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SPEED GRADIENT CONTROL AND PASSIVITY OF NONLINEAR OSCILLATORS

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Abstract. The general approach for adaptive and nonadaptive control design is proposed based on the speed-gradient method. The paper deals with the problems of control of nonlinear oscillators including the problem of swinging up the pendulum and adaptive sinchronization of two Duffing's systems. The link between speed-gradient and passivity approaches is discussed.

Key Words. Pendulum, conservative systems, chaotic systems, speed-gradient method, passivity.

1. INTRODUCTION

Problems of nonlinear control have drawn recently significant attention. Among those of particular interest are the problems of control of nonlinear oscillating and vibrating systems arising in various applied areas of mechanics, electronics, etc. Together with conventional control goals (regulation and tracking) some specific goals are of interest for oscillating systems, for example, excitation (swinging) and synchronization (Mori *et al.* 1976; Furuta &Yamakita, 1991; Wiklund *et al.*, 1993; Akulenko, 1993; Chernous'ko *et al*, 1980). However general approaches for achieving these specific goals have not been presented in the literature so far. Recently it was suggested (Fradkov, 1994) to apply the speed gradient method for these purposes.

The present paper is aimed to study in more details problems of excitation (stabilization of energy-based functionals for Hamiltonian systems) and synchronization (for Duffing's system) by means of speed-gradient (SG) approach. Besides the passivity of the closed-loop system is established based on the link between SG and passivity approaches studied in (Seron *et al.*, 1994).

The paper is organized as follows: the main ideas of the SG and passivity approaches are briefly exposed in the section 2 and 3 respectively. Section 4 is devoted to SG control of Hamiltonian systems. Examples of the SG control of pendulum and Duffing's system are worked out in section 5 and 6.

2. SPEED GRADIENT ALGORITHMS

Consider the controlled plant equation in the state space form

$$\dot{x} = F(x,u,t), \ t \geq 0 \tag{2.1}$$

where $x \in \mathrm{R}^n$ is a plant state vector, $u \in \mathrm{R}^m$ is an input vector, $F(\cdot):\mathrm{R}^{n+m+1} \to \mathrm{R}^n$ is continuously differentiable in x,u vector-function. Input variables may be of arbitrary nature: real control action on the plant, adjustable parameters, etc. Consider the problem of finding the control law $u(t)=U[x(s),u(s):0\leq s\leq t]$, ensuring the control aim:

$$Q_t \to 0 \ \text{ when } t \to \infty, \tag{2.2}$$

where Q_t is some objective functional $Q_t = Q[x(s),u(s):0 \leq s \leq t]$.

To design a speed-gradient algorithm for the typical case $Q_t = Q(x(t),t)$ where $Q(x,t)>0$ is scalar smooth objective function determine a function $\omega(x,u,t)$ as the speed of change of Q_t along the trajectories of the system (2.1): $\omega(x,u,t) = F^T(x,u,t)\nabla_x Q$. SG-algorithm changes the control action along the gradient of $\omega(x,u,t)$ in u. Its combined form looks as follows (Fradkov, 1979; Fomin *et al.*, 1981; Fradkov, 1990; Fradkov, 1991):

$$\frac{d}{dt}(u - u_* + \psi(x,u,t)) = -\Gamma\nabla_u \omega(x,u,t) \tag{2.3}$$

where $\psi(x,u,t)$ satisfies pseudogradient condition $\psi^T \nabla_u \omega(x,u,t) \geq 0$, $\Gamma = \Gamma^T > 0$ is $m*m$ gain matrix and u_* is a smooth bounded function. The equation (2.3) can be rewritten in integral form:

$u = -\psi(x,u,t) - \Gamma \int_0^t \nabla_u \omega(x,u,s)ds$. The main special cases of (2.3) are SG-algorithm in differential form :

$$\dot{u} = \dot{u}_* - \Gamma \nabla_u \omega(x,u,t) \qquad (2.4)$$

and SG-algorithm in the finite form :

$$u = -\psi(x,u,t), \qquad (2.5)$$

having in turn linear and relay versions:

$$u = -\Gamma \nabla_u \omega(x,u,t), \qquad (2.5a)$$

$$u = -\Gamma sign\{\nabla_u \omega(x,u,t)\}, \qquad (2.5b)$$

where components of vector sign{z} are signs of the corresponding components of vector z.

The following stability theorem can be proved for SG-system (2.1), (2.3) similarly to those of (Fradkov, 1990; Fradkov, 1991):

Theorem 1 (combined form). Assume that the right hand sides of the system (2.1),(2.3) are smooth functions in x,u which are bounded together with derivatives in any region where the function $Q(x,t)$ is bounded. Assume that $\omega(x,u,t)$ is convex in u and the following *stabilizability condition* is valid: there exists

$u_* \in R^m$ such that $\omega(x,u_*,t) \leq 0$ for all $x \in R^n$ (2.6)

Then $Q(x(t),t)$ is bounded along each trajectory of (2.1), (2.3).

Besides, if the *asymptotic stabilizability* condition is valid:

$$\omega(x,u_*,t) \leq -\rho(Q(x,t)). \qquad (2.7)$$

where $\rho(Q) > 0$ for $Q>0$, then the goal (2.2) is achieved for all trajectories of (2.1), (2.4).

The previous condition can be weakened:

$$\omega(x,u_*,t) \leq -\rho(x), \qquad (2.7a)$$

where $\rho(x) \geq 0$. In this case $Q(x(t),t)$ is bounded along each trajectory of (2.1), (2.3) and $\rho(x) \to 0$ as time increases to infinity.

The proof of theorem is based on Lyapunov function

$$V(x,u,t) = Q(x,t) + (u - u_*)^T \Gamma^{-1} (u - u_*) \qquad (2.8)$$

The combined form (2.3) is more convenient when $\dot{u}_* = 0$. In the case when it is difficult to find constant "ideal" control u_*, satisfying (2.6) or (2.7), SG-algorithms in finite form may be applied. Theorems about stability of the finite SG algorithm can be found in (Fradkov, 1990).

Moreover it can be shown that for the finite SG algorithms the goal (2.2) is still achieved under *weakened stabilizability conditions*: for some bounded $u_*(x,t)$ (2.6) is valid and there exist the sequence of time instances $t_k \to \infty$, k=1,2,... and the sequences of nonnegative numbers $\{\alpha_i\}_1^\infty, \{\rho_i\}_1^\infty$, such that

$$Q_{k+1} - Q_k \leq -\rho_k Q_k + \alpha_k, \ \sum_{k=1}^{\infty} \rho_k = \infty, \ \alpha_k / \rho_k \to 0 \quad (2.9)$$

where $Q_k = Q(x(t_k),t_k)$. Note also that Lyapunov function for the case of finite algorithms is just the objective function $Q(x,t)$.

Note that for affine systems

$$\dot{x} = f(x) + g(x)u \qquad (2.10)$$

speed gradient is just Lie derivative of Q:

$$\nabla_u \dot{Q} = (\nabla_x Q)^T g(x) = (L_g Q)^T.$$

3. SPEED GRADIENT AND PASSIVITY

Applicability of the SG algorithms is connected with the passivity of closed-loop system. Recall that the system (2.1) is called passive with respect to output $y = h(x)$ if there exists smooth nonnegative function $V(x)$ (storage function), such that $V(0) = 0$ and the following dissipation inequality (DI) is valid:

$$\int_0^t u^T(s)y(s)ds \geq V_t - V_0, \qquad (3.1)$$

where V_t and V_0 denote $V(x(t))$ and $V(x(0))$ respectively.

System is called output strict passive (OSP) if the strict DI holds for some $\rho > 0$:

$$\int_0^t u^T(s)y(s)ds \geq V_t - V_0 + \int_0^t \rho y^T(s)y(s)ds \quad (3.2)$$

DI in ifinitisemal form looks as follows:

$$\dot{V} \leq y^T(t)u(t) - \rho y^T(t)y(t) \qquad (3.3)$$

Suppose the free affine system (2.10) (for u=0) is Lyapunov stable and $Q(x)$ is its Lyapunov function, i.e.

$$(\nabla_x Q)^T f(x) \leq 0 \qquad (3.4)$$

Then for controlled system (2.10) we have

$$\dot{Q} = (\nabla_x Q)^T f + (\nabla_x Q)^T gu \leq (\nabla_x Q)^T gu,$$

i.e. system (2.10) is passive with respect to output $y = g^T \nabla_x Q$ which is just the speed gradient of the storage function $Q(x)$.

4. SPEED-GRADIENT ALGORITHMS FOR HAMILTONIAN SYSTEMS

Consider the controlled plant equations in the generalized Hamiltonian form:

$$\dot{p} = -\nabla_q H + B(p,q)u, \quad \dot{q} = \nabla_p H \qquad (4.1)$$

where $p,q \in R^n$ are generalized coordinates and momenta; $H=H(p,q)$ is Hamiltonian function (total energy of the system); $u=u(t)$ is input (generalized force), $B(p,q)$ is nonsingular $n*n$ matrix function: $\det(B(p,q))\neq 0$.

Formalize the control aim as approaching the given energy surface:

$$S = \{(p,q): H(p,q) = H_*\} \qquad (4.2)$$

The objective (4.2) can be reformulated as

$$H(p(t),q(t)) \to H_* = \text{const} \qquad (4.3)$$

or written in the form (2.2), where $x = (p,q)^T$, and

$$Q(x) = 1/2(H(p,q) - H_*)^2 \qquad (4.4)$$

To build SG-algorithm calculate $\dot{Q}$:

$$\dot{Q} = (H - H_*)\dot{H} = (H - H_*)(\nabla_p H)^T B(x)u \qquad (4.5)$$

The differential SG-algorithm (2.4) can be represented in the form:

$$\dot{u} = -\gamma(H - H_*)B^T(x)\nabla_p H \qquad (4.6)$$

where $\gamma>0$ is the gain coefficient.

The finite forms (2.5a), (2.5b) look as follows :

$$u = -\gamma(H - H_*)B^T(x)\nabla_p H \qquad (4.7)$$

$$u = -\gamma \mathrm{sign}[(H - H_*)B^T(x)\nabla_p H] \qquad (4.8)$$

To analyze the behavior of systems with algorithms (4.6)-(4.8) Theorems 1 can be used. It can be shown that the differential algorithm (4.6) satisfies conditions of the Theorem 1 with stabilizability condition in form (2.6) for the constant $u_* \equiv 0$. It follows from Theorem 1 that $H(p,q)$ is bounded along the trajectories of the system (4.1), (4.6) together with $Q(x)$. However the theorem does not ensure achievement of the initial goal (4.3). As a matter of fact the goal (4.3) is not achieved and simulation demonstrates complex behavior of the system (4.1),(4.6).

Algorithms (4.7),(4.8) give better convergence. Taking, e.g. $u_* = -(H - H_*)B^T(x)\dot{q}$ we obtain from (4.5) $\dot{Q} = -2Q[\dot{q}^T BB^T \dot{q}]$. It means that the condition (2.7) is not valid, because $\dot{q}$ may vanish in some instants $t_k > 0$, k=1,2,... . However it follows from LaSalle's invariance principle that each trajectory of the system (4.1),(4.7) converges either to the surface (4.2) (i.e. the goal (4.3) is achieved) or to the equilibrium point $\dot{q} = 0$ (stationary point of H). The arguments similar to (Shanidi, 1994) validate that the dimension of the set of initial conditions for which trajectories converge to the saddle points of H is less than n. On the other hand if the value of H in the point of maximum or minimum is different of H_*, then such a point also cannot be a limit point for (4.1),(4.7), because in this case the stabilizability condition in the difference form (2.9) is valid. Hence the goal (4.3) is achieved for almost all initial conditions.

5. EXAMPLE 1: CONTROL OF PENDULUM

Consider simple controlled pendulum equation

$$J \cdot \ddot{\varphi} + m \cdot g \cdot l \cdot \sin\varphi = u \qquad (5.1)$$

where φ is the angle of pendulum defined to be zero in its lower position, u is the controlling torque; J, m, l are the inertia, mass and length of the pendulum, correspondingly; g is the gravity acceleration. The pendulum energy is

$$H = 1/2\, J \cdot (\dot{\varphi})^2 + m \cdot g \cdot l \cdot (1 - \cos\varphi) \qquad (5.2)$$

Consider the problem of swinging the pendulum up to the magnitude with energy H_*. The achievement of the objective (4.3) for $H_* = 0$ means stabilization of pendulum in the lower position, i.e. supression of oscillations, while for $H_* > 2mgl$ it corresponds to the permanent rotation.

SG-algorithms (4.6), (4.7) in the differential and finite forms are

$$\dot{u} = -\gamma(H - H_*)\dot{\varphi} \qquad (5.3)$$

$$u = -\gamma(H - H_*)\dot{\varphi} \qquad (5.4)$$

It can be easily seen that the algorithm (5.3) satisfies conditions of Theorem 1, the stabilizability condition being valid in the form (2.6). Theorem 1 gives boundedness of the energy, i.e. $d\varphi/dt$ is bounded.

The simulation results were obtained for the parameter values m=1kg, l=1m, J=1kgm, $d\varphi/dt(0)$=0, $u(0)$=0. System with algorithm (5.3) for $\varphi(0)=\pi/4$ has oscillating behavior with the magnitude far from the desired one. In the contrary algorithm (5.4) swings the pendulum up to the desired magnitude even for small initial conditions $\varphi(0)$=0.5 deg. for control gain γ=0.1.

6 EXAMPLE 2: ADAPTIVE SYNCHRONIZATION OF TWO DUFFING'S SYSTEMS

Consider controlled forced Duffing equation

$$\ddot{x} + p\dot{x} + p_1 x + x^3 = q\cos\omega t + \theta, \quad p>0 \qquad (6.1)$$

that has become a traditional example of oscillating system with complex dynamics. In (6.1) θ is the control action.

As it has been shown in (Chen & Dong, 1993, b) when some parameters of the Duffing equation re varied, the solution trajectories of the equation display changes of dynamic behavior.

We will consider the problem of controlling a chaotic trajectories of the Duffing equation to one of periodic or chaotic solutions of the reference Duffing system that has desired dynamics.

By introducing $\dot{x}_1 = x_2$, equation (6.1) can be rewritten as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -p_1 x_1 - x_1^3 - p x_2 + q\cos\omega t + \theta \end{cases} \tag{6.2}$$

The problem is to choose the control action θ ensuring that the trajectories of the system (6.2) tend to the solutions of the reference model:

$$\begin{cases} \dot{x}_{1m} = x_{2m} \\ \dot{x}_{2m} = -p_1 x_{1m} - x_{1m}^3 - p x_{2m} + q_m\cos\omega t \end{cases} \tag{6.3}$$

where parameter q_m determines desired dynamics of the reference model.

Chen and Dong (1993, b) proposed controller of the following form to solve the above problem:

$$\theta = -K(x_1 - x_{1m}) + 3x_1x_{1m}(x_1 - x_{1m}),\ K > -p_1. \tag{6.4}$$

They showed that this control law ensures acheivement of the goal $\|x_1 - x_{1m}\| \to 0$ and $\|x_2 - x_{2m}\| \to 0$ when parameters of the system (6.2) concide to those of the reference model: $q = q_m$. When $q \neq q_m$ the goal is not achieved for any fixed K. The SG method discussed above can be employed to design adaptive control law for this problem when some parameters of the controlled system are unknown.

Let q be an adjustable parameter in the system (6.2): $q = q_0 + u$, where u is the adaptation variable. The law can be designed by the SG method. The control aim can be reformulated as finding $\theta(t)$ such that

$$Q(e(t)) \to 0 \text{ when } t \to \infty, \tag{6.5}$$

where $e = (x_1 - x_{1m}, x_2 - x_{2m})^T$ is error vector, and

$$Q(e) = \left((K + p_1)e_1^2 + e_1^4/2 + e_2^2\right)/2 \tag{6.6}$$

is the objective function.

Calculations give

$$\nabla_u \dot{Q}(e) = [0 \quad \cos\omega t]e = e_2\cos\omega t\ . \tag{6.7}$$

Choosing $\Gamma = \gamma I_2$ and $\psi(x,u,t) = \lambda\nabla_u\dot{Q}(x)$, one can obtain the SG-algorithm in combined form:

$$u = -\lambda(x_2 - x_{2m})\cos\omega t - \gamma\int_0^t (x_2 - x_{2m})\cos\omega s\, ds, \tag{6.8}$$

$\lambda > 0, \gamma > 0$.

It can be seen that all conditions of the Theorem 1 are satisfied. Indeed for $u_* = q_m - q_0$ we conclude that $\dot{Q}(e) = -pe_2^2$, or $\dot{Q}(e) = -\rho(e)$, where $\rho(e) = pe_2^2 > 0$ as it required in the condition (2.7a). Therefore according to Theorem 1 convergence $\rho(e) \to 0$ is established. Substructing (6.3) from (6.2) gives that $e_2 \to 0$ implies $e_1 \to \text{const}$ that in turn can be satisfied if $q \to q_m$ and $e_1 \to 0$. Thus we have established that the adaptive controller (6.4), (6.8) drives the trajectories of the system (6.2) to the periodic or chaotic solutions of the reference system (6.5). It follows from the results of section 3 that the overall system is passive with respect to output $y = e_2\cos\omega t$.

It is worth mentioning that considered case is rather simplified: we assumed that only one parameter of the controlled system is unknown. This is not a restriction of the SG algorithm and this assumtion was made in order to help the reader to catch the main idea of the proposed adaptive controller. It can be shown (Pogromsky, 1995) that the SG adaptive control can be designed for the Duffing's system when all parameters (except for ω) are unknown (including the phase uncertainty).

Proposed adaptive controller can be utilized to promote or eliminate chaos in controlled nonlinear oscillator. Indeed, it does not matter which dinamics is observed in oscillator before control is applied because the control law ensures convergence to the trajectory of the reference system whose behavior can be specified by appropriate choice of the parameter q_m. Extensive computer simulation demonstrates that the proposed algorithm works duly and the choice of coefficients K, γ, λ determines the speed of convergence of the objective function to zero.

7. CONCLUSION

The power of the speed-gradient approach has been demonstrated previously for various problems of stabilization and tracking. In this paper the speed-gradient approach is extended to the oscillating system using energy-based objective functions. Theoretical results establish stability of the closed-loop systems and simulations show good transcient processes. It is shown also that SG system possesses passivity properties

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