(1) = O(1) \quad \text{for} \quad \omega \to \pm\infty.$$

We must still show boundedness for $\omega \to \pm\infty$ of the right side of Eq. (1.11), which we denote by $\Psi_\chi(\omega)$. For $\chi > \chi_1$ the matrix function $\Psi_\chi(\cdot)$ is continuous and, consequently, is bounded on every finite interval. We show that for each χ there exists a finite limit $\lim_{\omega\to\pm\infty} \Psi_\varkappa(\omega)$. Let $B_\lambda = (\lambda I_n - A_\varkappa)^{-1}B$. Then $B_\lambda = B/\lambda + O(1/|\lambda|^2)$ for $\lambda \to \infty$, and it follows that $\lim_{\omega\to\infty} B_{i\omega}[G^*L^*B_{i\omega}]^{-1} = B[G^*L^*B]^{-1}$. Therefore there exists a finite limit

$$\lim_{\omega\to\pm\infty}\Psi_\varkappa(\omega)=\lim_{\omega\to\pm\infty}[B^*_{i\omega}LG]^{-1}B^*_{i\omega}RB_{i\omega}[G^*L^*B_{i\omega}]^{-1}=[B^*LG]^{-1}B^*RB[G^*L^*B]^{-1}$$

exists. Thus, condition b) of Lemma 5 is satisfied under $\varkappa > \varkappa_2$ for any $\varkappa_2 > \varkappa_1 > 0$.

Finally, the validity of condition c) under sufficiently large $\varkappa$ follows from the easily verifiable relations $\lim_{\omega\to\infty}\omega^2\Pi(i\omega) = -2\mathrm{Re}\,G^*L^*A_\varkappa B - B^*RB = -2\mathrm{Re}\,G^*L^*AB - B^*RB + 2\varkappa[G^*L^*B]^2$. The sufficiency of the conditions of the theorem is proved. We now demonstrate their necessity.

Let the relations (1.1)-(1.3) be satisfied for some $H_0 = H_0^* > 0$ and C_0. It follows from Lemma 5 that the polynomial $\delta_0(\lambda) = \det[\lambda I_n - A(C_0)]$ is Hurwitzian and $\mathrm{Re}\,G^*\chi_0(i\omega) > 0$ for any $\omega \in R^1$, where $\chi_0(\lambda) = L^*(\lambda I_n - A(C_0)]^{-1}B$. Due to the invariance of the polynomial $\varphi(\lambda)$

$$\varphi(\lambda)=\delta_0(\lambda)\det G^*\chi_0(\lambda). \tag{1.12}$$

Computing from both sides of (1.12) the increase in the argument under $\lambda = i\omega$, where ω varied from $-\infty$ to $+\infty$, we have $\Delta\arg\varphi(i\omega) = n\pi + \Delta\arg\det G^*\chi_0(i\omega)$. By Lemma 4, $\Delta\arg\varphi(i\omega) \geq (n-m)\pi$. But $\varphi(\lambda)$ is a polynomial of degree $n-m$ with leading coefficient $\lambda^{n-m}\det D$, where $D = G^*L^*B$ (cf. Lemma 1). Therefore $|\Delta\arg\varphi(i\omega)| \leq (n-m)\pi$, from which we obtain $\Delta\arg\varphi(i\omega) = (n-m)\pi$ and consequently $\varphi(\lambda)$ is Hurwitzian and $\det D > 0$. It remains to show that $D = D^* \geq 0$. By Lemma 5, $\lim_{\omega\to\infty}\omega^2\mathrm{Re}\,G^*\chi_0(i\omega) = \lim_{\omega\to\infty}\mathrm{Re}\,[i\omega G^*L^*B - G^*L^*A(C_0)B + O(1/|\omega|)] = \lim_{\omega\to\infty} i\omega[D - D^*] - \mathrm{Re}\,G^*L^*A(C_0)B > 0$, so that $D = D^*$. We now note that the relations (1.1), (1.2) obviously cease to be fulfilled if in these we change $A(C_0)$ to $A(C_0) - \varkappa I_n$, $\varkappa > 0$. By again applying Lemma 5, we obtain that $-G^*L^*A(C_0)B + \varkappa D > 0$ for any $\varkappa > 0$. Thus $D \geq 0$, which completes the proof of the theorem.

A special case of the theorem we have proved (for $m = 1$) is considered in [15].

COROLLARY [15]. Let A, B, L, G, R be complex matrices and vectors of orders $n\times n$, n, $n\times l$, l, $n\times n$, respectively, where $R = R^* \geq 0$, $B \neq 0$. We represent the vector function $\chi(\lambda) = L^*(\lambda I_n - A)^{-1}B$ in the form $\chi(\lambda) = \alpha(\lambda)/\delta(\lambda)$, where $\delta(\lambda) = \det(\lambda I_n - A)$. [In this connection, $\alpha(\lambda)$ is a vector of polynomials of degree not greater than $n-1$.] For the existence of an $n\times n$ matrix $H = H^* > 0$ and an l-dimensional vector C, satisfying the relations (1.1)-(1.3) and real in the real case, it is necessary and sufficient that the polynomial $\varphi(\lambda) = G^*\alpha(\lambda)$ be Hurwitzian and of degree $n-1$ with positive leading coefficient.

§ 2. PROOF OF THEOREMS 1, 2

As was already stated, the Lyapunov function constructed in Theorem 2 is used in the proof of Theorem 1. Therefore, we first prove Theorem 2.

Proof. For every $\xi \in \Xi$ let the matrices C^0, $H_j = H_j^*$, $j = 0, 1, \ldots, m$, exist such that the function (0.5) possesses properties (A), (B). It follows from (A) that $H_j > 0$, $j = 0, 1, \ldots, m$. We write out the expression for $\dot V_\xi(x, C)$ under $f_\xi(t) \equiv 0$

$$\dot V_\xi(x,C)=2x^*H_0\left(Ax+\sum_{j=1}^m b_jc_j^*y\right)+2\sum_{j=1}^m(c_j-c_j^0)H_jF_j(y), \tag{2.1}$$

where b_j, c_j, c_j^0, $F_j(y)$ $(j = 1, \ldots, m)$ are the columns of the matrices B, C, C^0, $F(y)$, respectively. Equation (2.1) can be rewritten in the form

$$\dot V_\xi(x,C)=x^*(H_0A_0+A_0^*H_0)x+2\sum_{j=1}^m(c_j-c_j^0)^*[H_jF_j(y)+(x^*H_0b_j)y], \tag{2.2}$$

where $A_0 = A + B[C^0]^*L^*$. Since the right side of (2.2) is linearly dependent on C, condition (B) is equivalent to the relations

$$H_0A_0+A_0^*H_0<0, \tag{2.3}$$
$$H_jF_j(y)+(x^*H_0b_j)y\equiv 0,\quad j=1,\ldots,m. \tag{2.4}$$

We will change the vector x so that $y = L^*x \equiv \mathrm{const} \neq 0$ [the existence of such an x follows from the condition $\chi(\lambda) \not\equiv 0$]. From (2.4) it is obvious that $x^*H_0b_j = \mathrm{const}$, and it follows that for some l-dimensional vectors g_j, $j = 1, \ldots, m$ the equalities $H_0b_j = Lg_j$ are valid. Thus, the adaptation algorithm must be given by the relation (0.4), in which $P_j = H_j^{-1}$, $j = 1, \ldots, m$. In addition, for any matrices C^0, H_0 inequality (2.3) and the relations $H_0b_j = Lg_j$, $j = 1, \ldots, m$, which after introducing the $l\times m$ matrix $G = \|g_1, \ldots, g_m\|$ have the form

$$H_0B=LG \tag{2.5}$$

must be valid. The necessity of the conditions of the theorem now follow from Theorem 3 (it is necessary to set $R = 0$, $\tau = I_m$).

Conversely, let the adaptation algorithm have the form (0.4). By Theorem 3 (for the real case and $R = 0$) for any $\xi \in \Xi$ matrices C^0, $H_0 = H_0^* > 0$ exist which satisfy relation (2.3) and relation (2.5) with the change of G to $G\tau_\xi$. Setting $H_j = \tau_j P_j^{-1}$, $j = 1, \ldots, m$, we determine a function $V_\xi(x, C)$ of the form (0.5). The constructed function $V_\xi(x, C)$ satisfies condition (A) (since $H_j > 0$, $j = 0, 1, \ldots, m$) and condition (B) [by (2.4)]. Theorem 2 is proved.

Remark 1. From the proof of the theorem it is clear that the derivative of the constructed function (0.5), by virtue of the system (0.1), (0.2), (0.4), has the form, for $f_\xi(t) \equiv 0$

$$\dot{V}_\xi(x, C) = -x^* Q_\xi x, \tag{2.6}$$

where $Q_\xi = Q_\xi^* > 0$ is a positive definite $n \times n$ matrix.

Remark 2. For $m = 1$ the polynomial $\varphi(\lambda)$, involved in the statement of the theorem, has the form $\varphi(\lambda) = g^* \alpha(\lambda)$, where $\alpha(\lambda) = \chi(\lambda)\delta(\lambda)$ is the ratio of the $l \times 1$ target matrix and g be an l-dimensional vector. The conditions of the theorem in this case are equivalent to the condition that $\varphi(\lambda)$ be a Hurwitzian polynomial with positive leading coefficient.

Proof of Theorem 1. By Theorem 2, for any $\xi \in \Xi$ matrices $C^0 = C^0(\xi)$, $H_j = H_j(\xi)$, $j = 0, 1, \ldots, m$ exist such that the function $V_\xi(x, C) = x^* H_0 x + \sum_{j=1}^{m} (c_j - c_j^0)^* H_j (c_j - c_j^0)$ possesses the properties (A), (B). Taking Remark 1 to Theorem 2 into account, we see that the derivative of the function $V_\xi(x, C)$ by virtue of the system (0.1), (0.2), (0.4) can be written in the form

$$\dot{V}_\xi(x(t), C(t)) \leqslant -x^*(t) Q_\xi x(t) + x^*(t) H_0 f_\xi(t),$$

where $Q_\xi = Q_\xi^* > 0$ is some $n \times n$ matrix. Consequently, numbers $\varkappa_1 > 0$, $\varkappa_2 > 0$ (depending, in general, on ξ) exist such that $\dot{V}_\xi(x(t), C(t)) \leq -\varkappa_1 \|x(t)\|^2 + \varkappa_2 \|x(t)\| \cdot \|f(t)\|$. Integrating the inequality obtained over the limits from zero to $t > 0$ and using the notation $\rho_t^2 = \int_0^t \|x(s)\|^2 ds$, $\eta^2 = \int_0^\infty \|f(t)\|^2 dt$, we have

$$\varkappa_1 \rho_t^2 - \varkappa_2 \rho_t \eta - V_\xi(x(0), C(0)) \leqslant -V_\xi(x(t), C(t)) \leqslant 0, \tag{2.7}$$

from which with the help of an obvious bound we obtain that $\rho_t \leq \eta\varkappa_2/\varkappa_1 + \sqrt{V_\xi(x(0), C(0))/\varkappa_1}$. Thus, the magnitude of $\rho^2 = \int_0^\infty \|x(t)\|^2 dt$ is finite. It follows from (2.7) that $V_\xi(x(t), C(t)) \leq V_\xi(x(0), C(0)) + \varkappa_2 \eta \rho$, i.e., the solution of the system of differential equations (0.1), (0.2), (0.4) is bounded and continuous on the interval $[0, \infty)$. Further, the right sides of Eqs. (0.4) are quadratic forms of the vector x. Therefore, there exists a finite limit $\lim_{t\to\infty} C(t)$, i.e., condition (II) of Definition 1 is fulfilled. To verify condition (I), we note that for any $t \geq 0$ the equality

$$\|x(t)\|^2 = 2 \int_0^t x^*(s)[Ax(s) + BC^*(s) L^* x(s) + f(s)] ds + \|x(0)\|^2 \tag{2.8}$$

is valid. Since the integral in the right side of (2.8) is absolutely convergent, the limit $\lim_{t\to\infty} \|x(t)\|^2 = \mu$ exists. But $\int_0^\infty \|x(t)\|^2 dt < \infty$, and therefore $\mu = 0$ and the adaptive design of the system is proved.

Remark 1. It is possible to show [by using (2.7)] that for any $t \geq 0$ the inequality

$$V_\xi(x(t), C(t)) \leqslant V_\xi(x(0), C(0)) + \varkappa_2^2 \times \eta^2/(4\varkappa_1), \tag{2.9}$$

is valid, giving a bound on the maximum deviation of the phase coordinate of the system (0.1), (0.2), (0.4) during the whole adaptive process.

Remark 2. Theorems 2 and 3 for the case $m = 1$ are proved in [15].

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