CONTROL OF CHAOS: METHODS AND APPLICATIONS

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1 PRELIMINARIES

1.1 Notion of Chaos

There are many possible definitions of chaos.
In fact, there is no general agreement within
the scientific community as to what
constitutes a chaotic dynamical system.
R.Devaney. A first course in chaotic
dynamical systems. Addison-Wesley, 1992.

Most of definitions interpret the idea that chaotic motion is locally unstable and globally bounded. It means that the solutions with close initial conditions will diverge to some finite distance after some time (so called "sensitive dependence on initial conditions").

We will call a solution $\bar{x}(t)$, $0 \le t < \infty$ of the system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{1}$$

with initial condition $\bar{x}(0) = \bar{x}_0$ *chaotic*, if it is Lyapunov unstable and all the solutions starting from some neighbourhood of $\bar{x}_0$ are bounded on $[0,\infty)$. Similarly for discrete-time system $x_{k+1} = f(x_k)$.

To define chaotic system we need the notion of *attractor*.

Definition 1. A set B_0 is called the *attracting set* for the system (1) if there exists an open set B, $B_0 \subset B$ such that

$$\lim_{t\to\infty} \operatorname{dist}(x(t), B_0) = 0 \tag{2}$$

for any solution $x(t)$ with $x(0) \in B$.

Definition 2. A closed attracting set B_0 is called the *attractor* if it is minimal, i.e. there is no smaller attracting subset of B_0. The set of initial conditions B for which (2) holds is called the basin of attraction.

Definition 3. An attractor B_0 is called *strange* or *chaotic* if it is bounded and all the trajectories starting on it are chaotic.

The system (1) is called *chaotic* if it possesses at least one chaotic attractor.

Important for control property of chaotic trajectories is *recurrence*: they return to any vicinity of any past value.

Definition 4. The function $x : \mathbb{R}^1 \to \mathbb{R}^n$ is called *recurrent* if for any $\epsilon > 0$ there exists $T_\epsilon > 0$ such that for any $t \ge 0$ there exists $T(t,\epsilon)$, $0 < T(t,\epsilon) < T_\epsilon$ such that $|x(t + T(t,\epsilon)) - x(t)| < \epsilon$.

Recurrent trajectories possess two important properties formulated in Pugh lemma and Anosov lemma and providing formal support of the claim that chaotic attractor is the closure of all the periodic trajectories contained in it.

Lemma 1 (Pugh). Let $\bar{x}(t), t \geq 0$ be the recurrent trajectory of the system (1) with smooth $f(x)$. Then for any $\varepsilon > 0$ there exists smooth function $f_1(x)$ such that $||f_1(x)||_\infty + ||Df_1(x)||_\infty < \varepsilon$ and the solution $x(t)$ of the system $\dot{x} = f(x) + f_1(x)$ with same initial condition $x(0) = \bar{x}(0)$ is periodic.

Lemma 2 (Anosov). Let $\bar{x}(t), t \geq 0$ be the recurrent trajectory of the system (1) with smooth $f(x)$. Then for any $\varepsilon > 0$ there exists x^* such that $||x^* - \bar{x}(0)|| < \varepsilon$ and the solution $x(t)$ of the system (1) with initial condition $x(0) = x^*$ is periodic.

The notion of attractor is related to the criterion of recurrence formulated by G. Birkhoff in 1927.

Theorem 1 (Birkhoff). Any trajectory contained in the compact minimal invariant set is recurrent. And any compact minimal invariant set is the closure of some recurrent trajectory.

It follows from Birkhoff theorem that any solution starting from its ω-limit set is recurrent. Under additional assumption that ω-limit set of $\bar{x}(t)$ is attractor, any chaotic trajectory, starting from its ω-limit set is recurrent.

Criteria of chaos. The most standard criterion of chaotic behavior is based on computation of upper Lyapunov exponent. For a linear system

$$\dot{x} = A(t)x \tag{3}$$

it is defined as follows

$$\varrho_L = \overline{\lim_{t\to\infty}} \frac{\ln|\Phi(t,t_0)|}{t-t_0}, \tag{4}$$

where $\Phi(t,\tau)$ is the fundamental matrix of the system (3) satisfying $x(t) = \Phi(t,\tau)x(\tau)$ for all $t, \tau \in \mathbb{R}^1$.

If the trajectory $\bar{x}(t)$ of (1) is bounded and $\varrho_L > 0$, where ϱ_L is Lyapunov exponent of the system (1), linearized along $\bar{x}(t)$ (i.e. $A(t) = \partial f(\bar{x}(t))/\partial x$)then $\bar{x}(t)$ is chaotic. The value of $\varrho_L > 0$ indicates the degree of exponential instability of the system.

Poincaré map. Poincaré map allows to consider discrete-time system instead of continuous-time one and to reduce by 1 its dimension. To define Poincaré map assume that $\bar{x}(t)$ be T-periodic solution (1) starting from x_0. i.e. $\bar{x}(t+T) = x(t)$ for all $t \geq t_0, x(t_0) = x_0$. Let S be a smooth surface (*transverse surface* or *cross-section* across x_0), defined by the equation $s(x) = 0$ where $s : \mathbb{R}^n \to \mathbb{R}^1$ is a smooth scalar function such that it intersects the trajectory in x_0 transversely, i.e. $s(x_0) = 0, \nabla s(x_0)^T F(x) \neq 0$. It can be shown that the solution starting from $x \in S = \{x : \; s(x) = 0\}$ close to x_0 will cross the surface $s(x) = 0$ again at least once. Let $\tau = \tau(x)$ be the time of the first return and $x(\tau) \in S$ be the point of the first return.

Definition 5. The mapping $P : x \mapsto x(\tau)$ is called the *Poincaré map* or *return map.*

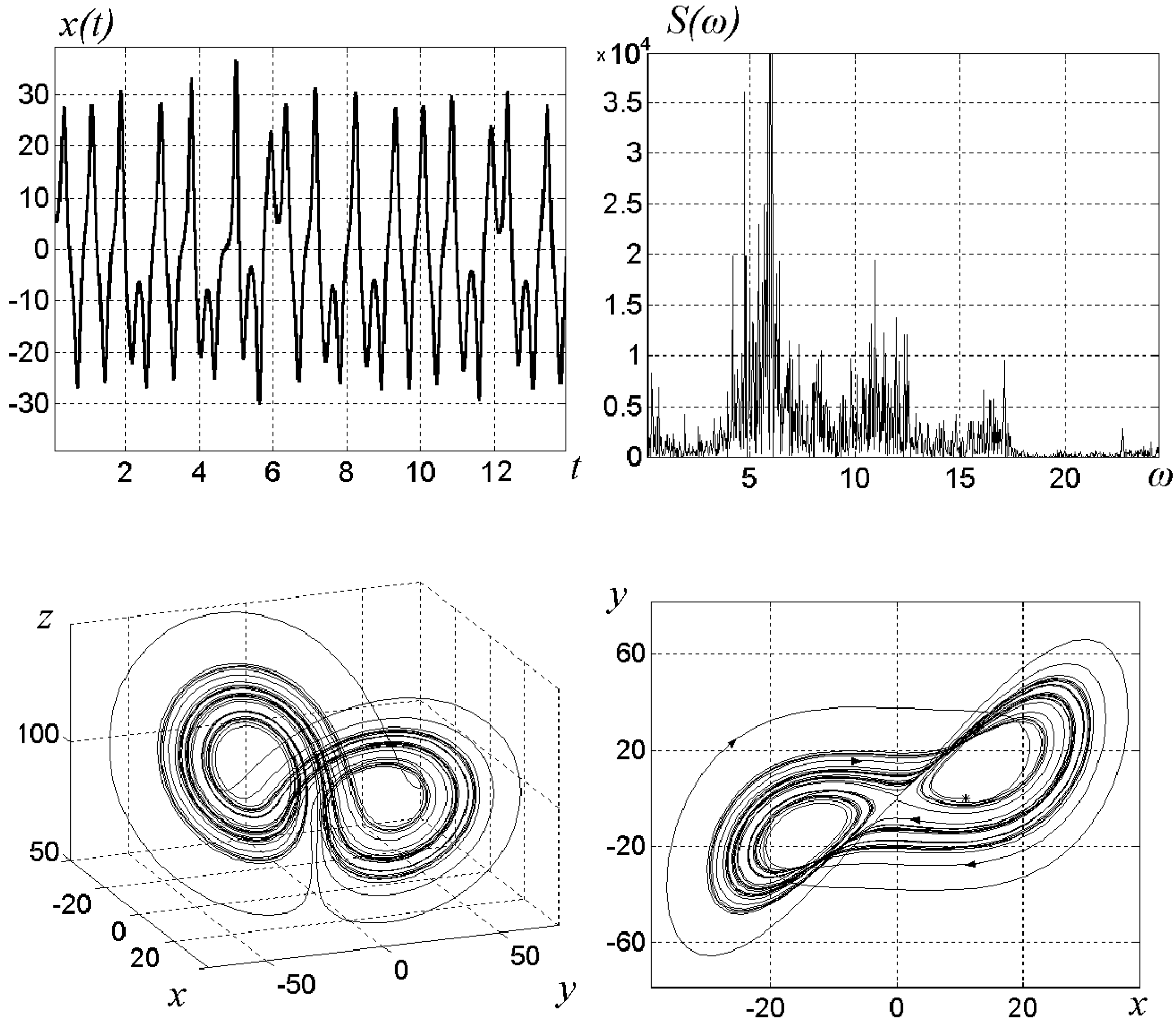
1.2 Examples of Chaotic Systems

1.2.1 Examples of continuous-time chaotic systems

Lorenz system

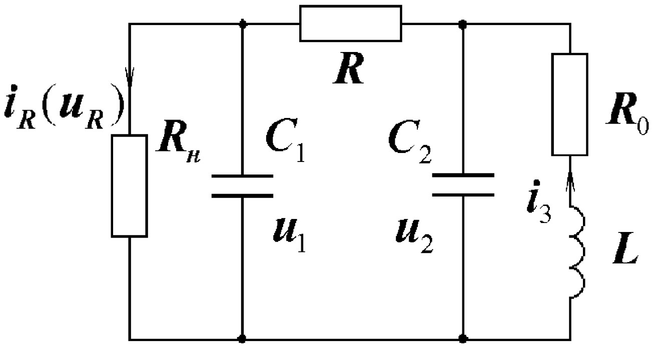
$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = rx - y - xz, \\ \dot{z} = -bz + xy. \end{cases}$$

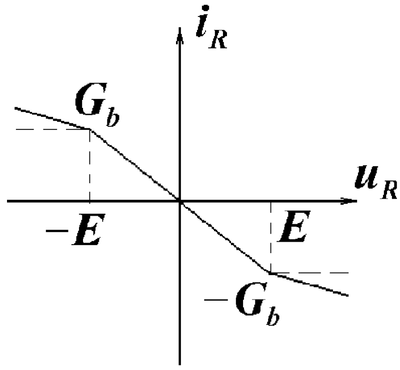
$$\sigma = 10, \quad r = 97, \quad b = 8/3.$$



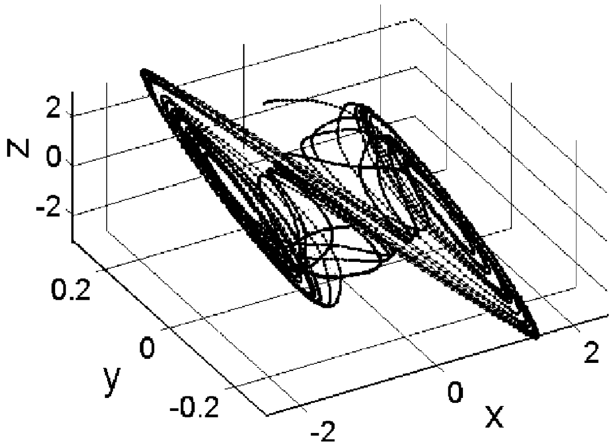
Chua circuit

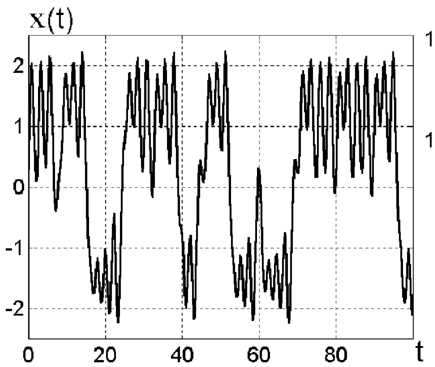
$$\begin{cases} \dfrac{du_1}{dt} = \dfrac{1}{C_1}\left(\dfrac{u_2 - u_1}{R} - i_R(u_1)\right), \\ \dfrac{du_2}{dt} = \dfrac{1}{C_2}\left(\dfrac{u_1 - u_2}{R} + i_3\right), \\ \dfrac{di_3}{dt} = -\dfrac{1}{L}\left(u_2 + R_0 i_3(u_1)\right). \end{cases} \qquad \begin{cases} \dot{x} = p\big(y - f(x)\big), \\ \dot{y} = x - y + z, \\ \dot{z} = -qy. \end{cases}$$

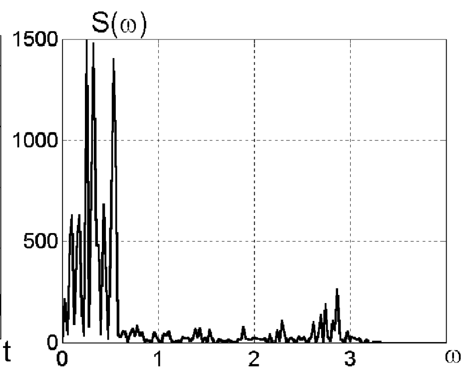




$$f(x) = M_1 x + 0.5(M_1 - M_0)(x + 1 - x - 1)$$
$$p = 9, \quad q = 14.3, \quad M_1 = -6/7, \quad M_0 = 5/7.$$







1.2.2 Examples of discrete-time chaotic systems

Example 1. Pseudorandom numbers generator:

$$x_{k+1} = \{Mx_k\},$$

$\{\ldots\}$ –fractional part of a number.

$M > 1$, irrational $\to$ chaos; ($M = 2$ – Bernoulli shift)

Example 2. Logistic map:

$$x_{k+1} = \lambda x_k(1 - x_k)$$

$3.57 < \lambda < 4 \to$ chaos.

Example 3. Tent map:

$$x_{k+1} = \begin{cases} rx_k, & 0 \le x_k < 0.5, \\ r(1 - x_k), & 0.5 \le x_k \le 1. \end{cases}$$

$1 < r < 2 \to$ chaos

Example 4. Hénon system:

$$\begin{cases} x_{k+1} = 1 - ax_k^2, \\ y_{k+1} = -Jx_k. \end{cases}$$

Example 5. Impact oscillator (bouncing ball):

($\alpha = 1$ – Chirikov (standard) map)

$$\begin{cases} v_{k+1} = \alpha v_k + K \sin \phi_k, \\ \phi_{k+1} = \phi_k + v_{k+1}. \end{cases}$$

v_k – velocity of the ball before kth impact,

ϕ_k – normalized time of kth impact.

1.3 Models of Controlled Systems

Several classes of models of chaotic systems are considered in the literature:

Time-invariant differential equations in the state space:

$$\dot{x} = F(x, u), \tag{5}$$

where x is n-dimensional vector of the state variables; $\dot{x} = d/dt$ stands for the time derivative of x; u is m-dimensional vector of inputs (control variables) and law of measurements:

$$y = h(x). \tag{6}$$

where y is the l-dimensional vector of output variables.

Time-varying differential models

$$\dot{x} = F(x, u, t). \tag{7}$$

Affine in control models

$$\dot{x} = f(x) + g(x)u. \tag{8}$$

Discrete-time state-space models

$$x_{k+1} = F_d(x_k, u_k), \tag{9}$$

where $x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^m, y_k \in \mathbb{R}^l$, the value of the state, input and output vectors at kth stage of the process. The model is specified by the map F_d.

Two physically different cases are possible:

A. The input variables represent some physical variables (forces, torques, intensity of electrical or magnetic fields, etc.)

B. The input variables represent change of physical parameters of the system, i.e. $u(t) = p - p_0$, where p_0 is the nominal value of the physical parameter p.

For example a model of an oscillator (pendulum) controlled by applying a torque to the rotation axis can be put into the form

$$J\ddot{\varphi} + r\dot{\varphi} + ml \sin\varphi = u(t), \tag{10}$$

where φ is the angle of deflection from vertical; J, m, l are physical parameters of the pendulum (inertia, mass, length); u(t) is a controlling torque. The description (10) is transformable into the form (5) with the state vector $x = (\varphi, \dot{\varphi})^T$.

Otherwise, let the pendulum be controlled by changing its length. Then the model, instead of (10) is

$$J\ddot{\varphi} + r\dot{\varphi} + m(l_0 + u(t)) \sin\varphi = 0, \tag{11}$$

where l_0 is initial length of the pendulum.
Obviously, (11) is a special case of (5).

1.4 Control Goals

STABILIZATION:

The typical goal is stabilization of an unstable reference trajectory $x_*(t)$:

$$\lim_{t\to\infty}[x(t) - x_*(t)] = 0 \tag{12}$$

PARTIAL STABILIZATION:

$$\lim_{t\to\infty}[y(t) - y_*(t)] = 0 \tag{13}$$

for any solution $x(t)$ of (5) with initial conditions $x(0) = x_0 \in \Omega$, where Ω is given set of initial conditions.

Typical control problems are: to find a control function in the form of *open loop (feed-forward) control*

$$u(t) = U(t, x_0), \tag{14}$$

or in the form of *state feedback*

$$u(t) = U(x(t)) \tag{15}$$

or in the form of *output feedback*

$$u(t) = U(y(t)) \tag{16}$$

to ensure the goal (12) or (13).

GENERATION (EXCITATION) OF OSCILLATIONS:

The goal trajectory may be specified only partially. Then it may be formulated as achieving the limit equality

$$\lim_{t\to\infty} G(x(t)) = G_* \tag{17}$$

or inequality

$$\underline{\lim}_{t\to\infty} G(x(t)) \geq G_*. \tag{18}$$

where $G(x)$ is given scalar goal function (e.g. total energy of mechanical or electrical oscillations).

SYNCHRONIZATION:

Concordance or concurrent change of the states of two or more systems or, concurrent change of some quantities related to the systems (e.g. equalizing oscillation frequencies):

$$\lim_{t\to\infty} [x_1(t) - x_2(t)] = 0 \tag{19}$$

In the extended state space $x = \{x_1, x_2\}$ of the overall system, relation (19) corresponds to convergence of the solution $x(t)$ to the diagonal set $\{x : x_1 = x_2\}$.
The goals (12), (13), (17), or (19) can be rewritten in terms of appropriate goal function $Q(x,t)$ as follows:

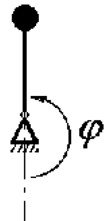
$$\lim_{t\to\infty} Q(x(t),t) = 0 \tag{20}$$

E.g. to reduce goal (19) to the form (20) one may choose $Q(x) = |x_1 - x_2|^2$.

2 METHODS OF CHAOS CONTROL

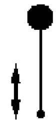
2.1 Feedforward Control by Periodic Signal

Example:
simple inverted
pendulum:



unstable

Kapitsa's
pendulum:



stable
if $A\omega > ml\sqrt{2gl}$

$$\frac{ml^2}{2}\ddot{\varphi} + \rho\dot{\varphi} + mgl\sin\varphi = u(t)\sin\varphi, \quad u(t) = A\omega^2 \sin\omega t$$

Stephenson (1908)
Kapitsa (1951)
Blekhman (1971)

Vibrational mechanics
(*I.I. Blekhman.* World Scientific, 2000)

Zames & Shneydor (1976) – Dither control

Meerkov (1980)
Bellman, Bentsman, Meerkov (1985) – Vibrational control

Drawback: control should be large!

Pettini (1988)
Lima, Pettini (1990)

Duffing system :
$$\ddot{\varphi} - \alpha\varphi + \beta\varphi^3 = -\delta\dot{\varphi} + \gamma\cos\omega_0 t$$

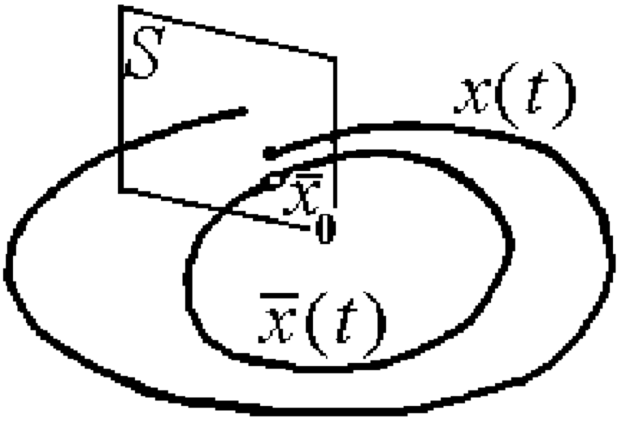
$$\beta \to \beta(1 + \eta\cos\omega t)$$

By Melnikov method and numerically region of η providing suppression of chaos is determined (η is moderate).

Open problem: How to suppress chaos by small feedforward control?

2.2 Linearization of Poincaré Map (OGY method)

E. Ott, C.Grebogi, J. Yorke (1990)
Controlling Chaos. Phys. Revi. Lett., v. 64, 1196–1199.



$$\dot{x} = F(x,u), \quad y = h(x)$$

$$x \in R^n, \quad u, y \in R^1$$

$\bar{x}(t)$ – T-periodic solution of (1) for u=0; $\quad \boldsymbol{\bar{x}(0) = \bar{x}_0}$.

$$\boldsymbol{S} = \{\boldsymbol{x} : \boldsymbol{s}(\boldsymbol{x}) = 0\}, \qquad \boldsymbol{s}(\boldsymbol{\bar{x}_0}) = 0.$$

Example : $\quad \boldsymbol{s}(\boldsymbol{x}) = \dfrac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}} \boldsymbol{F}(\boldsymbol{x},0) \quad (\dot{y}(t) = 0 \Rightarrow t$ – local extremum)

Linearization of Poincaré Map (OGY method)

Step 1: Introduce Poincaré Map and discrete system

$$x_{k+1} = P(x_k, u_k), \tag{21}$$

$x_k = x(t_k)$, $u(t) = u_k$, $t_{k-1} \le t < t_{k+1}$, t_k – time of kth crossing.
Step 2: Linearize (21) near $\bar{x}_0$, $u = 0$.

$$x_{k+1} = Ax_k + Bu_k. \tag{22}$$

Step 3: Estimate parameters A, B of (22), find a stabilizing feedback $u_k = Cx_k$ and choose control:

$$u_k = \begin{cases} Cx_k, & \text{if } \|x_k - \bar{x}_0\| < \Delta, \\ 0 & \text{else} \end{cases} \tag{23}$$

or

Step 3a: Introduce delay coordinates:

$$X(t) = \big[y(t), y(t-\tau), \dots, y\big(t-(N-1)\tau\big)\big]^T, \quad y_{k,i} = y(t_k - i\tau),$$

and choose control:

$$u_k = \begin{cases} U\left(y_k, y_{k,1}, \dots, y_{k,N-1}\right), & \text{if } |y_{k,i} - y_*| < \Delta, \quad i = 1, \dots, N-1 \\ 0 & \text{else} \end{cases} \tag{24}$$

Special case (OPF method):

$$\begin{matrix} y_k \text{ – local maxima,} \\ y_* = h(\bar{x}_0), \end{matrix} \qquad u_k = \begin{cases} K\left(y_k - y_*\right), & \text{if } |y_k - y_*| < \Delta, \\ 0 & \text{else} \end{cases} \tag{25}$$

Open problem: to justify adaptive control

Fradkov, Guzenko (1997): case $y_{k,i} = y_{k-i}$ + inner deadzone, + n-observability.

2.3 Delayed Feedback (Pyragas method)

K. Pyragas (1992), Phys. Lett. A, v. 170, 412–428.
To stabilize τ-periodic solution apply

$$u(t) = K\big[x(t) - x(t-\tau)\big],$$

where K is gain, τ is time delay.

Extended Pyragas method:

$$u(t) = K \sum_{j=1}^{N-1} c_j \,\big[y(t - j\tau) - y\big(t - (j+1)\tau\big)\big].$$

Open problem: when does it work?

1) τ – known, 2) τ – unknown,

$n = 1$: – necessary condition: $A < 1$, (A is a slope of Poincaré map $A = \frac{\partial F}{\partial x}\big(x(0)\big)$);
$n > 1$: – "odd numbers" limitation: ν is odd
(ν is number of real eigenvalues of A, exceeding 1)
(*Ushio*, 1996; *Nakajima*, 1997; *Just* et al., 1997)
Just et al., (1999) – approximate stability bounds for K.

Basso, *Genesio*, *Tesi* et al. (1997, 1998) suggested control

$$u(t) = G(p)\big[y(t) - y(t-\tau)\big],$$

where $G(s)$ is a transfer function of filter, and gave frequency domain condition of stability for Lur'e systems.

Applications of Pyragas method: lasers; magnitoelastic systems; cardiac conduction model; traffic models; PWM-controlled buck convertor.

2.4 Methods of Nonlinear Control

Many papers are devoted to demonstration of the applicability of standard control engineering notions and techniques to control of chaos.

Example: Open-plus-closed-loop (OPCL) method by Jackson and Grosu (1995); Aquirre and Torres (2000); Wang and Wang (1999); Tian et al (2000) for stabilization of the desired trajectory $x_*(t)$ for models $\dot{x}(t) = f(x(t)) + Bu$ with $\dim x = \dim u$:

$$u(t) = B^{-1}[\dot{x}_*(t) - f(x_*(t)) - K(x - x_*(t))]. \tag{26}$$

For $\dim x > \dim u$ – feedback linearization techniques: Yu (1997); Babloyantz et al(1997); Chen and Liu, (1999).

2.4.1 Feedback Linearization

Consider the systems affine in control:

$$\dot{x} = f(x) + g(x)u. \tag{27}$$

Definition 1. The system (27) is called *feedback linearizable in the open domain* $\Omega \in \mathbb{R}^n$ if there exist the smooth coordinate change $z = \Phi(x)$, $x \in \Omega$ and the feedback transformation

$$u = \alpha(x) + \beta(x)v \tag{28}$$

with smooth functions α, β such that Φ and β are smoothly invertible in Ω and the closed loop system (27), (28) is linear, i.e. there exist constant matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ so that

$$f(x) + g(x)\alpha(x) = A, \quad g(x)\beta(x) = B, \quad x \in \Omega. \tag{29}$$

Feedback linearizability of the system means that it is equivalent to the system

$$\dot{z} = Az + Bv, \tag{30}$$

where $z(t) \in \mathbb{R}^n$ is the new state vector and $v(t) \in \mathbb{R}^m$ is the new input.

Definition 2. The system (27) is said to have *relative degree r at the point $x_0 \in \mathbb{R}^n$ with respect to the output*

$$y = h(x), \tag{31}$$

if for any $x \in \Omega$, where Ω is some neighborhood of x_0, the following conditions are valid:

$$L_g L_f^k h(x) = 0, \quad k = 0, 1, \ldots, r-2, \quad L_g L_f^{r-1} h(x) \neq 0.$$

Recall that $L_\psi \phi(x) = \sum_{i=1}^n \frac{\partial \phi}{\partial x_i} \psi_i(x)$ stands for the Lie derivative of the vector function ϕ along the vector field ψ. Relative degree r is exactly equal to the number of times one has to differentiate the output in order to have the input explicitly appearing in the equation which describes the evolution of $y^{(r)}(t)$ in the neighborhood of x_0.

Criterion of feedback linearizability for single input–single output systems.

Theorem. The system (27) is feedback linearizable in the neighborhood Ω of the point $x_0 \in \mathbb{R}^n$ if and only if there exists a smooth scalar function $h(x)$ defined in Ω such that the relative degree r of (27), (31) is equal to n. In the case $r = n$ the state transformation $z = \Phi(x)$ and the feedback law reducing (27) to the chain of integrators (Brunovsky form) can be chosen as follows

$$\Phi(x) = \mathrm{col}(h(x), L_f h(x), \ldots, L_f^{n-1} h(x)) \tag{32}$$

$$u = \frac{1}{b(\Phi(x))} \left[-a(\Phi(x)) + v\right] \tag{33}$$

EXAMPLE. Feedback linearization of Lorenz system.

$$\begin{aligned}
\dot{x}_1 &= \sigma(x_2 - x_1),\\
\dot{x}_2 &= r x_1 - x_2 - x_1 x_3,\\
\dot{x}_3 &= -\beta x_3 + x_1 x_2 + u.
\end{aligned}$$

Let $y = x_1$. Then

$$\begin{aligned}
L_f y &= \dot{y} = \dot{x}_1 = \sigma(x_2 - x_1)\\
L_f^2 y &= L_f(L_f y) = \ddot{x}_1 = \sigma(\dot{x}_2 - \dot{x}_1) = \sigma\big[(r+1)x_1 - 2x_2 + x_1 x_3\big]
\end{aligned}$$

Relative degree $= 2$.

$$\begin{aligned}
z = \Phi(x): \quad & z_1 = x_1,\\
& z_2 = \sigma(x_2 - x_1),\\
& z_3 = \sigma\big[(r+1)x_1 + 2x_2 + x_1 x_3\big],\\
x = \Phi^{-1}(z): \quad & x_1 = z_1,\\
& x_2 = \frac{1}{\sigma} z_2 + z_1,\\
& x_3 = \frac{1}{x_1}\left[\frac{1}{\sigma} z_3 - (r-1) z_1 - \frac{2}{\sigma} z_2\right].
\end{aligned}$$

System is feedback linearizable for $x_1 \neq 0$.

2.4.2 Goal-oriented methods.

Goal-oriented methods: Methods based on reduction of the current value of the scalar goal (objective) function $Q(x(t), t)$.

Speed-gradient (SG) method (Fradkov, 1979): changing control u along the gradient in u of the speed $\dot{Q}(x)$.

General form of SG-algorithm:

$$u = -\Psi[\nabla_u \dot{Q}(x, u)], . \tag{34}$$

where $\Psi(z)$ is vector-function forming acute angle with z.

For affine controlled systems $\dot{x} = f(x) + g(x)u$ the algorithm (34) simplifies:

$$u = -\Psi[g(x)^T \nabla Q(x)]. \tag{35}$$

Special cases of (34) are proportional SG-algorithm

$$u = -\Gamma \nabla_u \dot{Q}(x, u). \tag{36}$$

where Γ is positive-definite matrix and relay SG-algorithm

$$u = -\Gamma \mathrm{sign}[\nabla_u \dot{Q}(x, u)]. \tag{37}$$

For adaptive systems SG-algorithms in differential form are used:

$$\dot{u} = -\Gamma \nabla_u \dot{Q}(x, u). \tag{38}$$

The SG-method is justified using Lyapunov function V constructed from the goal function: $V(x) = Q(x)$ for finite form algorithm and

$$V(x, u) = Q(x) + 0.5(u - u_*)^T \Gamma^{-1}(u - u_*),$$

for differential form algorithms, where u_* is desired (ideal) value of control variables.

Remarks: 1. VSS algorithms for the switching surface $h(x) = 0$ coincide with the speed-gradient algorithms for a goal function $Q(x) = |h(x)|$.

2. Models of chaotic systems often do not satisfy global Lipschitz condition owing to polynomial nonlinearities $x_1 x_2$, x^2, etc. Special attention should be paid to providing boundedness of the closed loop system solutions.

3. To respect the "small control" requirement the gain $\gamma > 0$ should be sufficiently small. An **outer** deadzone may be introduced in terms of the goal function, e.g.:

$$u(t) = \begin{cases} -\gamma \nabla_u \dot{Q}(x, u), & \text{if } \ |Q(x(t))| \leq \Delta, \\ 0, & \text{otherwise.} \end{cases} \tag{39}$$

Open problem: When does (39) work?

EXAMPLE. Stabilization of the equilibrium point of the thermal convection loop model

A. Fradkov and A.Pogromsky, “Control of Oscillations and Chaos” Singapore: World Scientific, 1998.

One of the simplest experimental setup which can demonstrate complex oscillatory behavior is the thermal convection loop. *Singer et al.*,(1991), *Wang* (1995) and others studied controlled thermal convection loop model:

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= -y - xz, \\ \dot{z} &= -z + xy - r + u, \end{aligned} \tag{40}$$

where u is the control variable which is a fluctuation in the heating rate superimposed on the nominal rate r, σ is the Prandtl number and r is the Raleigh number. (It can be obtained from the Lorenz system by replacing $z - r$ with z and assuming that $r =$ const and $b = 1$). For $u = 0$ and $0 < r < 1$ the system has one stable globally attracting equilibrium - $(0, 0, -r)$ that corresponds to the no-motion state of the thermal convection. At $r = 1$ two additional equilibrium points C_+ and C_- emerge: $x = y = \pm\sqrt{r-1}, z = -1$.

The convection equilibria lose their stability in the Andronov-Hopf bifurcation at $r = \sigma(\sigma + 4)/(\sigma - 2)$ and for greater values of the parameter r the system has no more equilibrium points.

Singer et al. (1991) suggested an on-off controller to stabilize the inherent unstable equilibrium point of this system:

$$u = -\gamma \mathrm{sgn}(z + 1). \tag{41}$$

Practical experimentation showed that once applied this controller stabilizes the thermal convection in a clockwise or counterclockwise direction that corresponds to the stabilization of one of the equilibria C_+ or C_-.

The controller (41) is a special case of the speed-gradient algorithm (37)for the objective function

$$Q(x, y, z) = (x - \sqrt{r-1})^2/\sigma + (y - \sqrt{r-1})^2 + (z + 1)^2.$$

It was shown in (*Fradkov and Pogromsky*, 1998) that any trajectory of the overall system tends to some rest point contained in the set of points (x, y, z) such that

$$\left\{ x = y, \quad \left|(x + \sqrt{r-1})(x - \sqrt{r-1})\right| \le \gamma, \quad z = -1 \right\}. \tag{42}$$

It yields convergence of the solution to the neighborhood of one of the inherent equilibrium points C_+ or C_-.

Other methods:

- Nonlinear observers. A survey of nonlinear observer techniques – see *Nijmeijer and Mareels* (1997) and for certain particular designs see *Morgul and Solak* (1997); *Grassi and Mascolo* (1997).
- Linear high-gain observer-based control for globally Lipschitz nonlinearities - Liao (1998).
- Centre manifold theory – *Friedel et al.*, (1997).
- Backstepping iterative design – *Mascolo and Grassi* (1997).
- Passivity based design (*Pogromsky*, 1998).
- Variable structure systems (VSS) design (*Yu*, 1997; *Konishi et al.*, 1998; *Fang et al.*, 2000; *Yau et al.*, 2000).
- Absolute stability theory (*Suykens et al.*, 1998); H_∞ control (*Curran et al.*, 1997; *Suykens et al.*, 1997);
- Combination of Lyapunov and feedback linearization methods (*Loria et al.*, 1998; *Lenz and Obradovic*, 1997).
- Adaptive control.
- Neural networks.
- Fuzzy control of chaos.

3 APPLICATIONS

3.1 Control of Turbulence

Wiener, R.J., *Dolby, D.C.*, *Gibbs, G.C.*, *Squires, B.*, *Olsen, T.*, *Smiley, A.M.* Control of chaotic pattern dynamics in Taylor vortex flow Physical Review Letters, V. **83**, 1999, 2340–2343.

3.2 Control of Friction

Elmer, F.J. Controlling friction Phys. Rev. E, V. **57**, 1998, R4903–R4906.

Rozman, M.G., *Urbakh, M.*, *Klafter, J.* Controlling chaotic frictional forces. Physical Review E, V. **57**, 1998, 7340–7343.

3.3 Control of Lasers

Dykstra R. et al. Experimental control of single-mode laser chaos by using continuous, time-delayed feedback. Phys. Rev.E., V. **57**, 1998, 6596–6598.

Glorieux, P. Control of chaos in lasers by feedback and nonfeedback methods. Intern. J. Bifurc. & Chaos, V. **8**, 1998, 1749–1758.

3.4 Attitude Control of Spacecrafts

Meehan, P.A. and *S.F. Asokanthan.* Control of Chaotic instabilities in a spinning spacecraft with dissipation using Lyapunov method. Chaos Solitons & Fractals, V. **13**, 2002, 1857–1869.

3.5 Control of Vibroformers in Aluminium Production

Paskota M. On modelling and the control of vibroformers in aluminium production. Chaos, Solitons & Fractals, V. **9**, (1/2), 1998, 323-335.

3.6 Control of Microeconomical Chaos

Holyst J.A, Hagel T, Haag G., Weidlich W. How to control a chaotic economy? J.Evol. Econ., 1996, 4, 31-42.

Holyst J.A, Hagel T, Haag G. Chaos, Solitons & Fractals, V.8, 1997, 1489-1505.

Behrens-Feichtinger model:

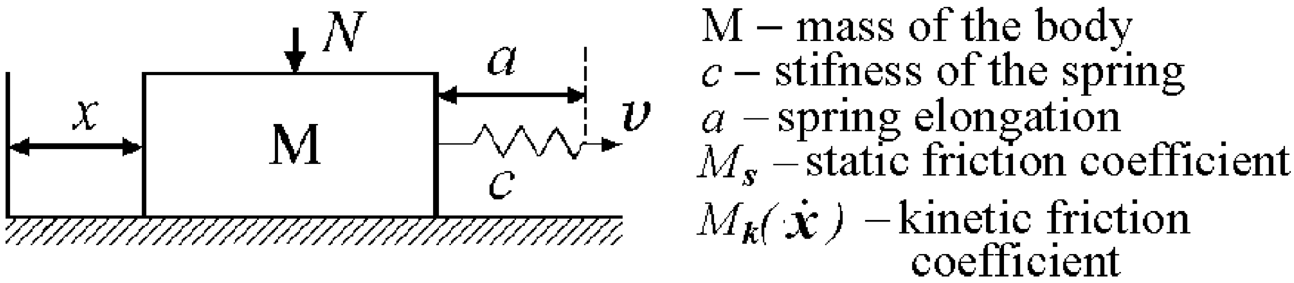
$$\begin{aligned} x_{k+1} &= (1-\alpha)x_k + \frac{a}{1+\exp(-c(x_k - y_k))}, \\ y_{k+1} &= (1-\beta)y_k + \frac{b}{1+\exp(-c(x_k - y_k))}, \end{aligned} \tag{43}$$

where $\alpha, \beta, 0 < \alpha, \beta < 1$ are the time rates of sales decay under zero investments; a, b describe the efficiencies or scales of investments; c is a measure of "elasticity' of the investment strategies. It is shown that competition in the control leads to 'parasitic' oscillations around the periodic orbit that can destroy the expected stabilization effect. OGY method is used for chaos control. It is shown that is one firm tries to eliminate chaos, it is possible for some values of parameters. If, however, both firms are trying to control market, chaos cannot be eliminated in many cases.

3.7 Control of Chaotic Frictional Forces

Elmer, F.J. Controlling friction Phys. Rev. E, V. **57**, 1998, R4903–R4906.

Rozman, M.G., *Urbakh, M.*, *Klafter, J.* Controlling chaotic frictional forces. Physical Review E, V. **57**, 1998, 7340–7343.



$$\begin{cases} M\ddot{x} = c\,a(t) - \mu_k\left(\dot{x}(t)\right)N, \quad \text{if } \dot{x}(t) \neq 0, \\ \dot{x}(t) = 0, \quad \text{if } |ca(t)| < \mu_s N, \\ \dot{a}(t) = \nu - \dot{x}(t) \end{cases} \tag{44}$$

Control:

$$\begin{aligned} &\text{a)} \quad \nu = \nu_0 + K_\nu\left[a(t) - a(t-\tau)\right], \\ &\text{b)} \quad N = N_0 + K_N\left[a(t) - a(t-\tau)\right], \end{aligned} \tag{45}$$

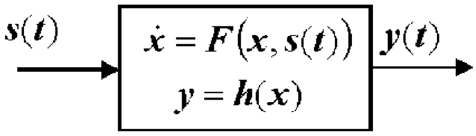
continuous-sliding state: $\dot{x} = \nu_0$, $a = \mu_k(\nu_0)N_0/c$.

3.8 Chaos-based Information Transmission

Fradkov, A.L., *Nijmeijer H.*, *Markov A.* Adaptive observer-based synchronization for communications. Intern. J.of Bifurcation and Chaos, 2000, V. **10**, N 12, pp. 2807–2814.

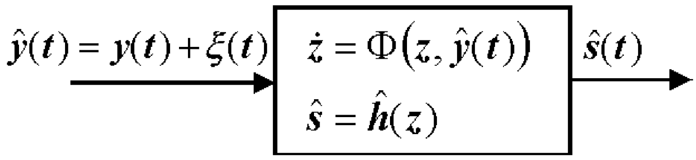
Andrievsky B.R., *A.L.Fradkov*, Information Transmission by Adaptive Synchronization with Chaotic Carrier and Noisy Channel, 39th IEEE CDC, 2000, 1025–1030; ECC'01, 2953–2957.

Transmitter:



$\dot{x} = F(x, s)$ – chaotic generator

Receiver:



$\xi(t)$ – noise.

Adaptive observer-based receiver:

$$\begin{cases} \dot{\hat{x}} = \hat{F}(\hat{y}, \hat{x}, \hat{\theta}), \\ \dot{\hat{\theta}} = \Theta(\hat{y}, \hat{x}, \hat{\theta}), \\ \hat{s} = \hat{h}(\hat{\theta}). \end{cases}$$

Goal: $\overline{\lim_{t\to\infty}} |\hat{s}(t) - s(t)| \leq \Delta$

Auxiliary goal: $\overline{\lim_{t\to\infty}} |\hat{x}(t) - x(t)| \leq \Delta_x$.

3.8.1 Transmitting signals by Chua circuit generator

Transmitter model in dimensionless form:

$$\begin{aligned}\dot{x}_{d_1} &= p(x_{d_2} - x_{d_1} + f(x_{d_1}) + sf_1(x_{d_1})) \\ \dot{x}_{d_2} &= x_{d_1} - x_{d_2} + x_{d3} \\ \dot{x}_{d_3} &= -qx_{d_2}\end{aligned} \tag{46}$$

$f(z) = M_0 z + 0.5(M_1 - M_0)f_1(z),\ f_1(z) = |z+1| - |z-1|$,
M_0, M_1, p, q are the transmitter parameters,
$s = s(t)$ is the message (to be estimated by receiver).
Assume: (a) transmitted signal is $y_d(t) = x_{d_1}(t) + \xi(t)$, where $\xi(t)$ is bounded irregular signal,
(b) values of the parameters p, q are known. **Receiver (adaptive observer):**

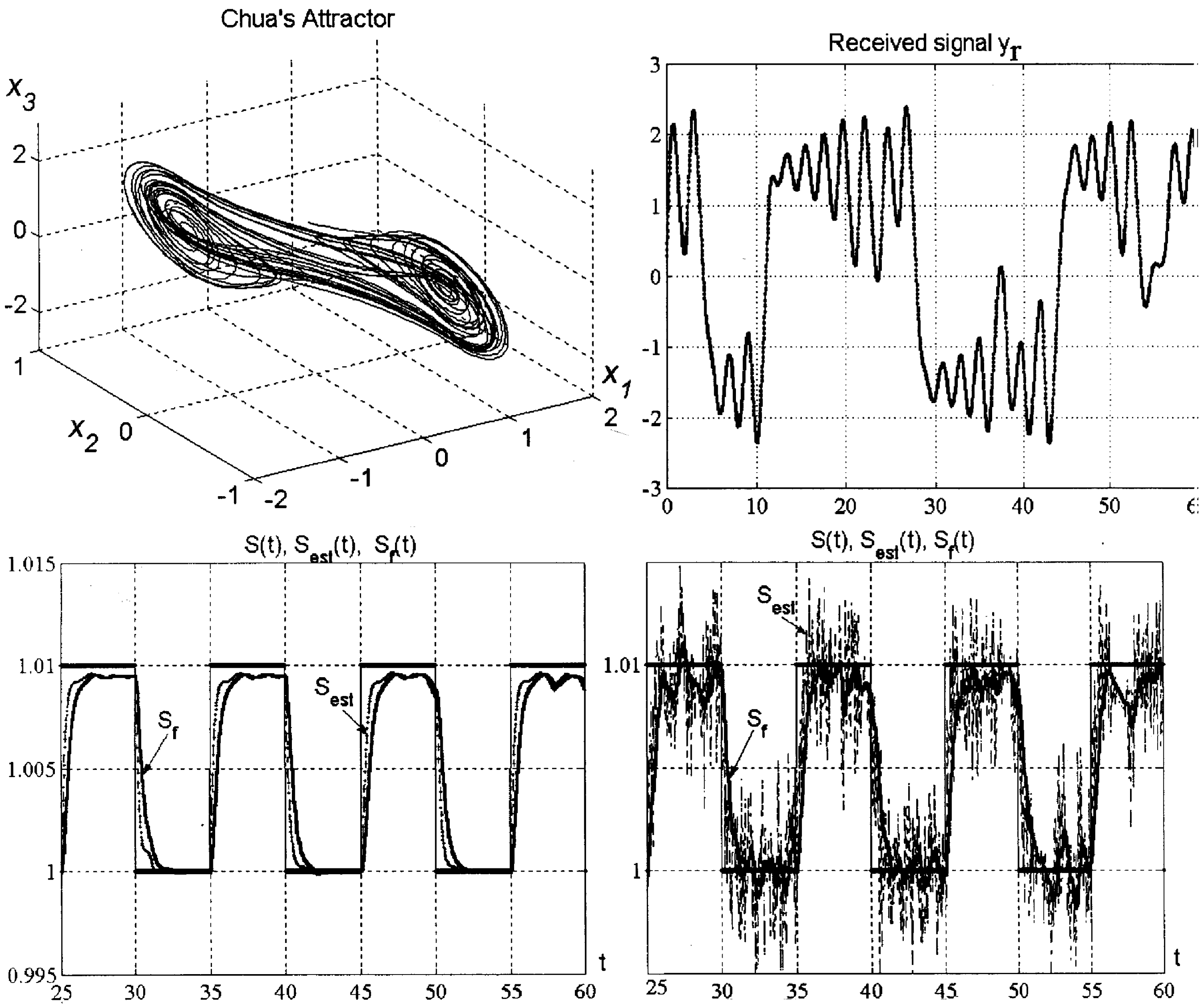
$$\begin{aligned}\dot{x}_1 &= p(x_2 - x_1 + f(y_d) + c_1 f_1(y_d) + c_0(x_1 - y_d)), \\ \dot{x}_2 &= x_1 - x_2 + x_3, \\ \dot{x}_3 &= -qx_2,\end{aligned} \tag{47}$$

where c_0, c_1 are the adjustable parameters.

Adaptation algorithm:

$$\begin{aligned}\dot{c}_0 &= -\gamma_0(y_d - x_1)^2 - \alpha_0 c_0, \\ \dot{c}_1 &= -\gamma_1(x_1 - y_d)f_1(y_d) - \alpha_1 c_1,\end{aligned} \tag{48}$$

γ_0, γ_1 - adaptation gains, α_0, α_1 - regularization gains, $c_1(t) = \hat{s}(t)$ - estimate of message.
(*Andrievsky B.R., A.L.Fradkov*, 39th IEEE CDC, 2000, 1025-1030; ECC'01, 2953-2957).



σ=0 σ=0.001

3.8.2 Future: Nonlinear theory of signals

Conventional ("linear") framework:

1. Signals are described by functions of time: $f(t)$.
2. Base signals are harmonic: $f(t) = A\ sin(\omega t + \alpha)$.
3. Signals are modulated by change of their parameters (e.g. A, ω, α)).
4. Signals are generated and transformed by systems.
5. Base systems are linear: $W(s)$.

"Nonlinear" framework:

1. Systems are described by models:

$$\dot{x} = F(x, u), \quad y = h(x)$$

   where $u(t)$ is input or parameter, $y(t)$ is output.

2. Signals are output functions of systems.
3. Signals are modulated by change of parameters of systems.

   (See *Fradkov A.L.* A nonlinear philosophy for nonlinear systems. *Proc. 39th IEEE Conf. Dec.Contr.* 2000, 4397-4402).

4 CONCLUSIONS

State-of-the-Art:

* A huge number of publications, see
  www.rusycon.ru (RUSYCON - Russian Systems and Control Archive):
  700 references of 1997–2000 at **www.rusycon.ru/chaos-control.html**

* A number of open problems in theory, see

  Fradkov A.L., Evans R.J. Control of Chaos: Some Open Problems. *Proc. 40th IEEE Conf. Dec. Contr.*, Orlando, 2001, pp. 698–703.

  Fradkov A.L., Evans R.J. Control of Chaos: Survey 1997–2000. *Proc. 15th IFAC Congress on Aut. Contr.*, Barcelona, July 2002 (see full text at **www.ccslab.nm.ru**)

* A variety of potential applications:

  – Chaotic mixing in chemical engineering (chaotic agitation of fluidized bed reactors, chaotic blending in polymer production, etc.)

  – Numerical analysis (stability analysis of fixed points via chaos control, stabilizing the Richardson eigenvector algorithm by controlling chaos)

  – Information storage (associative memory based on parametrically coupled chaotic elements.)

  – Encoding digital information using transient chaos.

  – etc.

Features of the field (conclusions of the book "Control of Oscillations and Chaos" by A.Fradkov and A.Pogromsky, Singapore: World Scientific, 1998):

1. **There is the great benefit of using the modern nonlinear and adaptive control theory.**
2. **There is no need to distinguish periodic and chaotic behavior. Accurate control is possible without accurate prediction.**
3. **There is no need to define chaos in order to control it.** (Recurrence property plays the key role)
4. **There is no need to use probability in order to control systems with seemingly random behavior.**