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Restricted Frequency Inequality is equivalent to Restricted Dissipativity

Tetsuya Iwasaki Shinji Hara Alexander L. Fradkov

Abstract— **A variety of powerful tools and results in systems and control theory rely on classical Kalman-Yakubovich-Popov-Zames results establishing equivalence between special frequency domain inequalities (FDIs), linear matrix inequalities (LMIs) and time domain inequalities (TDIs). Recent developments have addressed FDIs within (semi)finite frequency ranges to increase flexibility in the system analysis and synthesis. In this paper it is shown that validity of a general FDI within a restricted frequency range is equivalent to validity of the corresponding TDI under rate limitations specified by a matrix-valued integral quadratic constraint. The latter property of a system is termed "restricted dissipativity". Its special cases are "restricted passivity" and "restricted finite gain property". The equivalence between restricted FDI and restricted dissipativity is established for both continuous-time and discrete-time settings. The paper together with the previous developments extends Kalman-Yakubovich-Popov-Zames FDI-LMI-passivity results to FDIs specified within restricted frequency ranges.**

I. INTRODUCTION

An important chapter of the modern systems and control theory is formed by frequency domain methods. These methods have been used as a tool to bring mathematical formality to engineering specifications in terms of gain and phase in the frequency domain. As soon as such specifications are expressed in form of the frequency domain inequalities (FDIs) they can be transformed into linear matrix inequalities (LMI) using celebrated Kalman-Yakubovich-Popov lemma (KYP lemma). The KYP lemma establishes that validity of the FDI

$$\begin{bmatrix} (j\omega I-A)^{-1}B \\ I \end{bmatrix}^* \Pi \begin{bmatrix} (j\omega I-A)^{-1}B \\ I \end{bmatrix} \le 0 \qquad (1)$$

for all real ω such that $\det(j\omega I-A)\neq 0$ is equivalent to solvability of the LMI

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Pi \le 0 \qquad (2)$$

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T. Iwasaki is with the Department of Mechanical and Aerospace Engineering, University of Virginia, 122 Engineer's Way, Charlottesville, VA 22904-4746, USA, iwasaki@virginia.edu

S. Hara is with the Department of Information Physics and Computing, The University of Tokyo, 7-3-1 Hongo, Bunkyo, Tokyo 113-8656, Japan, Shinji_Hara@ipc.i.u-tokyo.ac.jp

A. L. Fradkov is with the Institute for Problems of Mechanical Engineering, Control of Complex Systems Laboratory, V.O., Bolshoy, 61, St. Petersburg, 199178, Russia, alf@control.ipme.ru

for a symmetric matrix P. In (1) and (2), matrices A and B define a linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t\in[0,\infty), \quad x(0)=0 \qquad (3)$$

with m input and n state variables; Π is a $(n+m)\times(n+m)$ symmetric matrix defining a quadratic constraint

$$\begin{bmatrix} x \\ u \end{bmatrix}^* \Pi \begin{bmatrix} x \\ u \end{bmatrix} \le 0.$$

Special cases of (1) correspond to important classes of positive real and bounded real systems.

The time domain interpretations of FDIs provide more flexibility in capturing various engineering requirements. They are intimately related to passivity and dissipativity concepts first introduced by Popov and Zames in the 1960s. Based on the KYP lemma it has been established that the FDI (1) is also equivalent to validity of the time domain inequality (TDI)

$$\int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^* \Pi \begin{bmatrix} x \\ u \end{bmatrix} dt \le 0 \qquad (4)$$

for all solutions of (3) with $u\in\mathcal{L}_2,\ x\in\mathcal{L}_2$. In particular, positive real and bounded real properties are equivalent to passivity and finite gain properties respectively, in the time domain [1], [2]. Similar results hold for discrete-time systems.

A drawback of the classical FDI framework is the fact that the FDI conditions are considered on the entire frequency range, while many of practical design specifications are given in terms of FDIs *within some restricted frequency ranges*. Some recent results [3]–[8] have addressed this issue and generalized the standard KYP lemma [9]–[11] to characterize FDIs within (semi)finite frequency ranges in terms of linear matrix inequalities (LMIs). These results are based on the idea of the S-procedure [12], [13], and are closely related to the literature on integral quadratic constraints [14], [15], indefinite linear quadratic control [16], and power distribution inequality [17].

The objective of this paper is to establish equivalence of the FDI and the TDI for restricted frequency range for both continuous-time and discrete-time setting, extending a prior result in [4] for the case of continuous-time, low frequency positive-realness. It will be shown that the frequency restriction in FDIs translates to a restriction on the class of input signals for which time domain inequalities (TDIs) are to hold. For instance, the bounded-real property holds for all ω within a low frequency range if and only if the corresponding TDI holds for all inputs $u \in \mathcal{L}_2$ that drive the system "slowly" (a precise definition for the "slowness" will be given).

We use the following notation. The set of square integrable (summable) signals on $[0, \infty)$ for continuous time is denoted by $\mathcal{L}_2$. The corresponding set for discrete time is denoted by ℓ_2. The Fourier transform of a signal x is denoted by $\hat{x}$. For a square matrix M, its Hermitian part is defined by $\mathrm{He}(M) := (M+M^*)/2$. For a Hermitian matrix M, its maximum eigenvalue is denoted by $\lambda_{\max}(M)$. The operators $\Re(\cdot)$ and $\Im(\cdot)$ take the real and imaginary parts of the arguments, respectively. The set of positive integers up to n is denoted by $\mathcal{I}_n$. The interior of a set $\mathbf{\Omega}$ is denoted by int $\mathbf{\Omega}$.

II. The continuous-time case

Let us first present a result instrumental to the proof of our main results. In particular, we present a special case of the generalized KYP lemma [5] that characterizes an FDI for a rational function on a segment (or segments) of the imaginary axis, in terms of an LMI. The procedure for specialization has already been outlined in [5] and hence the result is presented without a proof here.

Theorem 1: Let complex matrices A, B, Π, and real scalars ϖ_1, ϖ_2 be given. Let τ be $+1$ or -1, and define

$$\mathbf{\Omega} := \{ \ \omega \in \mathbb{R} \mid \tau(\omega - \varpi_1)(\omega - \varpi_2) \leq 0 \ \}. \qquad (5)$$

Suppose Π is Hermitian, (A, B) is controllable, and $\mathbf{\Omega}$ has a nonempty interior. Then the following statements are equivalent.

(i) The frequency domain inequality

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Pi \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \leq 0 \qquad (6)$$

holds for all $\omega \in \mathbf{\Omega}$ such that $\det(j\omega I - A) \neq 0$.

(ii) There exist Hermitian matrices P and Q such that $\tau Q \geq 0$ and

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} -Q & P + j\varpi_o Q \\ P - j\varpi_o Q & -\varpi_1\varpi_2 Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Pi \leq 0 \qquad (7)$$

hold, where $\varpi_o := (\varpi_1 + \varpi_2)/2$.

We mention that similar results have also been obtained in [3], [4]. Choosing the parameters $\varpi_1 = \varpi_2 = 0$ and $\tau = -1$, the set $\mathbf{\Omega}$ becomes the entire real numbers, and thus statement (i) becomes the FDI for all frequencies. In this case, the term associated with Q in the LMI (7) becomes positive semidefinite, and hence the best choice of Q for satisfaction of (7) is $Q = 0$. The resulting LMI with variable P is exactly the same as the one in [10], and thus Theorem 1 reduces to the standard KYP lemma.

We now provide a physical interpretation of the frequency domain inequality in terms of a time domain characterization of the input-to-state behavior. The following result gives a necessary and sufficient condition for a general FDI to hold within the frequency interval $\mathbf{\Omega}$, in terms of a TDI over a restricted class of input signals.

Theorem 2: Let complex matrices A, B, Π, and real scalars ϖ_1, ϖ_2 be given. Let τ be $+1$ or -1, and define $\mathbf{\Omega}$ by (5). Consider the system (3) where $x(t) \in \mathbb{C}^n$ is the state and $u(t) \in \mathbb{C}^m$ is the input. Assume that the system is asymptotically stable and controllable, Π is Hermitian, and $\mathbf{\Omega}$ has a nonempty interior. Then the following statements are equivalent.

(i) The frequency domain inequality (6) holds for $\omega \in \mathbf{\Omega}$.

(ii) The time domain inequality

$$\int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^* \Pi \begin{bmatrix} x \\ u \end{bmatrix} dt \leq 0 \qquad (8)$$

holds for all solutions of (3) with $u \in \mathcal{L}_2$ such that

$$\int_0^\infty \tau(\varpi_1 x + \jmath\dot{x})(\varpi_2 x + \jmath\dot{x})^* dt \leq 0. \qquad (9)$$

Proof: Suppose (i) holds. Then, from Theorem 1, inequality (7) holds for some $P = P^*$ and $Q = Q^*$ such that $\tau Q \geq 0$. Multiplying the inequality by $[\ x^* \ \ u^*\ ]$ from the left and by its conjugate transpose from the right, we have

$$\frac{d}{dt}(x^* P x) + \begin{bmatrix} x \\ u \end{bmatrix}^* \Pi \begin{bmatrix} x \\ u \end{bmatrix}$$
$$-\mathrm{tr}\ [\mathrm{He}((\varpi_1 x + \jmath\dot{x})(\varpi_2 x + \jmath\dot{x})^*)Q] \leq 0.$$

Integrating from $t = 0$ to ∞ using the stability property, we have

$$\int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^* \Pi \begin{bmatrix} x \\ u \end{bmatrix} dt \leq \mathrm{tr}\ [\mathrm{He}(S)Q] \qquad (10)$$

where

$$S := \int_0^\infty (\varpi_1 x + \jmath\dot{x})(\varpi_2 x + \jmath\dot{x})^* dt. \qquad (11)$$

By the Parseval's theorem [18], we have

$$S = \frac{1}{2\pi}\int_{-\infty}^\infty (\varpi_1 - \omega)(\varpi_2 - \omega)\hat{x}\hat{x}^* d\omega$$

and hence S is Hermitian and the bound on the right hand side of (10) becomes tr(SQ). Since $\tau Q \geq 0$, the bound tr(SQ) is nonpositive whenever $\tau S \leq 0$ holds, and thus we have statement (ii).

To show (ii) $\Rightarrow$ (i), let us first consider the single input case ($m = 1$). Suppose (i) is false. Then there exists $\omega_\star \in \mathbf{\Omega}$ such that $H(\omega_\star) > 0$ where $H(\omega)$ is defined to be the left hand side of (6). Since the system is stable, $H(\omega)$ is a continuous function of ω and hence a small perturbation of $\omega_\star$ does not alter the sign of $H(\omega)$. Thus we can assume that $\omega_\star$ belongs to int $\mathbf{\Omega}$, i.e., $\varpi_1 \neq \omega_\star \neq \varpi_2$ without loss of generality. Moreover, one can choose distinct frequencies ω_i ($i \in \mathcal{I}_n$) in the neighborhood of $\omega_\star$ so that $\omega_i \in$ int $\mathbf{\Omega}$ and $H(\omega_i) > 0$.

Define the input signal $u_\epsilon(t)$, where $\epsilon > 0$ is to be chosen later by

$$u_\epsilon(t) := \begin{cases} \sum_{i=1}^{n} e^{\jmath\omega_i t} & (0 \le t \le 1/\epsilon), \\ 0 & (1/\epsilon < t). \end{cases} \tag{12}$$

The corresponding state response $x_\epsilon(t)$ is given by

$$x_\epsilon(t) := \begin{cases} \sum_{i=1}^{n} \left(e^{\jmath\omega_i t} I - e^{At}\right) v_i & (0 \le t \le 1/\epsilon), \\ e^{A(t-1/\epsilon)}\zeta_\epsilon & (1/\epsilon < t) \end{cases} \tag{13}$$

where

$$v_i := (\jmath\omega_i I - A)^{-1}B, \quad \zeta_\epsilon := \sum_{i=1}^{n} \left(e^{\jmath\omega_i/\epsilon} I - e^{A/\epsilon}\right) v_i.$$

Let $\phi(\epsilon)$ be the left hand side of the time-domain inequality (8) evaluated along the process $x_\epsilon(t)$. Then

$$\begin{aligned} \phi(\epsilon) &:= \int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^* \Pi \begin{bmatrix} x \\ u \end{bmatrix} dt \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} \int_0^{1/\epsilon} e^{\jmath(\omega_j-\omega_i)t} dt + \mu(\epsilon) \end{aligned}$$

where $\mu(\epsilon)$ is the collection of terms related to e^{At} and

$$\sigma_{ij} := \begin{bmatrix} v_i \\ 1 \end{bmatrix}^* \Pi \begin{bmatrix} v_j \\ 1 \end{bmatrix}.$$

In view of Lemma 1 in Appendix, $\mu(\epsilon)$ is uniformly bounded over $\epsilon > 0$ due to stability of A. Noting that

$$\int_0^t e^{\jmath(\omega_j-\omega_i)\tau} d\tau = \begin{cases} t & (i = j), \\ \dfrac{e^{\jmath(\omega_j-\omega_i)t} - 1}{\jmath(\omega_j - \omega_i)} & (i \ne j) \end{cases}$$

the first term of $\phi(\epsilon)$ can be expressed as

$$\frac{1}{\epsilon}\sum_{i=1}^{n} \sigma_{ii} + \sum_{i \ne j} \frac{e^{\jmath(\omega_j-\omega_i)/\epsilon} - 1}{\jmath(\omega_j - \omega_i)} \sigma_{ij}.$$

Clearly, the second term above is uniformly bounded over $\epsilon > 0$. Therefore, $\phi(\epsilon)$ can be given by

$$\phi(\epsilon) = \frac{1}{\epsilon}\sum_{i=1}^{n} \sigma_{ii} + \varphi(\epsilon)$$

where $\varphi(\epsilon)$ is uniformly bounded. Noting that $\sigma_{ii} = H(\omega_i) > 0$, we now see that $\phi(\epsilon) > 0$ holds for sufficiently small $\epsilon > 0$.

Next we show that S defined in (11) is such that $\tau S \le 0$ for sufficiently small $\epsilon > 0$. Let us first split S into two pieces as follows:

$$S = S_1 + S_2, \quad S_1 := \int_0^{1/\epsilon} Z(t)dt, \quad S_2 := \int_{1/\epsilon}^\infty Z(t)dt,$$

$$Z := (\varpi_1 x + \jmath\dot{x})(\varpi_2 x + \jmath\dot{x})^*.$$

By direct substitution of (13) and its derivative, we have

$$S_1 = \sum_{i=1}^{n}\sum_{j=1}^{n} \int_0^{1/\epsilon} \xi_i(\varpi_1)\xi_j(\varpi_2)^* dt,$$

$$S_2 = \int_{1/\epsilon}^\infty \Phi(\varpi_1)\Phi(\varpi_2)^* dt,$$

where

$$\xi_i(\varpi) = \left((\varpi - \omega_i)e^{\jmath\omega_i t} I - (\varpi I + \jmath A)e^{At}\right) v_i,$$

$$\Phi(\varpi) := (\varpi I + \jmath A)e^{A(t-1/\epsilon)}\zeta_\epsilon.$$

As before, every term in S_1 associated with e^{At} is uniformly bounded over $\epsilon > 0$ due to stability of A (Lemma 1). The term associated with $e^{\jmath(\omega_i-\omega_j)t}$ will also be uniformly bounded if $i \ne j$; otherwise, it grows linearly with respect to $1/\epsilon$. It can be shown by Lemma 1 that $\|\zeta_\epsilon\|$ is uniformly bounded, and hence so is S_2. Therefore, S can be written as

$$S = \frac{1}{\epsilon} S_\star + S_o(\epsilon), \quad S_\star := \sum_{i=1}^{n} (\varpi_1 - \omega_i)(\varpi_2 - \omega_i) v_i v_i^*$$

where $S_o(\epsilon)$ is uniformly bounded over $\epsilon > 0$. To conclude that $\tau S \le 0$ for small $\epsilon > 0$, it suffices to show that $\tau S_\star < 0$. Now, substituting the definition of v_i, we have

$$\tau S_\star(\epsilon) = \sum_{i=1}^{n} \tau_i (\jmath\omega_i I - A)^{-1} BB^* (\jmath\omega_i I - A)^{-1*}$$

where

$$\tau_i := \tau(\varpi_1 - \omega_i)(\varpi_2 - \omega_i)$$

is strictly negative due to $\omega_i \in \mathrm{int}\, \mathbf{\Omega}$. Hence $\tau S_\star$ is negative definite due to controllability of (A, B); see Theorem 1 of Section 34 in [19]. This completes the proof of (ii) $\Rightarrow$ (i) for the single input case.

We now give a proof for the multiple input case ($m > 1$) using the result for the single input case. Suppose (i) is false. Then there exists $\omega_\star \in \mathrm{int}\, \mathbf{\Omega}$ such that $\lambda_{\max}[H(\omega_\star)] > 0$. By controllability of (A, B), there exists K such that $\mathcal{A} := A + BK$ has distinct eigenvalues (sufficiently close to those of A) and $\|K\|$ is small enough to make $\mathcal{A}$ stable and $\lambda_{\max}[\mathcal{H}(\omega_\star)]$ positive, where

$$\mathcal{H}(\omega) := \Gamma^*\Pi\Gamma, \quad \Gamma := \begin{bmatrix} I & 0 \\ K & I \end{bmatrix} \begin{bmatrix} (\jmath\omega I - \mathcal{A})^{-1}B \\ I \end{bmatrix}.$$

Let $\eta_\star$ be a vector such that $\eta_\star^*\mathcal{H}(\omega_\star)\eta_\star > 0$. Then, from Lemma 2, there exists a vector η such that $(\mathcal{A}, \mathcal{B})$ with $\mathcal{B} := B\eta$ is controllable and $\|\eta - \eta_\star\|$ is sufficiently small so that

$$\eta^*\mathcal{H}(\omega_\star)\eta = \begin{bmatrix} (\jmath\omega_\star I - \mathcal{A})^{-1}\mathcal{B} \\ 1 \end{bmatrix}^* \mathbb{I}\mathbb{I} \begin{bmatrix} (\jmath\omega_\star I - \mathcal{A})^{-1}\mathcal{B} \\ 1 \end{bmatrix}$$

is positive, where

$$\mathbb{I}\mathbb{I} := \begin{bmatrix} I & 0 \\ K & \eta \end{bmatrix}^* \Pi \begin{bmatrix} I & 0 \\ K & \eta \end{bmatrix}.$$

By the implication (ii) $\Rightarrow$ (i) for the single input case, there exists an input $\mathsf{u} \in \mathcal{L}_2$ that drives the system

$$\dot{\mathsf{x}} = \mathcal{A}\mathsf{x} + \mathcal{B}\mathsf{u}, \quad \mathsf{x}(0) = 0$$

such that

$$\phi := \int_0^\infty \begin{bmatrix} \mathsf{x} \\ \mathsf{u} \end{bmatrix}^* \amalg \begin{bmatrix} \mathsf{x} \\ \mathsf{u} \end{bmatrix} dt > 0,$$

$$\tau S := \int_0^\infty \tau(\varpi_1 \mathsf{x} + \jmath\dot{\mathsf{x}})(\varpi_2 \mathsf{x} + \jmath\dot{\mathsf{x}})^* dt \leq 0.$$

Let $u := \eta\mathsf{u} + Kx$. Then

$$\dot{x} = Ax + B(\eta\mathsf{u} + Kx) = \mathcal{A}x + \mathcal{B}\mathsf{u}, \quad x(0) = 0$$

and hence $x(t) \equiv \mathsf{x}(t)$. We now see that

$$\int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^* \Pi \begin{bmatrix} x \\ u \end{bmatrix} dt = \phi > 0$$

$$\int_0^\infty \tau(\varpi_1 x + \jmath\dot{x})(\varpi_2 x + \jmath\dot{x})^* dt = \tau S \leq 0.$$

Thus statement (ii) does not hold. ∎

The property specified by statement (ii) of Theorem 2 may be termed "restricted dissipativity." Its special cases are "restricted passivity" and "restricted finite gain property". Theorem 2 can be reduced to classical results such as TDI characterizations of bounded-real and positive-real transfer functions discussed in Section I. In particular, when $\varpi_1 = \varpi_2 = 0$ and $\tau = -1$, the constraint (9) is automatically satisfied and the set $\mathbf{\Omega}$ becomes the entire frequency range. Thus Theorem 2 states that FDI (6) holds for all frequency ω if and only if TDI (8) holds for all input $u \in \mathcal{L}_2$. To recover the bounded-real and positive-real results, one can choose, for $G(s) := C(sI - A)^{-1}B + D$,

$$\Pi = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* M \begin{bmatrix} C & D \\ 0 & I \end{bmatrix},$$

$$M = \begin{bmatrix} I & 0 \\ 0 & -\gamma I \end{bmatrix} \quad \text{(bounded real)},$$

$$M = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix} \quad \text{(positive real)}.$$

In Theorem 2, a general frequency interval $\mathbf{\Omega}$ is considered for the FDI, and this has translated to the input constraint described by (9). The physical meaning of this constraint is not clear in this general setting, but will become clear when special cases are considered as follows.

Corollary 1: Let real matrices A, B, Π, and a positive scalar ϖ be given. Suppose Π is symmetric and consider the system (3) where $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the input. Assume that the system is stable and controllable. The following statements are equivalent.

(i) The frequency domain inequality (6) holds for all ω such that $|\omega| \leq \varpi$.

(ii) The time domain inequality (8) holds for all $u \in \mathcal{L}_2$ such that

$$\int_0^\infty \dot{x}\dot{x}^\mathsf{T} dt \leq \varpi^2 \int_0^\infty x x^\mathsf{T} dt. \tag{14}$$

Moreover, the above two statements are equivalent when the two inequalities "$\leq$" are replaced by "$\geq$."

Proof: By specializing Theorem 2 for the case where $\varpi_1 = -\varpi_2 = \varpi$ and $\tau = 1$, we have the "$\leq$" result. Similarly, the choice of $\varpi_1 = -\varpi_2 = \varpi$ and $\tau = -1$ gives the "$\geq$" result. It should be noted, however, that the results obtained in this way would be directly valid only for systems with complex input/output signals. This is because the worst case input u in (12) is not real valued in the time domain and cannot be used here to prove (ii) $\Rightarrow$ (i). To fix the problem, one can modify u as

$$u(t) = 2\sum_{i=1}^n \cos\omega_i t = \sum_{i=1}^{2n} e^{j\omega_i t}, \quad (0 \leq t \leq 1/\epsilon)$$

where $\omega_{n+i} = -\omega_i$ ($i \in \mathcal{I}_n$). Note that $H(\omega_i) > 0$ and $\omega_i \in \text{int } \mathbf{\Omega}$ hold for all $i \in \mathcal{I}_{2n}$ because $H(\omega) = H(-\omega)$ and $\mathbf{\Omega}$ is "symmetric" about the origin, i.e., $\omega \in \mathbf{\Omega}$ implies $-\omega \in \mathbf{\Omega}$. Then, the rest of the proof is exactly the same as before. We just remark that $\eta_\star$ and η can always be chosen to be real, in the proof of the multiple input case. ∎

Roughly speaking, the first part of Corollary 1 states that the FDI in the low finite frequency range means that the system possesses the property (8) with respect to the input signal u that does not drive the states too quickly where the bound on the "quickness" is given by ϖ in the sense of (14). The second part of Corollary 1 makes a similar statement for the FDI in the high frequency range. A special case of Corollary 1 has been obtained in [4] where a time domain characterization is given for positive realness in the low frequency range.

It should be noted that condition (14) is a matrix-valued integral quadratic constraint, and is coordinate free, i.e. if (14) holds for some minimal realization of the system with states x, then it holds for all other minimal realizations with states Tx where T is an arbitrary nonsingular matrix.

III. THE DISCRETE-TIME CASE

In this section, we shall present TDI characterizations of FDIs in the discrete-time setting which parallel their continuous-time counterparts given in the previous section. The following result is a special case of the generalized KYP lemma [5], and provides an LMI condition for an FDI to hold on an arc of the unit circle.

Theorem 3: Let complex matrices A, B, Π, and real scalars $\bar{\vartheta}_1$, $\bar{\vartheta}_2$ be given. Suppose Π is Hermitian, (A, B) is controllable, and $0 < \bar{\vartheta}_2 - \bar{\vartheta}_1 \leq 2\pi$. Define

$$\mathbf{\Theta} := \{ \theta \in \mathbb{R} \mid \bar{\vartheta}_1 \leq \theta \leq \bar{\vartheta}_2 \}. \tag{15}$$

The following statements are equivalent.

(i) The frequency domain inequality

$$\begin{bmatrix} (e^{j\theta}I-A)^{-1}B \\ I \end{bmatrix}^* \Pi \begin{bmatrix} (e^{j\theta}I-A)^{-1}B \\ I \end{bmatrix} \leq 0 \quad (16)$$

holds for all $\theta \in \mathbf{\Theta}$ such that $\det(e^{j\theta}I-A) \neq 0$.

(ii) There exist Hermitian matrices P and Q such that $Q \geq 0$ and

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} -P & e^{j\bar{\vartheta}_o}Q \\ e^{-j\bar{\vartheta}_o}Q & P-\gamma Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Pi \leq 0 \quad (17)$$

hold, where $\bar{\vartheta}_o := (\bar{\vartheta}_1+\bar{\vartheta}_2)/2$, $\bar{\vartheta} := (\bar{\vartheta}_2-\bar{\vartheta}_1)/2$, and $\gamma := 2\cos\bar{\vartheta}$.

If we let $\bar{\vartheta}_1 = 0$ and $\bar{\vartheta}_2 = 2\pi$ in Theorem 3, we recover the standard discrete-time KYP lemma. In particular, we have $e^{\pm j\bar{\vartheta}_o} = -1$ and $\gamma = -2$, and the inequality in statement (ii) can be written as

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} -X+Q & -Q \\ -Q & X+Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Pi \leq 0$$

where $X := P+Q$. Since $Q \geq 0$, the term associated with Q is positive semidefinite. Hence, matrix X satisfies this inequality for some $Q \geq 0$ if and only if it does so for $Q = 0$. The condition obtained by letting $Q = 0$ is exactly the one appeared in the standard discrete-time KYP lemma in [10].

We now present a time-domain interpretation of the FDI in the discrete-time setting.

Theorem 4: Let complex matrices A, B, Π, and real scalars $\bar{\vartheta}_1$, $\bar{\vartheta}_2$ be given. Define $\mathbf{\Theta}$ by (15). Suppose Π is Hermitian and $0 < \bar{\vartheta}_2 - \bar{\vartheta}_1 \leq 2\pi$. Consider the system

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0,1,2,\ldots, \quad x_0 = 0, \quad (18)$$

where $x_k \in \mathbb{C}^n$ is the state, and $u_k \in \mathbb{C}^m$ is the input. Assume that the system is asymptotically stable and controllable. The following statements are equivalent.

(i) The frequency domain inequality (16) holds for $\theta \in \mathbf{\Theta}$.

(ii) The time domain inequality

$$\sum_{k=0}^{\infty} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^* \Pi \begin{bmatrix} x_k \\ u_k \end{bmatrix} \leq 0 \quad (19)$$

holds for all solutions of (18) with $u \in \ell_2$ such that

$$e^{j\bar{\vartheta}} \sum_{k=0}^{\infty} (x_{k+1} - e^{j\bar{\vartheta}_1}x_k)(x_{k+1} - e^{j\bar{\vartheta}_2}x_k)^* \leq 0 \quad (20)$$

where $\bar{\vartheta} := (\bar{\vartheta}_2 - \bar{\vartheta}_1)/2$.

Proof: The main idea for proving Theorem 4 is the same as its continuous-time counterpart. The implication (i) $\Rightarrow$ (ii) can be shown using the result of Theorem 3. To show the converse (ii) $\Rightarrow$ (i), one can use the worst case input of the form:

$$u_k := \begin{cases} \displaystyle\sum_{i=1}^{n} e^{j\theta_i k} & (0 \leq k < l), \\ 0 & (l \leq k) \end{cases} \quad (21)$$

with a sufficiently large integer $l > 0$. It can be shown that infeasibility of (i) implies existence of θ_i such that this input satisfies (20) but violates (19). ■

The following are special cases of Theorem 4 where FDIs in the low/high frequency ranges are considered.

Corollary 2: Let real matrices A, B, Π, and a positive scalar $\bar{\vartheta}$ be given. Suppose Π is symmetric and $0 < \bar{\vartheta} \leq \pi$. Consider the system (18) where $x_k \in \mathbb{R}^n$ is the state and $u_k \in \mathbb{R}^m$ is the input. Assume that the system is stable and controllable. The following statements are equivalent.

(i) The frequency domain inequality (16) holds for all $\theta \in [-\pi,\pi]$ such that $|\theta| \leq \bar{\vartheta}$.

(ii) The time domain inequality (19) holds for $x_0 = 0$ and all $u \in \ell_2$ such that

$$\sum_{k=0}^{\infty} (x_{k+1}-x_k)(x_{k+1}-x_k)^{\mathsf{T}} \leq \left(2\sin\frac{\bar{\vartheta}}{2}\right)^2 \sum_{k=0}^{\infty} x_k x_k^{\mathsf{T}}. \quad (22)$$

Moreover, the above two statements are equivalent when the two inequalities "≤" are replaced by "≥."

Proof: The result basically follows from Theorem 4 by choosing $\bar{\vartheta}_2 := \bar{\vartheta}$ and $\bar{\vartheta}_1 := -\bar{\vartheta}$ for the "≤" case and $\bar{\vartheta}_2 := 2\pi - \bar{\vartheta}$ and $\bar{\vartheta}_1 := \bar{\vartheta}$ for the "≥" case. As in the continuous-time case, the worst case input u_k can be made real-valued with the following modification:

$$u_k = 2\sum_{i=1}^{n} \cos\theta_i k = \sum_{i=1}^{2n} e^{j\theta_i k}, \quad (0 \leq k \leq l)$$

where $\theta_{n+i} = -\theta_i$ ($i \in \mathcal{I}_n$). The result can then be proved using this worst case input in a manner similar to the proof of Theorem 4. ■

A comment similar to its continuous-time counterpart applies to Theorem 4. In particular, it shows that the finite frequency condition in statement (i) is equivalent to requiring the property in (19) for any input u such that the rate of change of the state $x_{k+1} - x_k$ is bounded as in (22). As the frequency bound $\bar{\vartheta}$ reduces from π to 0, the rate bound $4\sin^2(\bar{\vartheta}/2)$ reduces from 4 to 0. When $\bar{\vartheta} = 0$, condition (22) becomes $x_{k+1} = x_k$ for all k. On the other hand, when $\bar{\vartheta} = \pi$, condition (22) holds for any square summable sequence x. This can readily be seen once we notice that

$$4\sum_{k=0}^{\infty} x_k x_k^{\mathsf{T}} - \sum_{k=0}^{\infty}(x_{k+1}-x_k)(x_{k+1}-x_k)^{\mathsf{T}} = \sum_{k=0}^{\infty}(x_{k+1}+x_k)(x_{k+1}+x_k)^{\mathsf{T}} \geq 0.$$

In this case, Corollary 2 reduces to the classical result that FDI (16) holds for all frequency θ if and only if TDI (19) holds for all input $u \in \ell_2$.

IV. Concluding remarks

This paper together with the results of the previous papers [3], [4] extends classical Kalman-Yakubovich-Popov-Zames FDI-LMI-passivity results for FDIs specified within restricted frequency ranges. In particular, we have shown that interpretations to FDIs on (semi)finite frequency ranges can be given in terms of TDIs valid under rate limitations specified by a matrix-valued integral quadratic constraint.

The results of the paper are also strongly related to the results on dynamic integral quadratic constraints (DIQCs) [20], [21]. Namely, the matrix-valued integral quadratic inequalities (9) and (20) can be interpreted as a special case of DIQCs in the time domain specifying rate limitations for the system solutions. Using these previous results, one may reduce control design problems with specifications in a restricted frequency range to an extended set of LMIs and then utilize numerically efficient methods to solve them.

Control design problems with specifications in a restricted frequency range appear in a variety of applications, ranging from the PID servo tracking, mixed sensitivity problem, digital filtering, to integrated design of structural/control systems [5]. The authors believe that the proposed approach opens a new avenue of research in systems and control theory. Its further development may extend the existing LMI-technology to a broad class of controller design problems with specifications in a restricted frequency range.

APPENDIX

Lemma 1: Let square matrices A and M be given. Suppose A is a Hurwitz matrix and define

$$P := \int_0^\infty e^{A^* t} e^{At} dt.$$

Then the following bounds hold for all $t \geq 0$:

(a) $\|e^{At}\| \leq \sqrt{\mathrm{cond}(P)}$

(b) $\left\| \int_0^t e^{A\tau} d\tau \right\| \leq \|A^{-1}\| \left(1 + \sqrt{\mathrm{cond}(P)}\right)$

(c) $\left\| \int_0^t e^{A^*\tau} M e^{A\tau} d\tau \right\| \leq \|P\| \dfrac{\|M + M^*\| + \|M - M^*\|}{2}$

where $\mathrm{cond}(\cdot)$ is the condition number. Similar bounds for discrete time case are obtained by substituting $e^A = \bar{A}$ and replacing integration with summation.

Lemma 2: Let matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$ be given. Suppose (A, B) is controllable and A has distinct eigenvalues. Then, for each $\epsilon > 0$ and $u_\star \in \mathbb{C}^n$, there exists $u \in \mathbb{C}^n$ such that (A, Bu) is controllable and $\|u - u_\star\| < \epsilon$.

Proof: The result basically follows from Proposition 2 and its proof in Section 34 of [19]. Since A has distinct eigenvalues, there exists a similarity transformation such that $\mathcal{A} := TAT^{-1}$ is diagonal. Since controllability is preserved under similarity transformation, $(\mathcal{A}, \mathcal{B})$ is controllable where $\mathcal{B} := TB$, which implies that every row of $\mathcal{B}$ is nonzero. It then follows from Lemma 3 that there exists a vector u such that every entries of $\mathcal{B}u$ are nonzero and $\|u - u_\star\| < \epsilon$. For such u, $(\mathcal{A}, \mathcal{B}u)$ is controllable, which in turn implies that (A, Bu) is controllable. ■

Lemma 3: Let a vector $u_\star \in \mathbb{C}^m$ and nonzero vectors $b_i \in \mathbb{C}^m$ $(i \in \mathcal{I}_n)$ be given. Then for each $\epsilon > 0$, there exists $u \in \mathbb{C}^m$ such that $\|u - u_\star\| < \epsilon$ and $b_i^* u \neq 0$ for all $i \in \mathcal{I}_n$.

Proof: We first show by induction that there is a $w \in \mathbb{C}^m$ such that $b_i^* w \neq 0$ for all $i \in \mathcal{I}_n$. Suppose we have $k \in \mathcal{I}_n$ and w_k such that $b_i^* w_k \neq 0$ for all $i \in \mathcal{I}_k$. Then $w_{k+1} := w_k + \delta b_{k+1}$ satisfies $b_i^* w_{k+1} \neq 0$ for all $i \in \mathcal{I}_{k+1}$ for sufficiently small δ. Hence, starting from $w_1 := b_1$ and applying the argument repeatedly, we have $w := w_n$ such that $b_i^* w \neq 0$ for all $i \in \mathcal{I}_n$. Now, choosing $u := u_\star + \varepsilon w$ with sufficiently small ε, we have the result. ■

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