SWINGING UP OF SIMPLIFIED FURUTA PENDULUM

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Abstract

The paper contains a detailed analysis of the application of the passification approach to the problem of rendering the hyperbolic upright equilibrium of the simplified model of the Furuta pendulum globally attractive. It is shown that any smooth feedback control passifying the system with the naturally defined storage functions does not provide the desired property. Based on the idea of the VSS-like modification of the speed-gradient method, stabilizing regulator is suggested and studied both theoretically and by simulations.

1 Introduction

The pendulum seems to be the simplest benchmark example which can demonstrate the complex nonlinear behaviour. One of the problems naturally arising with the pendulum is the stabilization of its stationary points, the downward and upright positions. The asymptotic stabilization of the downward position of the controlled pendulum can be achieved by the methods based on decreasing the energy of the system which are well studied for general Lagrangian and Hamiltonian systems, see [6, 7]. Recently similar in spirit approach based on the speed-gradient method and energy-like Lyapunov functions was proposed for swinging a Hamiltonian system (e.g. pendulum) up to the desired energy level [3, 4]. However the direct application of the energy-speed-gradient algorithm [3] for the stabilization of the upright position encounters some difficulties and, as it is shown in [9, 10], does not stabilize the desired upper equilibrium point even in a weak sense, the upright position of the pendulum will not always be a unique ω-limit point of the closed loop system. In [9, 10] the VSS-modification of the energy-speed-gradient method was suggested which renders the upright position to be a unique ω-limit point of the closed loop system.

The present paper is devoted to the swinging up problem for the pendulum-like system which was suggested in [1] as a simplified model of the so called Furuta pendulum [5]. The behaviour of the system crucially depends on some positive parameter a. In particular, if $a > 1$ the system has 4 equilibrium points, while $a \leq 1$ there are just 2 equilibriums. When $a = 0$ the system model coincides with the simple pendulum model. Although the suggested in [1] model does not describe the motion of the Furuta pendulum, it remains of interest due to the following reasons: the system is nonlinear and has a rich set of 'irregular motions': several hyperbolic equilibriums, homoclinic curves; this system serves as a nontrivial example for testing the different swinging strategies; it explicitly reflects the difficulties connected to the stabilization of the set rather than just the equilibrium point.

The contribution of the paper is twofold. First, it is shown that any smooth feedback control derived by the passification approach with the naturally defined storage function, does not result in the stabilization of the upright position of the simplified Furuta pendulum even in a weak sense. Second, discontinuous feedback control is suggested which renders the upright equilibrium globally attractive.

The paper is organized as follows. Section 2 contains the description of the simplified model of the Furuta pendulum and some important properties of this system. The main results of the paper are collected in section 3. The results of the computer simulation are presented in section 4 and some conclusions are given in section 5.

2 Preliminaries

The motions of the simplified Furuta pendulum are described by the equation

$$\ddot{\theta} - \sin\theta - a\cdot\sin\theta\cdot\cos\theta + b\cdot\dot{\theta} = u\cdot\cos\theta, \qquad (2.1)$$

where a, b are positive constants; u is a control function, see [1]. In the further analysis it will be assumed that the damping term in (2.1) is equal to zero, i. e. $b = 0$. After a change of variables

$$\begin{cases} q &= \theta - \pi \\ p &= \dot{\theta} \end{cases} \qquad (2.2)$$

the system (2.1) takes the form

$$\begin{cases} \dot{p} &= -\sin q + a\cdot\sin q\cdot\cos q + R(q,p)\cdot u \\ \dot{q} &= p \end{cases}, \qquad (2.3)$$

where $R(q,p) = -\cos q$. In the sequel we assume that $R(q,p)$ is an arbitrary smooth scalar function.

It is reasonable to take as a phase space of the system motions a cylindrical phase space with a unit circle at the base, i. e. $0 \le q < 2\pi$. Therefore the points (q_1, p) and (q_2, p) if $q_1 - q_2 = 2k\pi$ for some integer k will be considered as identical.

The full energy (or Hamiltonian function) $H_0(q,p)$ of the unforced system (2.3), i. e. with $u = 0$, has the form

$$H_0(q,p) = \frac{1}{2}p^2 + (1-\cos q) + \frac{a}{4}\cdot\cos 2q \qquad (2.4)$$

and is its conserved quantity.

The number of equilibrium points of the unforced system (2.3) depends on the value of the positive parameter a. If $a < 1$ then the unforced system (2.3) has two equilibrium points, the downward and upright positions which are stable (but not asymptotically) and hyperbolic correspondingly. If $a > 1$ there are four equilibrium points $[q,p] = [0,0]$, $[\pm\arccos\frac{1}{a}, 0]$ and $[\pi, 0]$. Moreover, both the downward and upright positions are hyperbolic, but the new equilibria $[\pm\arccos\frac{1}{a}, 0]$ are stable, see also [1]. It is useful to calculate the values of $H_0(q,p)$ corresponding to the equilibria of the unforced system (2.3),

$$H_0(\pm\arccos\frac{1}{a}, 0) = 1 - \frac{a}{4} - \frac{1}{2a}, \qquad (2.5)$$

$$H_0(0,0) = \frac{a}{4}, \qquad (2.6)$$

$$H_0(\pi, 0) = 2 + \frac{a}{4}. \qquad (2.7)$$

In particular, if $a > 1$, i. e. when there exist four equilibria, then

$$H_0(\pm\arccos\frac{1}{a}, 0) < H_0(0,0) < H_0(\pi,0).$$

Let H_* be any constant with

$$H_* \ge \frac{a}{4}, \text{ if } a < 1, \quad \text{and} \quad H_* \ge 1 - \frac{a}{4} - \frac{1}{2a}, \text{ if } a > 1.$$

Introduce the scalar functions

$$V(q,p) = \frac{1}{2}[H_0(q,p) - H_*]^2, \qquad (2.8)$$

$$y(q,p) = p\cdot R(q,p)\cdot[H_0(q,p) - H_*] \qquad (2.9)$$

These functions possess an important property. Namely, system (2.3) together with output function y is *passive* (see [2]) with storage function V.

3 Main Results

The upright position of the system (2.3) possesses an important property which at first glance can provide the stabilizability of this equilibrium. Namely, $[\pi, 0]$ is an unique ω-limit point of the energy level which it belongs to. More precisely

Proposition 3.1 *The set*

$$V^{(2+a/4)} = \left\{[q,p] : H_0(q,p) \equiv 2 + \frac{a}{4}\right\} \qquad (3.1)$$

consists of two homoclinic curves and the upright position of the unforced ($u = 0$) system (2.3). In particular, the set $V^{(2+a/4)} \backslash [\pi, 0]$ coincides simultaneously with the stable and unstable manifolds of $[\pi, 0]$.

Thus to stabilize the upright position one can attempt, using the passivity arguments, to stabilize the compact invariant set $V^{(2+a/4)}$. However this attempt fails as it is demonstrated by the following propositions.

Proposition 3.2 *Suppose $a > 1$ and the set $\{[q,p] : R(q,p) = 0\}$ does not contain any whole trajectory of the unforced system (2.3). Let $H_* = 2 + \frac{a}{4}$ and ϕ be any scalar smooth function with $z\phi(z) > 0\ \forall z \neq 0$, $\phi(0) = 0$. Take the regulator*

$$u = -\phi(y), \qquad (3.2)$$

where y is defined by (2.9). Then for any solution $[q(t), p(t)]$ of the closed loop system (2.3), (3.2), except the unstable equilibriums $[\pm\arccos\frac{1}{a}, 0]$, the following alternatives hold:

- *a). $\lim_{t\to+\infty} V(q(t), p(t)) = 0$. Moreover, for this case, if $[q(0), p(0)] \notin V^{(2+a/4)}$ then the set Ω of all ω-limit points of $[q(t), p(t)]$ coincides with either $[\pi, 0] \cup \Gamma_1$, or $[\pi, 0] \cup \Gamma_2$, or $[\pi, 0] \cup \Gamma_1 \cup \Gamma_2$, where Γ_1, Γ_2 are homoclinic curves of the unforced system (2.3), Γ_1, $\Gamma_2 \in V^{(2+a/4)}$;*

- *b). the trajectory $[q(t), p(t)]$ tends to the hyperbolic equilibrium point $[0, 0]$ as $t \to +\infty$.*

Proposition 3.3 *Suppose $0 < a < 1$ and the set $\{[q,p] : R(q,p) = 0\}$ does not contain any whole trajectory of the unforced system (2.3). Let $H_* = 2+\frac{a}{4}$ and ϕ be any scalar smooth function with $z\phi(z) > 0 \; \forall z \neq 0$, $\phi(0) = 0$. Take the regulator*

$$u = -\phi(y), \tag{3.3}$$

where y is defined by (2.9). Then for any solution $[q(t), p(t)]$ of the closed loop system (2.3), (3.2), except the unstable equilibrium $[0,0]$, the following limit relation

$$\lim_{t \to +\infty} V(q(t), p(t)) = 0$$

holds. Moreover, for this case, if $[q(0), p(0)] \notin V^{(2+a/4)}$ then the set Ω of all ω-limit points of $[q(t), p(t)]$ coincides with either $[\pi, 0] \cup \Gamma_1$, or $[\pi, 0] \cup \Gamma_2$, or $[\pi, 0] \cup \Gamma_1 \cup \Gamma_2$, where Γ_1, Γ_2 are homoclinic curves of the unforced system (2.3), $\Gamma_1, \Gamma_2 \in V^{(2+a/4)}$.

The global stabilization of the upright position of the system (2.3) **(recall that the stabilization of $[\pi,0]$ means to render this point as a *unique* ω-limit point of any solution of the closed loop system and this is different from the asymptotic stability of this point)** might be achieved by the following procedure: first, to construct the feedback control providing that any solution of the closed loop system achieves the set $V^{(2+a/4)}$ in *finite* time. As it was shown in proposition 3.1, the invariant set $V^{(2+a/4)}$ consists exactly of the upright position and two homoclinic curves, which are simultaneously the stable and unstable manifolds of the upright position, i. e. $[\pi, 0]$ is the unique ω-limit and α-limit point of any motion belonging to $V^{(2+a/4)}$. If the closed loop system trajectory achieves the set $V^{(2+a/4)}$ in finite time, then it is possible to turn off the feedback control at this time. The original motion of the unforced system (2.3) brings it to the upright position.

However this attempt, using only continuous feedback control, again fails.

Proposition 3.4 *Consider the controlled system (2.3) with $R(q,p) = -\cos q$. For any continuous feedback control $u = u(q,p)$ defined on $S^1 \times R^1$ the closed loop system has at least one equilibrium point which belongs to the set*

$$\left\{[q,p] : \; H_0(q,p) < 2 + \frac{a}{4}\right\}. \tag{3.4}$$

The next statement suggests the discontinuous feedback control, which result in the stabilization of the upright equilibrium provided $a \in (0,1)$.

Theorem 3.4 *Suppose $0 < a < 1$ and the set $\{[q,p] : R(q,p) = 0\}$ does not contain any whole trajectory of the unforced system (2.3). Let $\varepsilon \in (\frac{a}{4}, 2+\frac{a}{4})$ and ϕ be any scalar smooth function such that $z\phi(z) > 0 \; \forall z \neq 0$, $\phi(0) = 0$. Then the regulator*

$$u = \begin{cases} u^*, \; \textit{if } [q,p] = [0,0] \\ -\phi\left(R(q,p) \cdot p \cdot \left[H_0(q,p) - \left(2 + \frac{a}{4} + \varepsilon\right)\right]\right), \\ \qquad \textit{if } H_0(q,p) < 2 + \frac{a}{4} \\ 0, \; \textit{if } H_0(q,p) = 2 + \frac{a}{4} \\ -\phi\left(R(q,p) \cdot p \cdot \left[H_0(q,p) - \left(2 + \frac{a}{4} - \varepsilon\right)\right]\right), \\ \qquad \textit{if } H_0(q,p) > 2 + \frac{a}{4} \end{cases} \tag{3.5}$$

where u^* is any non-zero constant, globally renders the upright position of the system (2.3) as an unique ω-limit point of the closed loop system (2.3), (3.5).

4 Computer Simulations

In all the simulations[1] the feedback regulator was $\phi(z) = z$. Figure 1 shows the energy level stabilization of proposition 3.2 with $a = 2$. The initial values were $q(0) = \pi/180$, $\dot{q}(0) = 0$. Figure 2 shows the stabilizing of the upright position, using the regulator given in theorem 3.4 with $a = \frac{1}{2}$. The initial values were $q(0) = -\pi/3$, $\dot{q}(0) = 4$.

5 Conclusions

This paper contains a detailed analysis of the application of the passification approach with the objective to render the hyperbolic upright equilibrium of the simplified model of the Furuta pendulum to be globally attractive. It is shown that any smooth feedback control passifying the system with the naturally defined storage functions does not provide the desired property. Based on the idea of the VSS-like modification of the speed-gradient method, stabilizing regulator is suggested. Simulation demonstrates the good convergence rate of the closed loop system.

Note that the proposed regulator allows to achieve the desired goal by means of control having arbitrarily small level (if friction is neglected). It distinguish the passification-based energy control approach [3] from other regulators designs for conservative systems.

6 Appendix

Proof of Proposition 3.1. The linear approximation of the unforced system (2.3) in the equilibrium point $[q,p] = [\pi, 0]$ has the form

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & a+1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

[1]The simulations were made in Matlab 5.1/SIMULINK, a trademark of The MathWorks, Inc.

and

$$\det\left(\lambda I_2 - \left[\begin{array}{cc} 0 & a+1 \\ 1 & 0 \end{array}\right]\right) = \lambda^2 - (a+1).$$

These calculations show that the equilibrium $[\pi, 0]$ is hyperbolic and its stable manifold Γ_s and its unstable manifold Γ_u are 1-dimensional.

The function $H_0(q, p)$ is a conserved quantity of the unforced system (2.3), i. e. it preserves its value along any solution of the unforced system (2.3). One can easily check that the equilibrium $[\pi, 0]$ belongs to the set $V^{(2+a/4)}$. Due to hyberbolity of $[\pi, 0]$ and continuity of $H_0(q, p)$ the stable and unstable manifolds Γ_s, Γ_u belong to the set $V^{(2+a/4)}$. To complete the proof one should show that

$$\text{a). } \Gamma_s = \Gamma_u; \quad \text{b). } \Gamma_s = V_1^{(2+a/4)} \setminus [\pi, 0]. \tag{6.1}$$

Given $[q_0, p_0] \in \Gamma_u$, consider the solution $[q(t), p(t)]$ of the unforced system (2.3) stating in this point. Obviously $[q(t), p(t)] \subset V^{(2+a/4)}$ and $[q(t), p(t)]$ tends to $[\pi, 0]$ as $t \to -\infty$. The set $V^{(2+a/4)} \setminus [\pi, 0]$ does not contain any equilibrium of the unforced system (2.3) and is homeomorphic to two open intervals $(0, 1)$. Then $[q(t), p(t)]$ tends to $[\pi, 0]$ as $t \to +\infty$. But it is possible only if $[q(t), p(t)] \in \Gamma_s$ $\forall t$. The equalities (6.1) are proved. ■

Proof of Proposition 3.2. Given $[q_0, p_0]$, consider the solution $[q(t), p(t)]$ of the closed loop system (2.3), (2.9), (3.2) stating in this point. Along $[q(t), p(t)]$ the inequality

$$\dot{V}(q(t), p(t)) = -y \cdot \phi(y) \leq 0 \tag{6.2}$$

holds. The function V, see (2.8), is proper on the cylindrical state space. Therefore the inequality (6.2) implies that the trajectory $[q(t), p(t)]$ belongs to a compact set and has non-empty ω-limit set Ω. Take any point $[\hat{q}_0, \hat{p}_0] \in \Omega$ and consider the solution $[\hat{q}(t), \hat{p}(t)]$ of the closed loop system (2.3), (2.9), (3.2) with origin in this point. Taking advantage of the standard arguments, see [2, 8], one obtains that along the trajectory $[\hat{q}(t), \hat{p}(t)]$ the value of $y(\hat{q}(t), \hat{p}(t))$ is identically equal to zero. Therefore, see (2.9), either $\hat{p}(t) \equiv 0$ or $H_0(\hat{q}(t), \hat{p}(t)) \equiv 2 + \frac{a}{4}$. Obviously the first equality corresponds to the one of the equilibriums of the closed loop system (2.3), (2.9), (3.2), which in turn coincide with the equilibrium of the unforced (i. e. with $u = 0$) system (2.3): $[q, p] = [0, 0]$, $[\pm\arccos\dfrac{1}{a}, 0]$ and $[\pi, 0]$. The point $[\pi, 0]$ belongs to the set $V^{(2+a/4)}$. Thus the solution $[q(t), p(t)]$ may tend either to the set $V^{(2+a/4)}$ or the equilibriums: $[q, p] = [0, 0]$, $[\pm\arccos\dfrac{1}{a}, 0]$.

The linear approximation of the closed loop system (2.3), (2.9), (3.2) in these equilibriums has the form

$$\frac{d}{dt}\left[\begin{array}{c} y_1 \\ y_2 \end{array}\right] = \left[\begin{array}{cc} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial q} \\ 1 & 0 \end{array}\right]\Bigg|_{[q^*, 0]} \left[\begin{array}{c} y_1 \\ y_2 \end{array}\right], \tag{6.3}$$

where $[q^*, 0]$ is one of the equilibriums and

$$f(q, p) = -\sin q + a \cdot \sin q \cdot \cos q - R(q, p) \cdot \phi\left(y_1(q, p)\right).$$

The straightforward calculations show that

$$\begin{aligned} \frac{\partial f}{\partial p}(q^*, 0) &= -R(q^*, 0)^2 \cdot \dot{\phi}(0) \cdot \left[H_0(q^*, 0) - \left(2 + \frac{a}{4}\right)\right], \\ \frac{\partial f}{\partial q}(q^*, 0) &= -\cos q^* + a \cdot \cos 2q^*. \end{aligned}$$

Let $q^* = 0$ then the linear part of the closed loop system (2.3), (2.9), (3.2) has the form (6.3) with the matrix

$$\left[\begin{array}{cc} 2R(0, 0)^2 \cdot \dot{\phi}(0) & a - 1 \\ 1 & 0 \end{array}\right]$$

and its characteristic polynomial is

$$p(\lambda) = \lambda^2 - 2 \cdot R(0, 0)^2 \cdot \dot{\phi}(0) + (1 - a). \tag{6.4}$$

The value of $p(0) = 1 - a$ is negative hence $[0, 0]$ is hyperbolic equilibrium of the closed loop system.

Let $q^* = \pm\arccos\frac{1}{a}$, then the linear approximation (6.3) in this point has the matrix

$$\left[\begin{array}{cc} R(\pm\arccos\frac{1}{a}, 0)^2 \dot{\phi}(0)\left(1 + \frac{a}{2} + \frac{1}{2a}\right) & \frac{1}{a} - a \\ 1 & 0 \end{array}\right]$$

and its characteristic polynomial is

$$p(\lambda) = \lambda^2 - \lambda \cdot R(\pm\arccos\frac{1}{a}, 0)^2 \dot{\phi}(0)\left(1 + \frac{a}{2} + \frac{1}{2a}\right) + a - \frac{1}{a}.$$

Due to assumptions $a > 1$ and provided $\dot{\phi}(0) > 0$ the both roots of $p(\lambda)$ have positive real part. Therefore the points $[\pm\arccos\frac{1}{a}, 0]$ are unstable equilibriums of the closed loop system and there exist some neighbourhoods U_1, U_2 of these stationary points such that any solution with origin in U_1 or U_2 will eventually leave $U_1 \cup U_2$.

Another arguments show that $[\pm\arccos\frac{1}{a}, 0]$ is unstable even $\dot{\phi}(0) = 0$. Indeed, one can easily check that the function V attains the local maximum in these points and it does not increase along the closed loop system solution, see (6.2). The closed loop system does not have any other elements of ω-limit sets of the solutions close to these equilibriums. This means that they are unstable.

Thus it is shown that almost all solutions (except those which belong to the 1-dimensional stable manifold of $[0, 0]$) will tend to the set $V^{(2+a/4)}$. To complete the proof let us consider the linear part of the closed loop system in the equilibrium $[\pi, 0]$. It looks as (6.3) with the matrix

$$\left[\begin{array}{cc} 0 & a+1 \\ 1 & 0 \end{array}\right].$$

Obviously $[\pi, 0]$ is hyperbolic. This ends the proof. ■

Proof of Proposition 3.3 essentially repeats the arguments of the Proposition 3.2 proof. Taking advantage of the previous arguments one easily check that any solution tends either to one of the equilibriums or to the set $V^{(2+a/4)}$. Due to the assumption $0 < a < 1$ the closed loop system has only two equilibriums: $[0, 0]$ and $[\pi, 0]$.

Moreover, $[\pi, 0] \in V^{(2+a/4)}$. Calculating the linear approximation of the closed loop system (2.3), (2.9), (3.3) in equilibrium $[0, 0]$, one obtains that the characteristic polynomial of this approximations in $[0, 0]$ has the form (6.4). Due to the assumption $0 < a < 1$ the polynomial (6.4) is asymptotically unstable. This ends the proof. ■

Proof of Proposition 3.4. Let us take any continuous on $S^1 \times R^1$ feedback control $u = u(q, p)$. The point $[q^*, p^*]$ is an equilibrium of the closed loop system iff $p^* = 0$ and

$$f(q^*) = -\sin q^* + a \cdot \sin q^* \cdot \cos q^* - \cos q^* \cdot u(q^*, 0) = 0.$$

To establish the existence of such a point it is sufficient to show that there exist q_1 and q_2 such that $f(q_1) < 0$ and $f(q_2) > 0$. Then due to continuity of f on S^1 one can concludes that the closed loop system has at least two equilibriums. One of these equilibriums can coincide with the upright position $[\pi, 0]$, where $H_0(\pi, 0) = 2 + \frac{a}{4}$. But another equilibrium should belong to the set (3.4).

Let $q_1 \in (\frac{\pi}{2} - \varepsilon, \frac{\pi}{2})$ with $\varepsilon > 0$ then obviously there exists some small $\varepsilon > 0$ such that $f(q_1) < 0$. Let $q_2 \in (\frac{3\pi}{2} - \varepsilon, \frac{3\pi}{2})$ then again there exists $\varepsilon > 0$ such that $f(q_2) > 0$. Proposition 3.4 is proved. ■

Proof of Theorem 3.4. Given $[q_0, p_0]$ with $H_0(q_0, p_0) < 2 + \frac{a}{4}$, consider the solution $[q(t), p(t)]$ of the closed loop system (2.3), (3.5) with origin in $[q_0, p_0]$. Along this solution the relation

$$\frac{d}{dt}\left[H_0(q(t), p(t)) - \left(2 + \frac{a}{4} + \varepsilon\right)\right] = -y_1(t) \cdot \phi(y_1(t)) \leq 0, \tag{6.5}$$

where $y_1(q, p) = R(q, p) \cdot p \cdot \left[H_0(q, p) - (2 + \frac{a}{4} + \varepsilon)\right]$, is valid. Therefore $[q(t), p(t)]$ is bounded and has non-empty ω-limit set Ω. Moreover, due to standard arguments

$$y(q(t), p(t)) \to 0 \quad \text{as} \quad t \to +\infty.$$

Take any point $[\hat{q}_0, \hat{p}_0] \in \Omega$ and consider the solution $[\hat{q}(t), \hat{p}(t)]$ of the closed loop system (2.3), (3.5) stating in this point. Along this solution it holds

$$y(\hat{q}(t), \hat{p}(t)) = R(\hat{q}(t), \hat{p}(t)) \cdot \hat{p}(t) \cdot \left[H_0(\hat{q}(t), \hat{p}(t)) - (2 + \tfrac{a}{4} + \varepsilon)\right] \equiv 0.$$

By assumption R is not identically zero along any trajectory of the unforced system (2.3), then one concludes that the last identity is consistent with either $\hat{p}(t) \equiv 0$ or $H_0(\hat{q}(t), \hat{p}(t)) \equiv 2 + \frac{a}{4} + \varepsilon$.

The first case corresponds to one of equilibriums of the unforced system: $[0, 0]$ and $[\pi, 0]$. It is obvious that the function $[H_0(q, p) - (2 + \frac{a}{4} + \varepsilon)]^2$ attains the local maximum in the point $[0, 0]$, and it decreases along the closed loop system trajectories, see (6.5). This shows that $[0, 0]$ is unstable.

Thus one can concludes that either $[\hat{q}(t), \hat{p}(t)]$ is $[\pi, 0]$ or $H_0(\hat{q}(t), \hat{p}(t)) \equiv 2 + \frac{a}{4} + \varepsilon$. But the last identity implies that there exists $T > 0$ such that $H_0(q(T), p(T)) = 2 + \frac{a}{4}$, i. e. the solution $[q(t), p(t)]$ approaches the set $V^{(2+a/4)}$ in the time T, where the regulator (3.5) switches to zero.

One can easily repeat this arguments for $[q_0, p_0]$ with $H_0(q_0, p_0) > 2 + \frac{a}{4}$. ■

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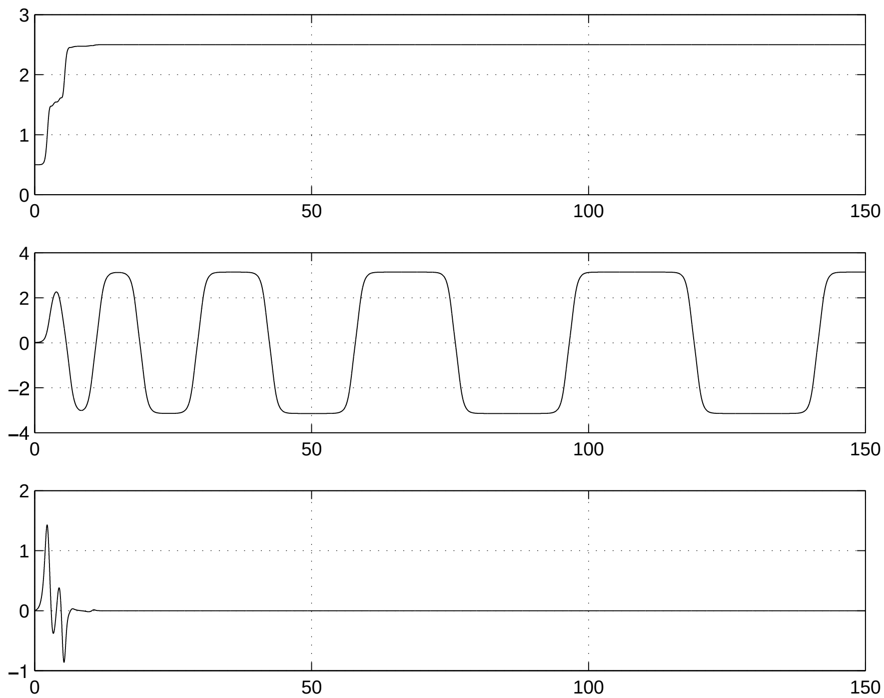

Figure 1: Energy level stabilization with $u = -y$

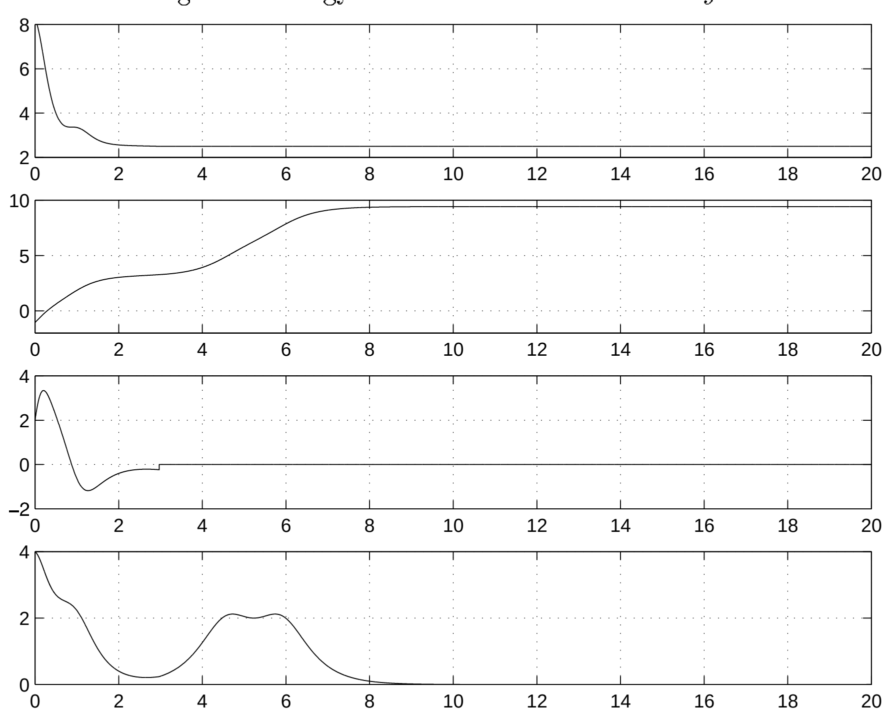

Figure 2: Stabilization of the upright position with regulator given in theorem 3.4