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THE S-PROCEDURE AND A DUALITY RELATIONS IN NONCONVEX PROBLEMS OF QUADRATIC PROGRAMMING

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Abstract. The following problem is considered. Real-valued functions $F(x)$ and $G_1(x), \dots, G_m(x)$ are given on a set X. It is required to clarify when the condition $F(x) \geqslant 0$ with $G_1(x) \geqslant 0, \dots, G_m(x) \geqslant 0$, $x \in X$, implies the existence of numbers $\tau_j \geqslant 0$, $j = 1, \dots, m$, such that $F(x) - \Sigma_1^m \tau_i G_j(x) \geqslant 0$ for all $x \in X$. If this is so, then the S-procedure is said to be *lossless* for the inequality $F(x) \geqslant 0$ subject to the constraints $G_1(x) \geqslant 0, \dots, G_m(x) \geqslant 0$. It is shown that the S-procedure is lossless if $m = 2$, X is a complex linear space, and $F(x)$, $G_1(x)$ and $G_2(x)$ are quadratic functionals on X, where $G_1(x_0) > 0$ and $G_2(x_0) > 0$ for some $x_0 \in X$. It is shown that the losslessness of the S-procedure, in general, is connected with the presence of duality in extremal problems. It is established that duality theorems hold in a number of nonconvex quadratic programming problems.

Bibliography: 5 titles.

Introduction

This paper is a continuation of [1]. We remind the reader of the problem that was considered in [1]. Real-valued functions $F(x)$ and $G_1(x), \dots, G_m(x)$ are given on an arbitrary set $X = \{x\}$. Let $\tau_1, \dots, \tau_m$ be real numbers and $\tau = \|\tau_j\|_1^m$. We set

$$S(x, \tau) = F(x) - \sum_{j=1}^{m} \tau_j G_j(x). \tag{0.1}$$

Consider the following two conditions:

$$F(x) \geqslant 0 \quad \text{for} \quad G_1(x) \geqslant 0, \dots, G_m(x) \geqslant 0, \ x \in X, \tag{0.2}$$

$$\exists \tau_j \geqslant 0, \ j = 1, \dots, m : S(x, \tau) \geqslant 0 \quad \forall x \in X. \tag{0.3}$$

Obviously, (0.3) implies (0.2). Under a number of additional conditions imposed on the functions $F(x)$ and $G_1(x), \dots, G_m(x)$, (0.2) implies (0.3), i.e., (0.2) and (0.3) are equivalent. In this case [1] the S-procedure is said to be *lossless* for the inequality $F(x) \geqslant 0$ subject to the constraints $G_1(x) \geqslant 0, \dots, G_m(x) \geqslant 0$. The losslessness of the S-procedure is defined similarly for the inequality $F(x) > 0$ or the equality $F(x) = 0$, subject to the constraints $G_j(x) > 0$ or $G_j(x) = 0$. (Any combinations are possible.) In applications, the functions $F(x)$ and $G_1(x), \dots, G_m(x)$ depend on certain "constructive" parameters, and the conditions (0.2) and (0.3) single out certain domains in the space of these parameters, say A for (0.2) and B for (0.3). We always have $A \subseteq B$. If the S-procedure is

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lossless, then $A = B$. The problem considered in [1] consisted in the clarification of conditions under which the S-procedure is lossless. In particular, it was shown in [1] that this is so for the inequality $F(x) \geqslant 0$ subject to one constraint $G(x) \geqslant 0$ when $F(x)$ and $G(x)$ are real quadratic forms(1) and X is a real linear space.(2) Earlier, it was established in [2] (with a different terminology) that the S-procedure is lossless when $F(x)$ and $G(x)$ are Hermitian forms and X is a complex linear space.(3) For two constraints (F, G_1, and G_2 are quadratic forms) the S-procedure is lossy [1] in the real case. Later we will show (Theorems 1.1 and 1.2) that for two regular (see §1) constraints in the complex case (F, G_1, and G_2 are Hermitian forms) the S-procedure is lossless. Moreover, we shall consider the question of the losslessness of the S-procedure if F, G_1, and G_2 are nonhomogeneous quadratic functionals. We shall also show that the losslessness of the S-procedure, in general, is closely connected with the validity of the duality theorem in the corresponding extremal problem. With the aid of the theorems on the losslessness of the S-procedure for quadratic functionals we shall establish that duality relations hold in certain nonconvex quadratic programming problems.

§1. The losslessness of the S-procedure for Hermitian forms (two constraints)

This section is devoted to the proof of the losslessness of the S-procedure for the inequality $F(x) \geqslant 0$ subject to the constraints $G_1(x) \geqslant 0$ and $G_2(x) \geqslant 0$ or $G_1(x) = 0$ and $G_2(x) = 0$, where F, G_1, and G_2 are Hermitian forms on a complex linear space X. The following lemma plays the central role in the proof..

LEMMA 1.1. *Let X be a complex linear space, and let F_i, $i = 1, 2, 3$, be arbitrary Hermitian forms on X. We define a mapping φ: $X \longrightarrow \mathbf{R}^3$ by the formula*

$$\varphi(x) = (F_1(x),\ F_2(x),\ F_3(x)),\ x \in X. \tag{1.1}$$

Then the set $\varphi(X)$ is convex.

REMARK. One can ask whether the set $\varphi(X)$ is convex under the mapping φ: $X \longrightarrow \mathbf{R}^k$ defined by $\varphi(x) = (F_1(x), \ldots, F_k(x))$ for any Hermitian forms $F_1, \ldots, F_k$.

(1)By a quadratic form $F(x)$ on a real linear space X we mean a functional of the form $F(x) = B(x, x)$, where $B(x, y)$ is a bilinear symmetric functional on $X \times X$. By a Hermitian form on a complex linear space we mean a functional $B(x, x)$, where $B(x, y)$ is a Hermitian-bilinear and Hermitian-symmetric functional (i.e., linear in the first argument, antilinear in the second argument, and satisfying $B(x, y) = \overline{B(y, x)}$ for all $x, y \in X$). In what follows, the arguments of quadratic and Hermitian forms will sometimes be omitted for brevity.

(2)In [1], this assertion was stated for a Euclidean space X. However, the proof goes over without any change to the case when X is an arbitrary real linear space. We remark that there is a mistake in the statement of this assertion in [1] (Theorem 1), namely, instead of "... the form (Gx, x) is not negative definite" it should read "... the form (Gx, x) is not nonpositive."

(3)This assertion is proved in another way in [1]. (The author of [1] did not know of [2].) We mention that this assertion for the complex case follows immediately from the same assertion for the real case, but the direct proof [2] is simpler.

The affirmative answer to this question for $k = 2$ follows immediately from Hausdorff's theorem on the convexity of the numerical image of an operator (see [3] and also [4]). Lemma 1.1 yields an affirmative answer for $k = 3$. It is easy to construct examples showing that the answer to this question for $k > 3$ is negative.

PROOF. We may assume that $\dim X > 1$, since otherwise the lemma is obvious. Let us show that the lemma holds under the assumption that $\dim X = 2$ ($X = \mathbf{C}^2$). An application of Lemma 3.3 of [1] to the case when $m = 2$, $X = \mathbf{C}^2$, and Φ is the set of Hermitian forms in two variables reduces the proof to showing that the set $M = \{x \in \mathbf{C}^2 \colon F_1(x) = \alpha_1, F_2(x) = \alpha_2\}$ is arcwise connected for any real α_1, α_2 and any $F_1, F_2 \in \Phi$. (The empty set is assumed to be arcwise connected.) It is sufficient to consider the case $F_1(x) \not\equiv 0$ and $F_2(x) \not\equiv 0$. Moreover, it may be assumed that $\alpha_1 \not\equiv 0$ and $F_2(x) \not\equiv 0$. Moreover, it may be assumed that $\alpha_1 = 0$ or $\alpha_2 = 0$ (if, e.g., $\alpha_1 \neq 0$, then we replace F_2 by $F_2 - \alpha_2 F_1/\alpha_1$). For definiteness, let $\alpha_2 = 0$. We set $x = (\xi, \eta)$, where $\xi \in \mathbf{C}^1$ and $\eta \in \mathbf{C}^1$, and reduce $F_2(x)$ to the canonical form. Then we may assume without loss of generality that M is given by the equation $\beta|\xi|^2 + \operatorname{Re}(\gamma\xi\bar{\eta}) + \delta|\eta|^2 = \alpha$, $|\xi|^2 - \epsilon|\eta|^2 = 0$. Here α, β, δ, and ϵ are real numbers, and γ is a complex number. We may also assume that $\gamma \neq 0$ and $\epsilon > 0$, since otherwise the linear connectedness of M can be established in an obvious way. Let $\gamma = \rho e^{i\kappa}$, where $\rho > 0$ and κ is real. We introduce polar coordinates $\xi = r_1 e^{i\varphi_1}$, $\eta = r_2 e^{i\varphi_2}$. Then the equations take the form

$$r_1^2 = \varepsilon r_2^2, \quad r_2^2(\beta\varepsilon + \delta + \sqrt{\varepsilon}\rho\cos(\varphi_1 - \varphi_2 + \varkappa)) = \alpha. \tag{1.2}$$

Note that the equations (1.2) do not change under the substitution $\varphi_1' = \varphi_1 + \psi$, $\varphi_2' = \varphi_2 + \psi$. Therefore, it is sufficient to establish that $M' = M \cap \{x \colon \varphi_1 + \varkappa = 0\}$ is arcwise connected. Indeed any point $x \in M$ with the polar coordinates $x = (r_1, \varphi_1, r_2, \varphi_2)$ can be connected by an arc in M with a point $x' \in M'$. In polar coordinates this arc can be given, e.g., by the formulas

$$x(t) = (r_1, \ \varphi_1 - t(\varkappa + \varphi_1), \ r_2, \ \varphi_2 - t(\varkappa + \varphi_1)), \quad t \in [0, 1].$$

The set M is given by the equations

$$r_1 = \sqrt{\varepsilon}\, r_2, \quad \varphi_1 = -\varkappa, \quad r_2^2(\beta\varepsilon + \delta + \rho\sqrt{\varepsilon}\cos\varphi_2) = \alpha. \tag{1.3}$$

Thus, it is sufficient to prove that the plane curve M'' with the polar coordinates (r_2, φ_2) given by the last equation in (1.3) is arcwise connected. For $\alpha = 0$, this curve is a point, a ray, or a pair of rays starting at the origin. For $\alpha \neq 0$, the domain of φ_2 is the circle or its part defined by the inequality

$$\alpha(\beta\varepsilon + \delta + \rho\sqrt{\varepsilon}\cos\varphi_2) > 0.$$

In this domain M'' can be given by the single-valued continuous dependence relation

$$r_2 = \sqrt{|\alpha|} / \sqrt{|\beta\varepsilon + \delta + \rho\sqrt{\varepsilon}\cos\varphi_2|},$$

i.e., M'' is arcwise connected. Thus, the lemma has been proved under the assumption that $X = \mathbf{C}^2$.

Let us now prove that $\varphi(X)$ is convex in the general case. Let $z_1, z_2 \in \varphi(X) \subset \mathbf{R}^3$, i.e., assume that there exist points $x_1, x_2 \in X$ such that $z_1 = \varphi(x_1)$ and $z_2 = \varphi(x_2)$. We claim that the entire interval with the end-points z_1 and z_2 also belongs to $\varphi(X)$. Consider the two-dimensional subspace $X_1 \subset X$ spanned by these vectors (if x_1 and x_2 are linearly dependent, then the lemma obviously holds because the forms F_1, F_2 and F_3 are homogeneous.) Let φ_1 be the restriction of φ to X_1. By what has been proved above, $\varphi_1(X_1)$ is a convex set containing z_1 and z_2 and contained in $\varphi(X)$. Therefore $\varphi(X)$ contains the interval with the end-points z_1 and z_2, i.e., $\varphi(X)$ is convex.

We shall need the so-called Slater regularity condition (see, for example, [5]), which is well-known in convex programming:

$$\exists x_0 \in X : G_j(x_0) > 0, \ j = 1, \ldots, m. \tag{1.4}$$

Moreover, in discussing constraints in the form of equalities we shall make use of a modified Slater condition:

$$\forall \tau = \| \tau_j \|_{j=1}^m, \ \tau_j \neq 0, \ \exists x_\tau \in X : \tau_j G_j(x_\tau) > 0, \ j = 1, \ldots, m. \tag{1.5}$$

For example, for $m = 1$ the condition (1.5) means that $G_1(x)$ is indefinite, i.e. assumes values of different signs on X. We say that the constraints $G_1(x) \geqslant 0, \ldots, G_m(x) \geqslant 0$ are *regular* if the condition (1.4) is satisfied. By regularity of constraints in the form of equalities $G_1(x) = 0, \ldots, G_m(x) = 0$ we mean that (1.5) is satisfied. Regularity of constraints in the form of an arbitrary set of equalities and inequalities is defined similarly.

THEOREM 1.1. *Let $G_1(x)$ and $G_2(x)$ be Hermitian forms on a complex linear space X. Then the S-procedure for the inequality $F(x) \geqslant 0$ subject to the regular constraints $G_1(x) \geqslant 0$ and $G_2(x) \geqslant 0$ is lossless if $F(x)$ is an arbitrary Hermitian form on X.*

PROOF. Consider the image $\varphi(X)$ of X under the mapping φ: $X \longrightarrow \mathbf{R}^3$ defined by the formula $\varphi(x) = (G_1(x), G_2(x), F(x))$. The condition (0.2) means that $\varphi(X)$ does not intersect the octant $Q = \{(z_1, z_2, z_3) \in \mathbf{R}^3, z_1 \geqslant 0, z_2 \geqslant 0, z_3 < 0\}$. Therefore, the set $\varphi(X)$, which is convex by Lemma 1.1, and the set $\bar{Q}$ do not have common interior points.(4) By the separation theorem, there exist numbers λ_1, λ_2, and λ_3, not all zero, such that $\lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 \geqslant 0$ for $(z_1, z_2, z_3) \in \overline{\varphi(X)}$ and $\lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 \leqslant 0$ for $(z_1, z_2, z_3) \in \bar{Q}$. Since $(0, 0, -1) \in \bar{Q}$, we have $\lambda_3 \geqslant 0$. Similarly, $\lambda_1 \leqslant 0$ and $\lambda_2 \leqslant 0$. Further, $\lambda_3 > 0$, since otherwise $\lambda_1 G_1(x) + \lambda_2 G_2(x) \geqslant 0$ for all $x \in X$, which contradicts (1.4). Setting $\tau_1 = -\lambda_1/\lambda_3$, $\tau_2 = -\lambda_2/\lambda_3$, we obtain the assertion of the theorem.

(4) $\bar{A}$ denotes the closure of a set A.

REMARK. The condition (1.4) is essential. It is easy to give examples of forms $G_1(x)$ and $G_2(x)$ not satisfying (1.4) for which the assertion of Theorem 1.1 is false. For example, one can take $X = \mathbf{R}^3 = \{(x_1, x_2, x_3)\}$, $G_1(x) \equiv -|x_1|^2$, $G_2(x) \equiv -|x_2|^2$. Then, for any $F(x)$ of the form

$$F(x) = f_1|x_1|^2 + f_2|x_2|^2 + \operatorname{Re}[f_{12}x_1\bar{x}_2 + f_{13}x_1\bar{x}_3 + f_{23}x_2\bar{x}_3]$$

with $|f_{13}| + |f_{23}| \neq 0$ the condition (0.2) is satisfied, but (0.3) is not. We have the same situation also for $m = 1$.

The following theorem is concerned with the conditions for the losslessness of the S-procedure with constraints in the form of equalities.

THEOREM 1.2. *Let G_1 and G_2 be two Hermitian forms satisfying* (1.5). *Then for any Hermitian form F such that $F(x) \geqslant 0$ when $G_1(x) = G_2(x) = 0$ there exist real numbers τ_1 and τ_2 such that $F(x) - \tau_1 G_1(x) - \tau_2 G_2(x) \geqslant 0$ for all $x \in X$. In other words, the S-procedure for the inequality $F(x) \geqslant 0$ subject to the regular constraints $G_1(x) = 0$ and $G_2(x) = 0$ is lossless for any Hermitian form F.*

PROOF. As in the proof of Theorem 1.1, we consider the mapping φ: $X \longrightarrow \mathbf{R}^3$ defined by $\varphi(x) = (G_1(x), G_2(x), F(x))$. By Lemma 1.1, the set $\varphi(X)$ is convex. By the condition of the theorem, $\varphi(X) \cap Q' = \varnothing$, where $Q' = \{(z_1, z_2, z_3) \in \mathbf{R}^3 \colon z_1 = z_2 = 0, z_3 < 0\}$. By the separation theorem, there exist numbers λ_1, λ_2, and λ_3, not all zero, such that $\lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 \leqslant 0$ for $(z_1, z_2, z_3) \in Q'$ (this means that $\lambda_3 \geqslant 0$) and $\lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 \geqslant 0$ for $(z_1, z_2, z_3) \in \varphi(X)$. Further, $\lambda_3 > 0$, for otherwise $\lambda_1 G_1(x) + \lambda_2 G_2(x) \geqslant 0$ for all $x \in X$, which contradicts (1.5). Setting $\tau_1 = -\lambda_1/\lambda_3$ and $\tau_2 = -\lambda_2/\lambda_3$, we arrive at the assertion of the theorem.

REMARK. The losslessness of the S-procedure for the inequality $F(x) \geqslant 0$ subject to the regular constraints $G_1(x) \geqslant 0$ and $G_2(x) = 0$ can be proved in the same way.

§2. The losslessness of the S-procedure for quadratic functionals

DEFINITION. A *quadratic functional* on a real linear space X is a mapping F: $X \longrightarrow \mathbf{R}^1$ defined by the formula $F(x) = A(x) + b(x) + c$, where $A(x)$ is a quadratic form, $b(x)$ is a linear functional on X, and c is a real number. A quadratic functional on a complex linear space X is a mapping F: $X \longrightarrow \mathbf{R}^1$ defined by the formula $F(x) = A(x) + \operatorname{Re} b(x) + c$, where $A(x)$ is a Hermitian form, $b(x)$ is a linear functional on X, and c is a real number.

THEOREM 2.1. *Let X be a complex or real linear space, and let F and G be quadratic functionals on X. Then the S-procedure for the inequality $F(x) \geqslant 0$ subject to the regular constraint $G(x) \geqslant 0$ or $G(x) = 0$ is lossless. (In the case of a real space X and the constraint $G(x) = 0$ it is assumed, in addition, that the functional G is nonlinear, i.e. its quadratic form does not vanish identically.)*

THEOREM 2.2. *Let X be a complex linear space, and let F, G_1, and G_2 be quadratic functionals on X. Then the S-procedure for the inequality $F(x) \geqslant 0$ subject to the regular constraints $G_1(x) \geqslant 0$, $G_2(x) \geqslant 0$ or $G_1(x) \geqslant 0$, $G_2(x) = 0$ or $G_1(x) = 0$, $G_2(x) = 0$ is lossless. (In the case $G_1(x) = G_2(x) = 0$ it is assumed in addition that at least one of the functionals G_1 or G_2 is nonlinear.)*

PROOF. Theorems 2.1 and 2.2 are proved in the same manner, by the reduction to the S-procedure for quadratic or Hermitian forms. Let us prove, e.g., Theorem 2.2. We consider the case of the constraints $G_1(x) \geqslant 0$ and $G_2(x) \geqslant 0$. Let $F(x) = A(x) + \mathrm{Re}\, b(x) + c$ and $G_j(x) = A_j(x) + \mathrm{Re}\, b_j(x) + c_j$, $j = 1, 2$. We may assume that the zero element of X satisfies the constraints, i.e. that $G_j(0) = c_j \geqslant 0$, $j = 1, 2$ (this can be achieved by a shift in X). We define a Hermitian form $F'(y)$, where $y = (x, \xi) \in X \times \mathbf{C}^1$, as follows:

$$F'(x, \xi) = \begin{cases} |\xi|^2 F(x/\xi), & \xi \neq 0, \\ A(x), & \xi = 0. \end{cases} \tag{2.1}$$

Obviously, $F'(x, \xi) = A(x) + \mathrm{Re}\,[b(x)\bar{\xi}\,] + c|\xi|^2$. We define the Hermitian forms $G_j'(x, \xi)$ from the functionals G_j, $j = 1, 2$, in a similar way; and we claim that $F'(x, \xi) \geqslant 0$ when $G_1'(x, \xi) \geqslant 0$ and $G_2'(x, \xi) \geqslant 0$, if $F(x) \geqslant 0$ when $G_1(x) \geqslant 0$ and $G_2(x) \geqslant 0$. Indeed, by (2.1) the assertion holds for $\xi \neq 0$. It remains to show that $F'(y) \geqslant 0$ when $G_1'(y) \geqslant 0$ and $G_2'(y) \geqslant 0$, where $y = (x, 0)$. Consider the family of vectors $y_\alpha \in X \times \mathbf{C}^1$ of the form $y_\alpha = (x + \alpha x_0, \alpha\xi_0(\alpha))$, where α is real. Let us show that we can always choose a vector $x_0 \in X$ and a complex function $\xi_0(\alpha)$ not vanishing and bounded in a neighborhood of $\alpha = 0$ such that, for a sequence $\alpha_n \to 0$, $n \to \infty$, the vectors y_{α_n}, $n = 1, 2, \ldots$, satisfy the constraints $G_1'(y_{\alpha_n}) \geqslant 0$, $G_2'(y_{\alpha_n}) \geqslant 0$, $n = 1, 2, \ldots$. Indeed, set $x_0 = 0$. Then

$$G_j(y_\alpha) = A_j(x) + \alpha\, \mathrm{Re}\,[b_j(x)\,\bar{\xi}_0(\alpha)] + \alpha^2 c_j\,|\xi_0|^2.$$

Since $A_j(x) = G_j'(x, 0) \geqslant 0$ and $c_j \geqslant 0$, $j = 1, 2$, the constraints $G_j'(y_\alpha) \geqslant 0$ are satisfied for all $\alpha \geqslant 0$ if $\xi_0(\alpha)$ satisfy the inequalities

$$\mathrm{Re}\,[b_j(x)\,\bar{\xi}_0(\alpha)] \geqslant 0, \quad j = 1, 2. \tag{2.2}$$

But each of the inequalities (2.2) defines a closed half-plane in the complex plane $\{\xi\}$; therefore the system (2.2) always has a solution $\xi_0(\alpha) \equiv \xi_0 \neq 0$. Thus, $F'(y_\alpha) \geqslant 0$ for $\alpha \geqslant 0$, and so

$$F'(x, 0) = \lim_{\alpha \to 0} F'(y_\alpha) \geqslant 0.$$

Further, the constraints $G_1'(x, \xi) \geqslant 0$ and $G_2'(x, \xi) \geqslant 0$ are regular, since $G_1(x) \geqslant 0$ and $G_2(x) \geqslant 0$ are. The application of Theorem 1.1 and the substitution $\xi = 1$ complete the proof of Theorem 2.2 for constraints in the form of inequalities. The proof for the remaining cases and also the proof of Theorem 2.1 differ only by the choice of sequences of vectors $y_{\alpha_n} \in X \times \mathbf{C}^1$ that satisfy the requirements indicated above and the corresponding constraints. It is not hard to check that under the assumptions of Theorems 2.1 and 2.2 such a choice is always possible.

REMARK. The requirements in Theorems 2.1 and 2.2 that the constraints are nonlinear are necessary. Namely, if $G(x) = b(x) + c$, where $b(x)$ is a linear functional on a real linear space X and c is a real number, then there always is a quadratic functional $F(x) \geqslant 0$ subject to the constraint $G(x) = 0$ is lossy. For example, one can set $F(x) = 1 - [G(x)]^2$. Similarly, in the complex case with $G_j(x) \equiv \operatorname{Re} b_j(x) + c_j$, $j = 1, 2$, the S-procedure for the inequality $F(x) \geqslant 0$ subject to the constraints $G_1(x) = 0$ and $G_2(x) = 0$ is lossy if one takes

$$F(x) = 1 - \operatorname{Re}(b_1(x) + c_1)\overline{(ib_2(x) + ic_2)}.$$

In the following section, the results of §§1 and 2 will be applied to prove duality theorems in a number of nonconvex extremal problems.

§3. The connection of the S-procedure with duality theorems in extremal problems

THEOREM 3.1. *Let $F(x)$ and $G_1(x), \ldots, G_m(x)$ be real-valued functions on a set X and suppose that the S-procedure for the inequality $F(x) \geqslant c$ subject to the constraints $G_1(x) \geqslant 0, \ldots, G_m(x) \geqslant 0$ is lossless for any real number c. Then the following duality relation holds in the extremal problem* $\inf\{F(x)\colon G_j(x) \geqslant 0, j = 1, \ldots, m\}$:

$$\inf_{G_j(x) \geqslant 0} F(x) = \sup_{\tau_j \geqslant 0} \inf_{x \in X} \left[F(x) - \sum_{j=1}^{m} \tau_j G_j(x) \right], \tag{3.1}$$

where the supremum on the right-hand side of (3.1) *is attained.*[5] *Conversely, if* (3.1) *holds and the supremum on the right-hand side of* (3.1) *is attained, then the S-procedure for the inequality $F(x) \geqslant c$, subject to the constraints $G_1(x) \geqslant 0, \ldots, G_m(x) \geqslant 0$ is lossless, where c is any real number.*

PROOF. To begin with we prove the first part of the theorem. It is easy to see that the right-hand side of (3.1) never exceeds the left. Indeed, for any $\tau_j \geqslant 0, j = 1, \ldots, m$, the following relation holds:

$$\inf_{\substack{G_j(x) \geqslant 0 \\ j=1,\ldots,m}} F(x) \geqslant \inf_{\substack{G_j(x) \geqslant 0 \\ j=1,\ldots,m}} \left[F(x) - \sum_{j=1}^{m} \tau_j G_j(x) \right] \geqslant \inf_{x \in X} \left[F(x) - \sum_{j=1}^{m} \tau_j G_j(x) \right]. \tag{3.2}$$

It remains to take the supremum on both sides of (3.2) over all m-tuples $\|\tau_j\|_{j=1}^{m}$, $\tau_j \geqslant 0$, $j = 1, \ldots, m$. Let us show that the losslessness of the S-procedure guarantees the reverse inequality. Let

$$\inf\{F(x) : G_j(x) \geqslant 0,\ j = 1, \ldots, m\} = l.$$

If $l = -\infty$, then (3.1) follows from (3.2). Therefore, we may assume that l is a

[5] Here and later, we use the natural extension of the definition of supremum to functions assuming infinite values. Namely, if $\Phi(\tau) \equiv -\infty$, $\tau \in T$, then we set $\sup_{\tau \in T} \Phi(\tau) = -\infty$.

finite number. Then $F(x) - l \geqslant 0$ for $G_1(x) \geqslant 0, \ldots, G_m(x) \geqslant 0$. The losslessness of the S-procedure for the function $F(x) - l$ means that there exist numbers $\tau_j \geqslant 0$, $j = 1, \ldots, m$, such that $F(x) - \Sigma_1^m \tau_j G_j(x) \geqslant l$ for all $x \in X$. Hence follows the first assertion of the theorem.

Conversely, let $F(x) - c \geqslant 0$ for $G_1(x) \geqslant 0, \ldots, G_m(x) \geqslant 0$. Then

$$\sup_{\tau_j \geqslant 0} \inf_{x \in X} S(x, \tau) = \inf_{G_j(x) \geqslant 0} F(x) \geqslant c.$$

Therefore there exist numbers $\tau_j \geqslant 0$, $j = 1, \ldots, m$, such that $S(x, \tau) \geqslant c$ for all $x \in X$, i.e.

$$F(x) - c - \sum_{j=1}^{m} \tau_j G_j(x) \geqslant 0,$$

which was to be proved.

A duality theorem for constraints in the form of equalities can be proved in a similar way.

THEOREM 3.2. *Let $F(x)$ and $G_1(x), \ldots, G_m(x)$ be real-valued functions on a set X, and suppose that the S-procedure for the inequality $F(x) \geqslant c$ subject to the constraints $G_1(x) = 0, \ldots, G_m(x) = 0$ is lossless for any real number c. Then the following duality relation holds in the problem* $\inf\{F(x)\colon\ G_j(x) = 0, j = 1, \ldots, m\}$:

$$\inf_{G_j(x) = 0} F(x) = \sup_{\tau} \inf_{x \in X} \left[F(x) - \sum_{j=1}^{m} \tau_j G_j(x) \right], \tag{3.3}$$

where the supremum on the right-hand side of (3.3) *is attained. Conversely, if* (3.3) *holds and the supremum is attained, then the S-procedure is lossless for the inequality $F(x) \geqslant c$ subject to the constraints $G_1(x) = 0, \ldots, G_m(x) = 0$, where c is any real number.*

The following assertions, which provide a number of examples of true duality relations in nonconvex extremal problems, follow directly from Theorems 2.1, 2.2, 3.1, and 3.2.

THEOREM 3.3. *Let $m = 1$, let X be a complex or real linear space, and let $G(x)$ be an indefinite quadratic functional on X, nonlinear in the real case. Then* (3.3) *holds for any quadratic functional $F(x)$ on X.*

THEOREM 3.4. *Let $m = 2$, and let $G_1(x)$ and $G_2(x)$ be quadratic functionals on a complex linear space X. Then the following relations hold for any quadratic functional $F(X)$ on X:* (3.1) *if the constraints $G_1(x) \geqslant 0$ and $G_2(x) \geqslant 0$ are regular;*

$$\inf_{\substack{G_1(x)\geqslant 0\\ G_2(x)=0}} F(x) = \sup_{\substack{\tau_1\geqslant 0\\ \tau_2\in R^1}} \inf_{x\in X} [F(x) - \tau_1 G_1(x) - \tau_2 G_2(x)], \tag{3.4}$$

if the constraints $G_1(x) \geqslant 0$ *and* $G_2(x) = 0$ *are regular; and* (3.3) *if the constraints* $G_1(x) = 0$ *and* $G_2(x) = 0$ *are regular and at least one of the functionals* $G_1(x)$ *or* $G_2(x)$ *is nonlinear.*

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