Pergamon

PII: S0005-1098(97)00230-6

Automatica, Vol. 34, No. 6, pp. 697–703, 1998

0005-1098/98 $19.00 + 0.00

Brief Paper

Exponential Feedback Passivity and Stabilizability of Nonlinear Systems*

ALEXANDER L. FRADKOV† and DAVID J. HILL‡

Key Words—Exponential stability; exponential passivity; nonlinear systems.

Abstract—Motivated by N. Krasovskii's characterisation of exponential stability, the concept of exponential passivity is introduced. It is shown that to make a nonlinear system with factorisable high-frequency gain matrix exponentially passive via either state or output feedback, exponential minimum phaseness and invertibility conditions are necessary and sufficient. These conditions also guarantee exponential output feedback stabilisability. This result extends previous results concerning linear systems. © 1988 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

In a series of papers by Byrnes and Isidori (1984, 1985, 1991), significant results were obtained extending to nonlinear systems some cornerstone concepts of linear control theory: normal forms, stabilisability, zero dynamics, minimum phaseness. The connection of these concepts with another fundamental property, namely passivity (dissipativity), was studied in Byrnes *et al.* (1991). In particular, it was shown in Byrnes *et al.* (1991) that an affine system possesses the property of state feedback passivity (meaning the existence of smooth state feedback rendering the system passive) if and only if it is weakly minimum phase and invertible. The case of strict passivity was also investigated in Byrnes *et al.* (1991), for which a similar result is valid with weak minimum phaseness replaced by minimum phaseness. The above-mentioned results establish clear and useful links between some of the main concepts of nonlinear control theory as well as generalising some known results for linear systems (Saberi *et al.*, 1990).

However, these known results are related only to the case of state feedback passivity. In view of the "input–output" nature of the passivity concept itself, it seems useful to establish relations between output feedback counterparts of stabilisability and passivity. The present paper aims at establishing such relations. A concept of exponential passivity plays a major role; this property bounds energy storage and dissipation functions quadratically.

When investigating nonlinear systems, an important role is played by semiglobal versions of system properties, i.e. properties valid in every given compact region of the system state space. The reason is twofold. On the one hand, achieving globality usually requires bounding the growth of nonlinearities in excessively restrictive ways from a theoretical point of view. On the other hand, good performance of the system in any bounded region is often quite satisfactory from a practical point of view.

To establish the main result of the paper, a semiglobally defined (i.e. depending on initial conditions) feedback law is used. Therefore semiglobal versions of the corresponding passivity properties are to be defined first.

In Section 2 exponential passivity and other related concepts are introduced. The necessary "exponential" version of the nonlinear Kalman–Yakubovich lemma (see Moylan, 1974; Hill and Moylan, 1976; Byrnes *et al.*, 1991) is also given.

The equivalence results of output feedback exponential passivity and exponential minimum phaseness plus invertibility is proven in Section 3 for the class of affine systems having globally defined normal form and factorisable high-frequency gain.

A side result is that state and output versions of feedback exponential passivity coincide for the above mentioned class of systems. To establish passivity and stabilisability of the closed-loop system, high gain arguments are used.

In Section 4, the interrelations between the obtained result (which can be considered as a nonlinear version of feedback Kalman–Yakubovich lemma) and its linear prototypes (Fradkov, 1974, 1976) are discussed.

For the reader's convenience we reproduce in Appendix the necessary result from Fradkov (1974, 1976) for the case $m = 1$, with the proof being a slightly modified version of Fradkov (1974).

*Received 17 January 1994; revised 4 July 1997. This work was performed while the authors were with the University of Newcastle. This paper was recommended for publication in revised form by Associate Editor Y. Yamamoto under the direction of Editor Ruth F. Curtain. *Corresponding author* Professor D. J. Hill. Tel. 0061 2 9351 4647; Fax 0061 2 93515132; E-mail: davidh@ee.usyd.edu.au.

†Academy of Sciences of Russia, Institute for Problems of Mechanical Engineering, 61 Bolshoy Av. V.O., 199178 St. Petersburg, Russia

‡Department of Electrical Engineering, University of Sydney, N.S.W. 2006, Australia

2. DEFINITIONS

Consider the nonlinear time-invariant affine in the control system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x), \end{aligned} \tag{2.1}$$

where $x \in R^n$, $u \in R^m$, $y \in R^m$; $f(x)$, $g(x)$ are C^1-vector fields and $h(x)$ is a C^1 vector-function.

The set of admissible inputs is assumed to be all locally square integrable R^m-valued functions $\mathscr{L}_{2e}(R^m)$. We assume the outputs in $\mathscr{L}_{2e}(R^m)$.

Definition 1 (Byrnes *et al.*, 1991; Willems, 1972). System (2.1) is C^r strictly passive if there exists a C^r, $r \geq 0$, nonnegative function $V(0) = 0$, and a positive-definite function $S(x)$ such that for all $u \in \mathscr{L}_{2e}(R^m)$ and all $t \geq 0$

$$V(x(t)) - V(x(0)) = \int_0^t y(\tau)^{\mathrm{T}} u(\tau)\,\mathrm{d}\tau - \int_0^t S(x(\tau))\,\mathrm{d}\tau. \tag{2.2}$$

This equality expresses an abstract energy balance for system (2.1). For linear systems, the interesting storage functions, V, and dissipation functions, S, are quadratic (Willems, 1972). It is of interest to study the class of nonlinear systems which are close to this situation in the sense that some (V, S) are quadratically bounded.

Definition 2. A strictly passive system (2.1) is called *C^r-exponentially passive* if there exist positive numbers α_1, α_2, α_3 such that the following inequalities hold:

$$\alpha_1 |x|^2 \leq V(x) \leq \alpha_2 |x|^2, \tag{2.3}$$

$$\alpha_3 |x|^2 \leq S(x) \tag{2.4}$$

for any $x(t)$-solution to equation (2.1).

The motivation of Definition 2 comes from a classicial result due to Krasovskii (1959).

Lemma 1. For exponential stability of the system

$$\dot{z} = F(z), \quad z \in R^n \tag{2.5}$$

with C^1-smooth right-hand side, i.e. the inequality

$$\| z(t) \| \leq C(\| z(0) \|) \exp(-\varepsilon t) \tag{2.6}$$

holds for some $\varepsilon > 0$, some continuous function $C(\cdot)$ such that $C(0) = 0$, $C(r) > 0$ for $r > 0$ and any solution $z(t)$ to equation (2.5), it is necessary and sufficient that there exist a C^1-smooth function $V(z)$ and positive numbers $\alpha_1, \ldots, \alpha_4$, satisfying quadratic type inequalities

$$\alpha_1 \| z \|^2 \leq V(z) \leq \alpha_2 \| z \|^2, \tag{2.7}$$

$$\dot{V}(z) \leq -\alpha_3 \| z \|^2, \tag{2.8}$$

$$\| \nabla V(z) \| \leq \alpha_4 \| z \|, \tag{2.9}$$

where $\dot{V}(z) = \nabla V(z)^T F(z)$.

It is then easily seen that an exponentially passive system for $u(t) \equiv 0$ is exponentially stable. The following lemma can be derived which provides one more version of "nonlinear Kalman–Yakubovich Lemma" (see Moylan, 1974; Hill and Moylan, 1976).

Lemma 2. System (2.1) is C^r-exponentially passive, $r \geq 1$, if and only if there exists a C^r-function $V(x)$ and positive numbers α_1, α_2, α_3 such that equation (2.3) is valid as well as the following relations:

$$\nabla V(x)^{\mathrm{T}} f(x) \leq -\alpha_3 \| x \|^2, \tag{2.10}$$

$$\nabla V(x)^{\mathrm{T}} g(x) = h(x)^{\mathrm{T}}. \tag{2.11}$$

We will need the following definition of a local form of passivity similar to one given in Pota and Moylan (1990).

Definition 3. System (2.1) is called *C^r-strictly passive in the region* $\Omega \subset R^n$, if there exist nonnegative C^r-function $V: \Omega \to R$ and positive-definite C^r-function $S: \Omega \to R$, such that inequality (2.2) is valid for all $t \geq 0$ and all $u(\cdot) \in L_{2e}(R^m)$ which ensure that $x(\tau) \in \Omega$ for any $\tau: 0 \leq \tau \leq t$.

If, moreover, the inequalities (2.3) and (2.4) are valid then system (2.1) is called C^r-exponentially passive in the region Ω.

Definition 4. System (2.1) is called *C^r-output feedback exponentially passive*, if there exists C^r-smooth output feedback

$$u = \alpha(y) + \beta(y)v, \tag{2.12}$$

where $v \in R^m$ is a new input such that the closed-loop systems (2.1), (2.12) is exponentially passive.

Definition 5. System (2.1) is called *C^r-output feedback exponentially stabilisable* if there exists C^r-smooth output feedback

$$u = \alpha(y) \tag{2.13}$$

such that the closed-loop system (2.1), (2.13) is exponentially stable.

We are also interested in formulating (see also Byrnes *et al.*, 1991) a state-feedback version of Definition 3.

Definition 6. System (2.1) is called *C^r-state-feedback exponentially passive*, if there exists C^r-smooth state feedback

$$u = \alpha(x) + \beta(x)v \tag{2.14}$$

such that the closed-loop system (2.1), (2.14) is C^r-exponentially passive.

Definition 7. System (2.1) is called *C^r-semiglobally output feedback exponentially passive* if for any compact set $\Omega \in R^n$ there exists C^r-smooth feedback

$$u = \alpha_\Omega(y) + \beta_\Omega(y)v, \tag{2.15}$$

such that the closed-loop system

$$\begin{aligned} \dot{x} &= f_\Omega(x) + g_\Omega(x)v, \\ y &= h(x), \end{aligned} \tag{2.16}$$

where $f_\Omega(x) = f(x) + g(x)\alpha_\Omega(h(x))$, $g_\Omega(x) = g(x)\beta_\Omega(h(x))$, is C^r-exponentially passive in the region Ω, and $V_{\Omega_1}(x) = V_{\Omega_2}(x)$, for $x \in \Omega_1 \cap \Omega_2$ where $V_\Omega(x)$ is a storage function ensuring passivity in Ω.

To formulate minimum phaseness properties, we need the notion of zero dynamics. Our results relate only to the systems in canonic (normal) form. Hence we will formulate the necessary properties only for the systems in normal form. General definitions can be found in Byrnes and Isidori (1991) and Isidori (1989).

Suppose system (2.1) has *normal form* in some neighbourhood of the origin:

$$\begin{aligned} \dot{z} &= q(z, y), \\ \dot{y} &= a(z, y) + b(z, y)u, \end{aligned} \tag{2.17}$$

where $z \in R^{n-m}$. (Conditions under which system (2.1) can be converted into form (2.17) by smooth coordinate change, can be found in Byrnes and Isidori (1991) and Isidori (1989).)

The equation

$$\dot{z} = q(z, 0) \tag{2.18}$$

provides the *zero dynamics* of system (2.1). System (2.17) is called *locally invertible* (or having local relative degree $(1, 1, \ldots, 1)$ if its high-frequency gain $m \times m$-matrix $b(0, 0)$ is nonsingular.

Finally, system (2.17) is called *exponentially minimum phase*, if there exist a C^1-smooth function $V_0(z)$ and positive numbers $\alpha_1, \ldots, \alpha_4$, satisfying quadratic type inequalities:

$$\alpha_1 \|z\|^2 \le V_0(z) \le \alpha_2 \|z\|^2, \tag{2.19}$$

$$\nabla V_0(z)^T q(z, 0) \le -\alpha_3 \|z\|^2, \tag{2.20}$$

$$\|\nabla V_0(z)\| \le \alpha_4 \|z\|. \tag{2.21}$$

From Lemma 1 we see that an exponentially minimum phase system has exponentially stable zero dynamics. For linear systems, exponential minimum phaseness reduces to being minimum phase.

Normal forms and all the related concepts can also be defined globally (see Byrnes and Isidori, 1991; Isidori, 1989). Particularly, system (2.17) is called invertible, if matrix $b(z, y)$ is nonsingular for any z, y.

3. MAIN RESULTS

Theorem 1. Suppose system (2.1) has globally defined normal form (2.17) with factorised high-frequency gain

$$b(z, y) = b_0(z)b_1(y), \tag{3.1}$$

where $b_0(z), b_1(y)$ are smooth $m \times m$-matrices, $b_0(z) = b_0(z)^T > 0$, and $b_1(y)$ is invertible.

Then the following statements are equivalent:

1. System (2.17) is C^r-semiglobally output feedback exponentially passive ($r \ge 1$).
2. System (2.17) is C^r-state feedback exponentially passive ($r \ge 1$)
3. System (2.17) is exponentially minimum phase.

If any of the statements (1), (2) or (3) is valid, then for any compact set Ω of initial conditions the system (2.17) is output-feedback exponentially stabilisable by means of feedback

$$u = -[b_1(y)]^{-1}[b_0(0)^{-1}a(0, y) + \chi y] \tag{3.2}$$

where $\chi > \chi_\Omega$ is a scalar gain.

Proof of Theorem 1. Equivalence of (2) and (3) is proved similarly to Theorem 4.7 (Byrnes *et al.*, 1991), using Lemma 2 instead of the standard nonlinear version of Kalman–Yakubovich lemma.

(1) ⇒ (3): Note that given compact set Ω, there exists feedback (2.15), rendering system (2.17) exponentially passive with storage function $V_\Omega(z, y)$, which can be chosen independently of Ω, i.e., $V_\Omega = V$ over any compact Ω. In view of Lemma 2, for $y = 0$ the following inequality holds:

$$\nabla_z V(z, 0)^T q(z, 0) \le -\alpha_3 \|z\|^2,$$

assuring, together with equation (2.3), the exponential minimum phaseness of (2.17). Function $V(z, 0)$ is a Lyapunov function for zero dynamics.

(3) ⇒ (1). In view of condition (3), the equation

$$\dot{z} = q(z, 0) \tag{3.3}$$

describing zero dynamics of system (2.17) is exponentially stable, i.e., there exists C^1-smooth function $V_0(z)$ satisfying quadratic type inequalities (2.7)–(2.9). Choose storage function candidate of form

$$V(z, y) = V_0(z) + \tfrac{1}{2} y^T [b_0(z)]^{-1} y \tag{3.4}$$

and take output feedback as

$$u = -[b_1(y)]^{-1}[b_0(0)^{-1}a(0, y) + \chi y] + [b_1(y)]^{-1}v, \tag{3.5}$$

for $y \neq 0$ and $u = 0$ for $y = 0$, where v is a new input and the number χ is to be determined. Note that feedback (3.5) is well defined since $\det b_1(y) \neq 0$ due to invertibility of system (2.17). The properties (2.19)–(2.21) of $V_0(z)$ and smoothness and boundedness of $b_0(z), b_0(z)^{-1}$ on any compact subset of R^{n-m} imply inequalities (2.3) in the definition of exponential passivity. To verify relations (2.10)–(2.11) of Lemma 2, consider the equations of the closed-loop system (2.17), (3.5):

$$\begin{aligned} \dot{z} &= q(z, y), \\ \dot{y} &= a(z, y) - b_0(z)[b_0(0)^{-1}a(0, y) + \chi y] \\ &\quad + b_0(z)v. \end{aligned} \tag{3.6}$$

Comparing this with system (2.1), it is convenient to define $f_\chi(z, y)$ and $g_\chi(z, y)$ for the closed-loop system (3.6) as follows:

$$f_\chi(z, y) = \begin{bmatrix} q(z, y) \\ a(z, y) - b_0(z)[b_0(0)^{-1}a(0, y) + \chi u] \end{bmatrix},$$

$$g_\chi(z, y) = \begin{bmatrix} 0 \\ b_0(z) \end{bmatrix}$$

Condition (2.11) holds since

$$\nabla V(z, y)^{\mathrm{T}} g_\chi(z, y) = y^{\mathrm{T}}[b_0(z)]^{-1} b_0(z) = y^{\mathrm{T}}.$$

To verify equation (2.10), note that

$$\begin{aligned} &\nabla V(z, y)^{\mathrm{T}} f_\chi(z, y) \\ &= \nabla V_0(z)^{\mathrm{T}} q(z, 0) + \nabla V_0(z)^{\mathrm{T}}[q(z, y) - q(z, 0)] \\ &\quad + \frac{1}{2} y^{\mathrm{T}} b_0(z)^{-1} \frac{\partial b_0(z)}{\partial z} b_0(z)^{-1} y q(z, y) \\ &\quad + y^{\mathrm{T}}[b_0(z)]^{-1} a(z, y) \\ &\quad - y^{\mathrm{T}} b_0(0)^{-1} a(0, y) - \chi \|y\|^2. \end{aligned} \tag{3.7}$$

We now aim to fix some compact set $\Omega \subset R^{n-m} \times R^m$ and determine number $\chi_\Omega > 0$ such that the right-hand side of equation (3.7) is nonpositive as $\chi > \chi_\Omega$. Using smoothness of the right-hand side of equations (3.6), and invertibility of equation (2.17) we have for $(z, y) \in \Omega$

$$\|q(z, y) - q(z, 0)\| \leq C_1 \|y\|,$$

$$\|b_0(z)^{-1}a(z, y) - b_0(0)^{-1} a(0, y)\| \leq C_2 \|z\|,$$

$$\|b_0(z)^{-1}\| \leq C_3, \quad \|\partial b_0(z)/\partial z\| \leq C_4,$$

$$\|q(z, y)\| \leq C_5$$

for some positive constants $C_1, C_2, \ldots, C_5$. Taking into account exponential minimum phaseness, we have

$$\begin{aligned} \nabla V^{\mathrm{T}} f_\chi &\leq -\alpha_3 \|z\|^2 + C_1\alpha_4 \|y\| \cdot \|z\| + C_2 \|y\| \cdot \|z\| \\ &\quad + \tfrac{1}{2} C_3^2 C_4 C_5 \|y\|^2 - \chi \|y\|^2 \\ &= -\alpha_3 \|z\|^2 - (\chi - \tfrac{1}{2} C_3^2 C_4 C_5) \|y\|^2 \\ &\quad + (C_1\alpha_4 + C_2) \|y\| \cdot \|z\| \end{aligned}$$

Hence, choosing χ_Ω to satisfy the inequality

$$\chi_\Omega \geq \frac{\alpha_3}{2} + \tfrac{1}{2} C_3^2 C_4 C_5 + \frac{2}{\alpha_3}(C_1\alpha_4 + C_2)^2, \tag{3.8}$$

we just obtain

$$\nabla V(x, y)^{\mathrm{T}} f_\chi(z, y) \leq -(\alpha_3/2)(\|z\|^2 + \|y\|^2)$$

for $\chi > \chi_\Omega$. So, statement (1) is proven.

To prove the statement about exponential stabilisability, note that an exponentially passive system with zero input is exponentially stable. □

Remark 1. It is easy to show that, if the global Lipschitz condition for the right-hand side of (2.17) and uniform positivity of $b_0(z)$, i.e., $b_0(z) \geq \mu I > 0$ are imposed, then the achieved output feedback exponential passivity becomes global.

Remark 2. The passifying control law can be taken in simplified form

$$u = \chi[b_1(y)]^{-1} y + [b_1(y)]^{-1} v \tag{3.9}$$

(instead of equation (3.5)) which has the advantage that for $b_1(\cdot)$ constant, the law (3.9) is linear.

However, the lower bound χ_Ω for gain χ ensuring passivity of the closed-loop system (2.17), (3.9) in this case increases. Indeed, the right-hand side of the inequality (3.8) in this case should be enlarged by the quantity $C_3 C_6$ where C_6 is the Lipschitz constant of $a(z, y)$ with respect to y at the set Ω.

4. COMPARISON WITH THE CASE OF LINEAR SYSTEMS

An unexpected consequence of the result obtained above is that under conditions of Theorem 1 (i.e. for normal form systems with factorisable high-frequency gain matrix) there is a close relationship between output feedback and state feedback exponential passivity.

An explanation of this phenomenon lies in the study of the nature of the exponential passive property itself implying the "strong" stability of zero dynamics. Indeed, many arguments can be invoked to show that exponential passivity and exponential stability are the natural counterparts of strict passivity and asymptotic stability of linear systems. As to linear systems, the equivalence of output and state feedback strict passivity follows from comparison of results in Fradkov (1976), where the case of output feedback strict passivity was considered, and (Saberi *et al.*, 1990; Picci and Pinzoni, 1992), where necessary and sufficient conditions of state feedback passivity were obtained. The results of such a comparison are formulated in the following theorems.

Theorem 2. Consider the linear system

$$\dot{x} = Ax + Bu, \qquad y = Cx \tag{4.1}$$

where $x \in R^n$, $y \in R^m$, $u \in R^m$ and $\operatorname{rank} B = m$.

Then the following three statements are equivalent:

1. System (4.1) can be made strictly passive by means of linear output feedback

$$u = Ky + Lv \tag{4.2}$$

   for some K, L where $v \in R^m$ is a new input, $\det L \neq 0$.
2. System (4.1) can be made strictly passive by means of linear state feedback

$$u = Mx + Lv \tag{4.3}$$

   for some M, L where $\det L \neq 0$, $v \in R^m$.
3. System (4.1) is minimum phase and $\det CB \neq 0$.

To simplify the proof of Theorem 2, another result of a similar kind (for fixed L) is used, which is interesting in its own right.

Theorem 3. Let $\operatorname{rank} B = m$ for system (4.1). Then the following statements are equivalent.

1. System (4.1) can be made strictly passive by means of output feedback (4.2) with given L, $\det L \neq 0$.
2. System (4.1) can be made strictly passive by means of state feedback (4.3) with given L, $\det L \neq 0$.
3. System (4.1) is minimum phase, and $CBL = (CBL)^{\mathrm{T}} > 0$.

Remark 1. Bearing in mind that a storage function of a linear passive system can always be taken as quadratic, i.e.

$$V(x) = x^{\mathrm{T}}Px \tag{4.4}$$

we can say that Theorems 2 and 3 give necessary and sufficient conditions for existence of $P = P^{\mathrm{T}} > 0$, K, L, such that

$$PA_K + A_K^{\mathrm{T}}P < 0, \qquad PBL = C, \qquad A_K := A + BKC \tag{4.5}$$

(for output feedback case) or such that

$$PA_K + A_K^{\mathrm{T}}P < 0, \qquad PBL = C, \qquad A_K := A + BM \tag{4.6}$$

(for state feedback case).

If matrices K and L were fixed, the solvability of equations (4.5) and (4.6) would be established by the Kalman–Yakubovich lemma. Hence, it is natural to call Theorems 2 and 3 and their nonlinear relative at Theorem 1 by "feedback Kalman–Yakubovich lemma".

Remark 2. The solvability conditions of (4.5) for $L = I$ were found in Fradkov (1974) (for $m = 1$) and in Fradkov (1976) for the general case. Since references Fradkov (1994, 1976) are of limited access (although published in English) we give an independent proof of the Feedback Kalman–Yakubovich Lemma for $m = 1$ in the Appendix. The solvability conditions of the "nonstrict" version of equation (4.6) (with inequality $PA_K + A_K^TP \leq 0$) were obtained in Saberi *et al.* (1990) and extended to the case of systems with feedthrough in Santosuosso (1993), where the case of strict passivity also was considered.

Remark 3. The additional requirement of symmetry and positivity of the high-frequency gain matrix, appearing in statement 3 of Theorem 3, can be satisfied by proper choice of L, e.g., $L = (CB)^{-1}$. If, however, L is fixed, e.g., $L = I$, then this requirement is necessary. Similarly, in the nonlinear case, $L = L(y)$ while high-frequency-gain matrix depends on both z and y. Hence, the requirement of symmetry and positivity of the factor $b_0(z)$ cannot be removed.

Let us now prove Theorems 2 and 3, starting with Theorem 3.

Proof of Theorem 3. (1) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (3) Substitute equation (4.3) into equation (4.1) to obtain the closed-loop equations

$$\dot{x} = (A + BM)x + BLv, \qquad y = Cx$$

The change

$$BL = \tilde{B}, \qquad L^{-1} = M = \tilde{M} \tag{4.7}$$

reduces the problem to the case $L = I$ considered in Saberi *et al.* (1990). Obvious modification of the proof of Proposition 2 in Saberi *et al.* (1990) (replacing $PA_K + A_K^TP \leq 0$ by strict inequality $PA_K + A_K^TP < 0$) gives directly statement (3).

Finally, to derive (1) from (3) we need to prove that there exist K and $P = P^T > 0$, satisfying equation (4.5). For $L = I$ it is just a result of Fradkov (1976) and the general case can be reduced to the case $L = I$ by the change (4.7)

Proof of Theorem 2. Implication (1) $\Rightarrow$ (2) is trivial.

To prove (2) $\Rightarrow$ (3), fix some matrix L, $\det L \neq 0$. Then it follows from Theorem 3 that systems (4.1) is minimum-phase, $(CBL) = (CBL)^T > 0$ and hence $\det CB \neq 0$.

To prove (3) $\Rightarrow$ (1), it suffices to establish solvability of equation (4.5). But it again follows from Theorem 3 for choice $L = (CB)^{-1}$.

Theorems 2 and 3 show that, for rendering a system strictly passive, output feedback and state feedback give the same results. Eamples show that this is not the case for rendering a systems "nonstrictly" passive.

It is interesting to note that Example 4.1 from Byrnes and Isidori (1991) (see also Byrnes and Isidori, 1989) shows that straight replacement of passivity by strict passivity and weak minimum phaseness by minimum phaseness (asymptotic stability of zero dynamics) is not enough to provide equal stabilising capabilities of both output and state feedback for nonlinear systems.

5. CONCLUSIONS

It was established in this paper that to render a system in normal form with factorisable high-frequency gain exponentially passive by means of either output or state feedback, it is necessary and sufficient that it is exponentially minimum phase and invertible. The same property for linear systems is derived from the results of Fradkov (1974, 1976), Saberi *et al.* (1990) and Picci and Pinzoni (1992).

It is shown that the "exponential" properties and symmetry of the state dependent factor of the high- frequency gain cannot be simply removed from the formulation of the Theorem 1. However, the question of necessity of these properties remains open.

Finally, it is worth noticing that the main contribution of the paper can be interpreted as a bringing together of different "linear" versions of the feedback Kalman–Yakubovich lemma and establishment of some new "exponential" nonlinear version of it. Other nonlinear versions of the feedback Kalman- Yakubovich lemma can be found in Byrnes *et al.* (1991) and Byrnes and Isidori (1989).

Acknowledgements—The main part of this paper was written while the authors were with the University of Newcastle.

The authors would like to thank Z-P. Jiang for his helpful comments.

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APPENDIX: FEEDBACK KALMAN–YAKUBOVICH LEMMA

Theorem. For the existence of the matrix $P = P^T > 0$ and $\theta \in R^{\ell}$, satisfying

$$PA_\theta + A_\theta^T P < 0, \tag{A.1}$$

$$PB = C^T g, \tag{A.2}$$

$$A_\theta = A + B\theta C, \tag{A.3}$$

it is necessary and sufficient that the function $g^T W(\lambda)$, where $W(\lambda) = C(\lambda I - A)^{-1}B$ is hyper-minimum phase, i.e. the polynomial $N(\lambda) = \det(\lambda I_n - A) g^T W(\lambda)$ is Hurwitz, $\deg N(\lambda) = n - 1$, $N(0) > 0$.

To prove the theorem, we need the two following lemmas.

Lemma 1 (*Yakubovich*, 1962). For the existence of the matrix $P = P^T > 0$, s.t.

$$PA + A^T P < 0, \qquad PB = d$$

it is necessary and sufficient that

(a) $\det(\lambda I_n - A)$ is Hurwitz,
(b) $\mathrm{Re}\, W(j\omega) > 0$,
(c) $\lim_{\omega\to\infty} \omega^2 \mathrm{Re}\, W(j\omega) > 0$,

where $W(\lambda) = d^T(\lambda I - A)^{-1}B$.

The above lemma contains the first published formulation of the Kalman–Yakubovich lemma.

Lemma 2. Let $P(\lambda) = \lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0$, $Q(\lambda) = q_{n-1}\lambda^{n-1} + \cdots + q_0$ be polynomials and $Q(\lambda)$ is Hurwitz, $\deg Q(\lambda) = n - 1, q_{n-1} > 0$. Then $\varepsilon P(\lambda) + Q(\lambda)$ is Hurwitz for all sufficiently small $\varepsilon > 0$.

Proof. Let $\lambda_i(\varepsilon)$, $i = 1, \ldots, n$ be zeros of $P_\varepsilon(\lambda) = \varepsilon P(\lambda) + Q(\lambda)$, ($\lambda_i(0)$ are zeros $Q(\lambda)$). Then $\lambda_i(\varepsilon) \to \lambda_i(0)$, $i = 1, \ldots, n-1$ and $\lambda_n(\varepsilon) \sim -q_{n-1}/\varepsilon \to -\infty$ as $\varepsilon \to 0$. This follows since $P_\varepsilon(\lambda)$ can be written as $P_\varepsilon(\lambda) = \lambda^{n-1}(\varepsilon\lambda + q_{n-1}) + R_{n-2}(\varepsilon, \lambda)$.

Proof of the Theorem. Introduce the notation $\delta(\lambda) = \det(\lambda I - A)$, $\delta_\theta(\lambda) = \det(\lambda I - A_\theta)$. Then $\delta_\theta(\lambda) = \delta(\lambda) - \theta^T\alpha(\lambda)$, $W_\theta(\lambda) = \alpha(\lambda)/\delta_\theta(\lambda)$, $N(\lambda) = g^T\alpha(\lambda) = q_{n-1}\lambda^{n-1} + \cdots + q_0$:

1. *Sufficiency.* We will show that (A.1)–(A.3) are valid for $\theta = -\mu g$, where $\mu > 0$ is sufficiently large. (A.1) follows from the relation $\delta_\theta(\lambda) = \delta(\lambda) + \mu g^T\alpha(\lambda)$ and Lemma 2. To prove (A.2) note that $N(j\omega) \neq 0$ and $\delta(j\omega) + \mu N(j\omega) \neq 0$ for all $\omega \in R^1$, if μ is sufficiently large. Therefore, equation (A.2) is equivalent to

$$\mathrm{Re}\,[g^T W_\theta(j\omega)]^{-1} > 0.$$

$$\mathrm{Re}\,[g^T W_\theta(j\omega)]^{-1} = \mathrm{Re}\frac{\delta(\lambda) + \mu g^T\alpha(\lambda)}{g^T\alpha(\lambda)}\bigg|_{\lambda = j\omega} = \mu + \mathrm{Re}\frac{\delta(j\omega)}{N(j\omega)}.$$

Hence, it suffices to show that $|\mathrm{Re}\,\delta(j\omega)/N(j\omega)|$ is bounded for $\omega \to \pm\infty$.

For $\omega \to \pm\infty$, we have

$$\mathrm{Re}\frac{\delta(j\omega)}{N(j\omega)} = \frac{\mathrm{Re}\, j\omega + a_{n-1} + 0(1/j\omega)}{q_{n-1} + 0(1/j\omega)} = 0(1)$$

and equation (A.2) is proved. Finally, the validity of equation (A.3) follows from the relations:

$$\lim_{\omega\to+\infty} \omega^2 \mathrm{Re}\, g^T W_\theta(j\omega) = -g^T C A_\theta B = -g^T CAB + \mu q_{n-1}.$$

2. *Necessity.* Calculate the increment of the argument of $N(j\omega)$ for ω varying from $-\infty$ to $+\infty$. Since $N(j\omega) = g^T W_\theta(j\omega)\cdot\delta_\theta(j\omega)$ we have $\Delta\arg N(j\omega) = \Delta\arg g^T W_\theta(j\omega) + \Delta\arg\delta_\theta(j\omega)$.

By virtue of the Mikhailov stability criterion it follows from equation (A.1) that $\Delta\arg\delta_\theta(j\omega) = n\pi$ and it follows from equation (A.2) that $|\Delta\arg g^T W(j\omega)| \leq \pi$. Hence $\Delta\arg N(j\omega) \geq (n-1)\pi$. Note that $N(\lambda)$ is a polynomial of degree $n - 1$, therefore $\Delta\arg N(j\omega) \leq (n-1)\pi$, i.e., $\Delta\arg N(j\omega) = (n-1)\pi$.

This means that the polynomial $N(\lambda)$ is Hurwitz, its degree is $n - 1$ and all coefficients are positive. The theorem is proved.