ISSN 0005-1179, Automation and Remote Control, 2009, Vol. 70, No. 10, pp. 1–62. © Pleiades Publishing, Ltd., 2009.
Original Russian Text © Andrievskii, Matveev, Fradkov, 2009, published in Avtomatika i Telemekhanika, 2009, No. 10, 34–99.

IN THE WORLD OF WISE MEN

Control and Estimation under Constrained Information: Toward a Unique Theory of Control, Computation, and Communication[1]

B. R. Andrievskii*, **A. S. Matveev****, **and A. L. Fradkov*****

*Institute of Problems of Mechanical Engineering, Russian Academy of Sciences, St. Petersburg, Russia
**St. Petersburg State University, St. Petersburg, Russia
***Institute of Problems of Mechanical Engineering, Russian Academy of Sciences, St. Petersburg, Russia and St. Petersburg State University, St. Petersburg, Russia

Received November 20, 2008

Abstract—An attempt was made to review the actively developing area of research devoted to analysis and design of the control systems with due account taken of the constraints imposed by the available finite-capacity digital communication channels. The problem's prehistory dating back to the 1960–1970's, as well as the new approaches that appeared during the last decade were analyzed. Much attention was paid to various versions of the now popular *data rate theorem*. Consideration was given to the problems of control through the communication networks and some results obtained for the nonlinear systems. The basic application areas were listed in brief.

PACS

1. INTRODUCTION

Systems where the control units and sensors are far apart and connected by limited-capacity digital information channels become more and more common. This situation is characteristic of the distributed sensor networks where quite a few sensors are connected to the central control unit, as well as of the systems for observation of moving objects, control of agricultural systems, mobile robots such as autonomous flight and underwater apparatuses, transportation vehicles, micromechanical flying insects, nano-satellites, and other moving devices. Limited capacity of the communication channels between individual system elements may have a pronounced effect on control and navigation. Even the fundamental characteristics of the control systems such as controllability and observability need reconsideration and revision because the conditions for controllability and observability change substantially if the information constraints are taken into account. This gives rise to the fundamental problem of determining the boundaries of the data transmission rates under which the desired aim of control is attainable in principle. Its solution gives an exhaustive, in a sense, description of the possibilities of the channel as part of the control system and also the requirements on the channel that follow from the stated aim.

Over the last decade, various modifications of this problem were given great attention and in its solution an essential progress has been made. A new stage in the development of the control

[1] This work was supported by the Russian Foundation for Basic Research, project nos. 08-01-00775 and 09-08-00803, Intersectional Program for Basic Research of the Branch of Power Engineering, Machine Building, Mechanics, and Control Processes, Russian Academy of Sciences, 2. "Problems of Control and Safety in Power Engineering and Technical Systems," and the Russian Federal Special-purpose Program "Scientific and Pedagogical Personnel of the Innovative Russia" (contract no. 02.740.11.5056.

theory, its convergence with the information theory is discussed more and more frequently [1–3]. The additional requirement for allowing for the computational constraints gives rise to a number of problems described as the *pursuit of a unique theory of control* $\times$ *communication* $\times$ *computing* $= C^3$ [4]. It goes without saying that the discussion of these crucial problems was not started today, but dates back to much earlier times when practicalization of the digital systems was just starting. However, the recent works are distinguished for formulating their results in the informational terms which seems more natural. Additionally, the recent works were devoted mostly to analysis of the errors in the existing circuits, whereas at the new stage the problems of system design, including the optimal design, with regard for the informational and computational constraints are posed.

At that, new sounding is found in the tested instruments such as state observers, zooming coders, switching controllers turning systems into hybrid ones, and so on. In the control loop, the limiting system possibilities for the given channel capacity are established and the control errors are estimated with the aim of optimizing the use of the communication channel. The impact of uncertainty which is characteristic of the dynamics of the majority of the technical systems is overcome using the methods of adaptive and robust control.

The traditional information theory underlies the communication engineering and defines the boundaries of the information transmission rate, as well as the error probability that is made attainable using the optimal coding. The corresponding results serve as a compass directing the development of the theory and practice of coding and a yardstick to measure the passed and remaining path. This theory uses criteria that are mostly abstracted from the final aims of information transmission. Considering the communication channel as a composite part of the control complex, it is only natural to change the landmarks and follow the criteria of the control theory. This approach is confessed by many participants of the new control-theoretical "boom."

The present work attempts to overview this booming field. In the first part (Secs. 1 and 2), the problem's prehistory is analyzed beginning from the works of the 1960–1970's. In the mid-1990's, the efforts of researchers were concentrated mostly on the possibility of stabilizing an unstable plant by a control system whose components are interconnected by a limited-capacity network. The case of linear plant and simplest network interconnecting the measurer (transducers, sensors), controller and actuator (drive) via two channels of *observation* and *control* offers a natural starting point. The data from the sensor to the controller and from the controller to the actuator are transmitted, respectively, through these channels. Even in this case some important results were established which demonstrated that, in particular, allowance for the bit rate of information transmission leads to basically new effects as compared with the classical theory. These results are discussed in Sec. 4 where appreciable space was devoted to different versions of the already famous "data rate theorem." We notice that presentation of Sec. 4 is more detailed and its style is more formal. Section 5 is devoted to the problems of control through the communication networks. The number of publications devoted to these problems is not great. Some results obtained for the nonlinear systems are given in Sec. 6 where the well-known results either are local or give only the sufficient conditions. Other approaches to the problems of control with constrained communication are discussed in Sec. 7. Section 8 characterizes briefly the applied studies on the subject of this review. The necessary technical material is compiled in the Appendix. Since the review is structured in terms of subjects, the works concerned with more than one subject are usually discussed in several sections.

We notice that selection of the material undoubtedly was influenced by the scientific interests of the present authors who adduce an excuse to the colleagues whose contribution was outside the main focus of the review.

The following notation is used below:

$\mathbb{Z}$ is the set of integers,

$\mathbb{R}$ is the set of real numbers,

$\mathbb{C}$ is the set of complex numbers,

$\Re z$ and $\Im z$ are, respectively, the real and imaginary parts of the complex number z,

$:=$ stands for "equal by definition,"

$\{a_1, \ldots, a_k\}$ is the set made up by the aforementioned elements,

$\{a\colon \mathcal{P}(a)\}$ is the set of element for which the assertion $\mathcal{P}(a)$ is true,

$A \setminus B$ is the difference of the sets A and B, that is, the totality of all elements of A that are not present in B,

$\emptyset$ is an empty set,

$|A|$ is the number of elements of the set A,

$[m\colon n] := \{j \in \mathbb{Z} : m \leqslant j \leqslant n\}$,

$\det A$ is the determinant of the matrix A,

$\|A\|$ is the spectral norm of the matrix A (in a special case where A is a column vector, $\|A\|$ coincides with the Euclidean norm of the vector A),

$\mathbf{diag}\,(a_1, \ldots, a_k)$ is the $k \times k$ diagonal matrix with the elements $a_1, \ldots, a_k$ on the diagonal,

$\boldsymbol{P}$ is the probability,

$\boldsymbol{P}[A|B]$ is the conditional probability of the event A under the assumption of the event B,

$\boldsymbol{E}$ is the expectation, and

$\wedge$ is the logical AND.

2. EARLIER WORKS: CONTROL AND OBSERVATION WITH QUANTIZATION

In the earlier works on the effect of quantization by the signal level in the digital control systems, the quantization unit ("*quantifier*," "*coding unit*" or "*coder*") was usually regarded as a source of independent random discrete process (noise) acting additively on the system. This assumption allows one to simplify appreciably the study of system with quantization, especially for the control plants obeying linear models (see, for example, [5–10]). This assumption, however, is too rough if the quantization step is commensurable with the range of variations of the transmitted variable [9, 11–15]. In the closed-loop discrete-time system, quantization by level may give rise of oscillatory processes similar to the self-oscillations in the continuous nonlinear systems. Analytical determination of the parameters of such processes is possible only in the simplest cases. The approximate numerical-analytical method of harmonic linearization that was extended to the discrete systems in [9, 16] enables one to simplify the study.

Apart from analysis, consideration was given to the problems of design, primarily, to minimization of the errors caused by quantization in the control loop. These problems are usually formulated as optimization of some integral objective functional (loss function). The earliest publications in this area are represented by [17–19]. Design of an optimal control system for a discrete linear plant with quantization of the *control action* was discussed in [19]. Consideration was given to the *static* coders following the characteristic $\tilde{z} = q(z)$, where z is the scalar input, $z \in \mathbb{R}$, and $\tilde{z}$ is the transformed (quantized) coder output. It is the static characteristic of the coder $q(\cdot)$ that is to be optimized. It is assumed that $q(\cdot)$ is a monotone odd function with $N+1$ different values, $q\colon \mathbb{R} \to \{c_0, c_1, \ldots c_N\}$. The values $c_i \in \mathbb{R}$ $(i = \overline{0, N})$ are the optimized parameters. The problem of optimization of a quantizer in an open loop was separately considered in [19] under the assumption that the static characteristics of the input z are know and defined by its one-dimensional distribution density $p(z)$. The expectation of some selected objective function $g(z, q(z))$ is the criterion for coder optimization. Such optimization problem is solved by the standard methods of mathematical programming. Next, consideration is given to the closed-loop dynamic system comprising the control plant, measurer, state observer, controller, and quantizer. It is assumed that the control plant is subject to external stochastic perturbations. The stochastic measurement noise is also taken into consideration. For the quadratic performance functional, linear control plant, Gaussian perturbations, and measurement noise, the *separation theorem* was formulated in [19] according to which the problems of optimization of controller, observer, and coder may be solved independently. The two first problems are solved using the well-known methods of optimal linear-quadratic control disregarding quantization, and the coder is optimized using the mathematical programming procedure established for the open-loop system.

Subsequent discussion [20–22] established that the results of [19] give rise to a *variable-parameter* coder ("*nonstationary coder*"), which essentially complicates their application in real-time systems. For the vector control signal, these results were developed in [21] where it was noticed that [17–19, 22, 23] considered the problems of optimization of coders of a *given* structure. An algorithm of the optimal quantizer was derived in [21] from the solution of the optimization problem for the closed-loop system using the linear-quadratic criterion with Gaussian noise (the so-called *LQG-problem*). It was found that for a given number of the quantization levels the optimal controller can be determined by solving separately the following subproblems:

1. design of the optimal Kalman filter for state estimation under stochastic actions;
2. design of the optimal controller from the estimates obtained;
3. optimal quantization of the continuous minimal-distortion control determined from the last item.

It was noticed in [21] that if it is possible to calculate in advance the quantizer parameters for a system with scalar control, then in the case of vector control these parameters must be determined in the course of system operation depending on the current realization of the process. The current value of the loss function is the sum of values of this function for the problem of optimal control without quantization and additional addend representing the quantization error.

Allied results were presented in [23] which considered the problem of optimal stabilization of a stochastic discrete linear plant with quadratic objective functional, provided that the *measurer* obeys an arbitrary nonlinear static characteristic including the nonlinear characteristic of the coder at the plant output $\tilde{y} = q(y)$. In particular, a structure was proposed in [23] where the "*innovation*," that is, the residual z between the plant output and its conditional mean value generated by the Kalman filter, rather than the plant output y, arrives to the coder. The following coder equations correspond to this structure:

$$\tilde{z}_k = q(z_k, \nu_k), \quad z_k = y_k - \bar{y}_k, \tag{2.1}$$

where y is the plant output, $\bar{y}$ is its conditional mean generated by the Kalman filter, ν is the output measurement noise (error), $q(\cdot)$ is the static characteristic of the coder, and $k = 0, 1, \dots$ is the discrete time.

Therefore, to employ the separation theorem at quantization of the control signal, one must use, according to [19, 21], a nonstationary coder *in the direct circuit*, and at quantization of the measurement signal the coder must be *feedbacked* according to [23]. It deserves noting that the transformation method (2.1) proposed in [23] anticipates the results of much later works such as [24–28]) where it is the innovation signal that is coded and transmitted through the communication channel.

Design of the optimal Kalman filters with allowance for the roundoff errors caused by the finite length of the computer word was discussed in [29]. Consideration was given to the effect of the errors of representation of the sensor signals, as well as the errors of realization of the parameters of optimal filter. The impact of such errors was described in [29] as an additive random process with a distribution density uniform over a given interval. A solution was suggested which is a trade-off between the quantization frequency and computer word length.

Design of the so-called *adaptive* quantizers where the signal transformation range is varied automatically in the course of operation was considered in [30–35]. The forecast obtained at transmission of the preceding block of signals is used in [35] to adjust the range of transformation (the upper and lower saturation levels of the quantizer). These levels are varied independently so as to embrace the forecasted range of signals. In [30] the width of the quantization range is varied at coding each signal. We notice that the aforementioned works were devoted only to the problem of signal transmission. Use of the adaptive quantizers in the *closed-loop* systems of control and estimation was discussed in a number of subsequent works such as [36–39])

The problem of minimizing the sensitivity of the discrete controllers to the quantization errors caused by the finite computer word width was posed in [40]. It was assumed that the controller coefficients were represented precisely, there were no effects of overflow and quantization of the results of measurements, and an error occurs in the computer prior to or after multiplication. It was desired to use the given controller transfer function to determine a realization in the form of the state equations such that the influence of the roundoff errors is minimal. As in many other works [7, 8, 29, 41], the roundoff error was represented by an external additive bounded noise. The H_2 and H_∞ norms were used for optimization. The above problem was shown to be reducible to that of convex optimization of a linear objective function under affine matrix constraints of the inequality type.

Occurrence of chaotic oscillations caused by level quantization was considered in [42–45].

Conclusions. The problems of control and estimation in the conditions of both level and time quantization attracted attention concurrently with the introduction of the digital technology. Owing to the nonlinearity of the systems with quantization in the closed loop, their analytical study proved to be a complex problem that is commonly simplified using the linearization procedure. The quantization errors are usually represented as bounded additive noise acting on the linear system. Another, more rare, approach lies in using a procedure of harmonic linearization. Yet, the level quantization in all cases was considered as an *approximation* of a real number by a finite number of digits which is the source of the system error. An alternative point of view based on the ergodic theory of the dynamic systems and regarding the quantized measurement $q(x)$ of a continuous variable x as an object carrying some incomplete information about x (see [46]) appeared with time. The following problem was also formulated in [46]: "the question we wish to answer ... may be phrased as follows: under what circumstances, and in what sense, can we *stabilize* an unstable discrete-time linear system by choosing a feedback control which depends only on a *quantized* measurement of the system's state?" Numerous publications devoted to its solution are reviewed in Sec. 4.

3. PROBLEMS OF CONTROL AND OBSERVATION UNDER CONSTRAINED CAPACITY OF THE COMMUNICATION CHANNEL

The paper [46] was among the first publications that indicated to the informational constraints in the communication channel between the sensors and observer in the problem of estimating the state of a dynamic system. Consideration was given to a closed-loop stabilization system consisting of a linear discrete control plant

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \dots, \tag{3.1}$$

and a feedback controller

$$u_k = f_k(q(x_0), \dots, q(x_k)), \tag{3.2}$$

where $x_k \in \mathbb{R}^n$ is the plant state, $u_k \in \mathbb{R}^m$ is the control action, A and B are the matrices of corresponding dimensions, and $q(\cdot)$ is the coder function. As can be seen from Eq. (3.2), the control signal may depend on the entire prehistory of measurements from the zero to the current kth time instants, which is of principal importance. D.F. Delchamps [46] emphasizes that "The significance of our allowing u_k to depend on the entire measurement history $\{\mathbf{q}(x(l)) : l \leqslant k\}$ should not be underestimated. ... quantization $\mathbf{q}$ makes the 'present' measurement $\mathbf{q}(x(k))$ somewhat more like a *partial observation* of the state $x(k)$ whose role as an input to the controller is analogous to that of a *linear* partial observation $e(k) = Cx(k)$ in the linear control theory." More detailed results can be found in [12].

The problem of state estimation under constrained capacity of the communication channel was formulated in [47,48]. In contrast to the classical approach of the estimation theory where the observation signal is represented by a continuous process distorted by additive noise, it was additionally assumed in [47, 48] that the signal is coded and transmitted via a limited-capacity communication channel. The plant state is represented by the stochastic process $\{x(t)\}_{t=0}^{\infty}$ with a distribution function that is known a priori and the first and second moments known at each time instants. The plant output is measured continuously, but owing to remoteness of the plant, the measurement data arrive to the estimation system via a communication channel. It was noticed in [48] that the classical information theory minimizes the asymptotic error of data transmission via the communication channel assuming that the readings of the input are independent and identically distributed. This approach leads to nonrecursive coding-decoding procedures. However, as was noted in [47,48], from the point of view of estimation of the states of dynamic systems the above assumptions about the properties of the input sample and the nature of the coding procedure are unacceptable.

In [47, 48], consideration is given to $\mathcal{Y}_t$, the set of possible realizations of the process x by the time t, $\mathcal{Y}_t = \{x(s), 0 \leqslant s \leqslant t\}$. The code word of length l_i transmitted at the ith step is denoted by c_i; τ_i denotes a time instant such that c_i is calculated from the trajectory of x by the instant τ_i, that is,

$$\begin{cases} c_i = h_i(y(\tau_i)), \quad y(\tau_i) \in \mathcal{Y}_{\tau_i}, \\ \tau_i = \tau_{i-1} + l_i\delta, \quad \tau_0 = 0, \quad i \geqslant 1, \end{cases} \tag{3.3}$$

where $\delta = 1/R$ is the time of transmission of one data bit through the communication channel. Correspondingly, the rate of data transmission through the channel $R = 1/\delta$ (bit/s).

In the search for analogs of the classical Shannon theorem [49] on the channel capacity, the problem of determining the lower bound of the rate of data transmission through the communication channel with which it is possible to provide convergence of the plant state estimation procedure was formulated in [50]. Such problems arise at measurement quantization in digital control systems and also in the decentralized observation systems where the sensor data are represented by coded (frequently, binary) signals that are transmitted through a discrete communication channel to the "data merge center" where they are processed and a decision is made. The problem of estimating the stochastic process $x(t)$ satisfying the first-order equation $\dot{x}(t) = ax(t)$, where $a > 0$ is a certain constant parameter, is considered in [50] (it is namely the problem of estimating an unstable process that is of interest). The initial state x_0 has a certain distribution density in the form of the *shifted Laplace distribution* $f(x) = \lambda e^{-\lambda|x-\mu|}/2$ with certain expectation μ and variance $\sigma^2 = 2/\lambda^2$. The measurement results $x(t)$ are transformed by a binary coder and transmitted at the discrete time instants t_k through the communication channel without distortions and losses. The instants t_k obey the relation $t_k = k\delta$, where $k = 0, 1, \ldots$ and δ is the time interval for transmission of one bit through the communication channel. The magnitude of $R = 1/\delta$ is called the *data transmission rate*. A recursive algorithm to obtain the estimate $\hat{x}_k$ of the values $x_k = x(t_k)$ from the transmitted binary sequence

$$h(x_k) = \begin{cases} 0, & x_k \in \mathbf{A}, \\ 1, & x_k \in \mathbb{R} \backslash \mathbf{A}, \end{cases} \tag{3.4}$$

was described in [48, 50]. It is based on the recurrent change of the *characteristic set* $\mathbf{A}$ of the coding function (3.4). According to this algorithm, the estimate $\hat{x}_k$ is calculated for the set $\mathbf{A}_k$ determined at the kth step from the criterion for minimum of the variance. At the initial step, the characteristic set $\mathbf{A}_0 = (-\infty, \mu]$. It was proved in [50] that for the considered stochastic process x_k the following condition for the data transmission rate

$$R > \frac{2a}{\ln 2}. \tag{3.5}$$

is the sufficient condition for root-mean-square convergence of the coding-estimation procedure of [48, 50]. In the subsequent works a similar condition was obtained for a larger class of systems.

Estimation of the state of a remote system through a limited-capacity digital communication channel was considered also in [51] where the problem was posed of determining the minimal capacity of a communication channel under which the desired precision of estimation may be provided. Closeness of the considered problem to the study of digital systems with level quantization [9, 11–15, 17–19, 22, 23] was noticed in [51] where attention was also drawn to some basic distinctions. First, in the system with level quantization the converter characteristic (*quantization step*) is usually fixed, whereas at the data transmission through the communication channel the coder is constrained only by the number of symbols in the output alphabet. The coding function may also be *time-variant* and dependent on all preceding measurements. This additional degree of freedom changes basically the essence of the problem at hand because the estimator becomes capable to select effectively the values to be measured by purposeful variation of the coding function. Second, at data transmission through the communication channel there is a delay growing linearly with the number of positions of the code words. Consequently, an increase in the number of coder positions beginning from some value leads in fact to a *reduction* in the accuracy of estimation because of obsolescence of the transmitted data, which casts some doubt on utility of the methods of the rate distortion theory and the multiterminal data compression [52–54] that are based on combining the information about the process in larger blocks before coding for the real-time systems. In distinction to [47, 48, 50] where consideration was given to the stationary first-order linear systems, the results of [51] refer to the nonlinear systems like

$$x_{k+1} = f_k(x_k), \quad k = 0, 1, \ldots, \tag{3.6}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $n \geqslant 1$ is the order of the system, the vector function $f_k(\cdot)$ for each $k = 0, 1, \ldots$ satisfies the Lipschitz condition with the constant L_k. It is assumed that the entire state vector can be measured precisely at the side of the data source, and the problem lies in transmitting the results of measurements via the communication channel of capacity of R bits in a step of discrete time. Therefore, the coder alphabet has $M = 2^R$ symbols. At each step k the communication channel transmits without distortions and delays one symbol s_k from this alphabet, the value of s_k being dependent on the entire process prehistory $x_0, \ldots, x_k$. Relying on the information obtained, the decoder constructs the estimate $\hat{x}_{k+1}$. The problem lies in selecting the coding-decoding functions in terms of the minimum of the limit rms estimation error $\lim\limits_{k\to\infty} \boldsymbol{E}\|x_k - \hat{x}_k\|^2$, where $\boldsymbol{E}$ is the operation of expectation over the set of the initial values $\{x_0\}$). Consideration is also given to the related problem of determining the least value of R providing the desired accuracy of estimation. The coding-decoding functions for various kinds of the distribution density $p(x_0)$

of the initial value x_0 were proposed in [51]. For example, for $p(x_0)$ with a compact support the procedure is based on recurrent decomposition into M^k, $k = 0, 1, \dots,$ equal hypercubes of the domain where x_k may stay beginning from a certain domain including the initial state x_0. The index of the corresponding hypercube s_k is transmitted through the communication channel. In this case, for the rms estimation error the following upper boundary was obtained in [51]:

$$\boldsymbol{E}\|x_k - \hat{x}_k\|^2 \leqslant \phi^2 \left(\frac{\prod\limits_{j=0}^{k} L_j}{M^{\frac{k}{n}}} \right)^2, \tag{3.7}$$

where ϕ a k-independent constant. Whence it follows that satisfaction of the condition

$$\lim_{k\to\infty} \frac{\prod\limits_{j=0}^{k} L_j}{M^{\frac{k}{n}}} = 0 \tag{3.8}$$

suffices for getting the zero limit error: $\lim\limits_{k\to\infty} \boldsymbol{E}\|x_k - \hat{x}_k\|^2 = 0$. Expression (3.8) shows that if the capacity of a communication channel exceeds some threshold value, then asymptotically precise estimation of state can be provided for the given sequence $\{L_k\}$. Additionally, it follows from estimate (3.7) that the rate of convergence is determined from $M = 2^R$; therefore, the greater values of R bring about higher rate of estimation. For the stationary linear system, $L_k \equiv L$, where L is the norm of the matrix A, satisfied is $f(x_k) = Ax_k$. Then, (3.8) leads to the inequality

$$R > n \log_2 L. \tag{3.9}$$

Condition (3.9) was later developed in the form of the data rate theorem considered in the following section.

The problem of optimization in the integral quadratic criterion of the linear stochastic systems under informational constraints due to the coding of measurements of the plant output was considered in [55]. The principle of separation was shown to retain validity; therefore, the linear controller is optimal in estimating the plant state. To attain the desired result, instead of the observation process, coded and transmitted trough the communication channel is the innovation process which, in distinction to the observation process, is a Gaussian process with independent and identically distributed values whose statistical characteristics are control-independent. This fact allows one to make use of the well developed procedures for design of the optimal vector quantizers [56–58]. Additionally, transmission of the innovation process enables one to take into consideration at the decoder side only the current values of the received signal and not its prehistory as it was the case at transmission of the measurement signal. The centroid property of the optimal vector quantizer is another basic feature of the proposed approach which enables one to consider the quantized variable as the conditional expectation of the original random value relative to an appropriate σ-subalgebra. This property can be conveniently used with the least squares method.

The linear discrete plant

$$x_{k+1} = Ax_k + Bu_k + w_k, \quad k = 0, 1, \dots, \tag{3.10}$$

is considered in more detail in [55]. Here, $x_k \in \mathbb{R}^d$ is the state vector; $u_k \in \mathbb{R}^m$ is the control vector; $w_k \in \mathbb{R}^d$ is the Gaussian white noise with the zero mean and the covariance matrix Q; and A and B are matrices of appropriate dimensions. The following nonanticipative condition is satisfied for

w_k: the sequence $\{w_j, j \geqslant k\}$ is independent of $\{x_j, u_j, w_{j-1}, j \leqslant k\}$ for all $k \geqslant 0$. Consideration is given to the problem of determining the optimal control $\{u_k\}$ minimizing the objective function

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \boldsymbol{E}\left(x_k^{\mathrm{T}} G x_k + u_k^{\mathrm{T}} R u_k\right) \tag{3.11}$$

with the given positive semidefinite matrices G and R. The standard conditions for controllability of the pair (A, B) and observability of the pair $(A, G^{\frac{1}{2}})$ are assumed to be satisfied, and additionally the condition $\lambda_{\max}(A^{\mathrm{T}}A) < 1$, where $\lambda_{\max}(\cdot)$ is the greatest eigenvalue of the argument matrix, is imposed on the spectral norm of the matrix A. The last assumption is used in the procedure for optimization of the code word length ensuring a trade-off between the precision and communications delay. This procedure includes solution of the discrete Lyapunov equation with the matrix A^N, where N is the communications delay depending on the code word length and data transmission rate along the communication channel. We notice that by the process of innovations [55] understands the forced component of the response of system (3.10) to the action w_k. Since it was assumed in the paper that the state vector x_k is measured without errors, this component can be calculated from the results of measurements.

The problem of linear-quadratic optimal control with an objective function like (3.11), $G = G^{\mathrm{T}} \geqslant 0$, $R = R^{\mathrm{T}} > 0$, for plant (3.10) with Gaussian perturbations w_k and initial states x_0 having the zero mean and covariance matrix K_{x_0} was considered in [24, 25]. In distinction to the classical formulation, it is assumed following [59] that the power of the signal transmitted through the communication channel between the plant output and the controller is limited. For such systems, the separation principle was shown to be valid, and the linear controller, to be optimal. The optimal relation between the costs of control and data transmission through the communication channel was established in [24, 25]. Instability of the control plant was also shown to be closely related with the requirements on the communication channel capacity: if the rate of data transmission through the communication channel is below some boundary, then the loss function inevitably goes into infinity and the system even cannot be stabilized. A communication channel with analog additive white Gaussian noise (AWGN) was considered in [24]. The communication channel obeys $b_k = a_k + v_k$, where $a_k, b_k \in \mathbb{R}^d$ are the vector input and output, respectively, and v_k is the additive noise. It is assumed that $\{v_k\}$ is a Gaussian process with independent and identically distributed values, zero mean, and a certain matrix of variances K_v. A *power constraint* on the signal transmitted through the channel is introduced in the form of $\boldsymbol{E}\|a\|^2 < d \cdot P$, where P is a given value. According to the Shannon theorem, the attainable data transmission rate R in such channel follows the expression

$$R = \max_{\operatorname{tr}(K_{a,k}) \leqslant d\cdot P} \frac{1}{2d} \log_2 \frac{\|K_{a,k} + K_v\|}{\|K_v\|}, \tag{3.12}$$

where $K_{a,k}$ is the covariance matrix of the transmitted signal a. Therefore, the values P and R are related uniquely. The control process is optimized using the *dynamic* ("*predicting*") coders-decoders (*devices with memory*) where the process values are predicted on the basis of the past measurements. The predictive encoder has been known for a long time and is widely used for transmission of the serial data collections [60]. In these devices, it is the *signal of error* between the true state of the process (plant) and its best prediction generated by the decoder that is encoded and transmitted through the communication channel. The equi-memory condition according to which both the coder and decoder make decisions on the basis of the same information is assumed to be met [60]. This condition is important because it enables the coder to follow the error between the true state of the plant (known at the side of the coder) and its estimate generated by the decoder. It is required in [24] that the state of the coder include the value of the decoder output at the preceding step y_k. Therefore, the coder needs to dispose of substantial information about the decoder state, that is, there should be some feedback between their outputs. One of the possibilities is offered by calculation of y_k on the coder side. For that, the coder needs precise values of the control action u_k, and the control law must be invertible. There are situations where the equi-memory condition proves to be excessive because "the plant itself can act as a connection" from the controller to the coder [24].

These studies were continued in [61] which considered the stochastic control systems with a communication channel between the measurement devices and the controller. The task lies in designing coder, decoder, and controller so as to satisfy the desired aims of control. In particular, consideration is given to the influence of the communication channel for the classical Gaussian linear-quadratic communication channel. Conditions were presented under which the problems of state estimation and control are separable for the optimal design. The boundaries of the attainable system performance were established, and the trade-off was demonstrated between the control and the data transmission costs which is characteristic of the problem at hand. In particular, the optimal quadratic loss function was shown to be decomposable into two terms: the loss function under complete information and the function of losses due to the limited rate of data transmission through the channel.[2]

Feedback-based stabilization of the discrete linear stationary determinate plant under quantized measurements with saturation was studied in [26]. Its authors indicate that it is usually assumed in the literature on control with quantization that given is a *fixed* quantizer describing the effect of the finite precision of data representation on the behavior of the control system [12, 63, 64]. They, however, keep to another point of view: fixed is only the *number of quantization levels*, the rest of the quantizer parameters being variable in the course of system operation. It is namely this approach that enables asymptotic stabilization of the system's state of equilibrium. In [26] used is a static quantizer $g\colon \mathbb{R} \to \mathbb{Z}$ with *sensitivity* $\Delta > 0$ and *saturation level* $M \in \mathbb{Z}$:

$$q(x) = \begin{cases} M, & \text{if} \quad x > (M+1/2)\Delta, \\ -M, & \text{if} \quad x \leqslant -(M+1/2)\Delta, \\ \lfloor x/\Delta + 1/2 \rfloor, & \text{if} \quad -(M+1/2)\Delta < x \leqslant (M+1/2)\Delta, \end{cases} \tag{3.13}$$

where $\lfloor x \rfloor = \max\{k \in \mathbb{Z}\colon k \leqslant x\}$ is the down round-off function. The quantization function of the vector variable $q\colon \mathbb{R}^n \to \mathbb{Z}^n$ is understood in the componentwise sense, that is, it is assumed that for each component of the state vector x_i there exists a quantizer $q_i(x_i)$ with sensitivity Δ_i $(i = 1, \ldots, n)$. Availability in quantizer (3.13) of saturation M distinguishes it from that of [12].[3] *Uniform* quantizers having identical values of $\Delta_i = \Delta$ are introduced.

The work focuses on the possibility of varying the quantizer sensitivity—rather than the level of saturation—on the basis of current data. The example of a photocamera with zooming and a fixed number of pixels is given as a motivation for this study. Following [65], consideration was given to the problem of control with information constraints in the sense that the system state is not known completely and known is only the block from the fixed set of the quantization blocks (hyperparallelepipeds) to which the plant state belongs at the given time instant. The control strategy suggested in [26] is divided into two stages. At the first stage where the initial plant state is unknown, the scale is zoomed out, that is the parameter Δ is increased so as to

[2] See also [62] where the notions of message epsilon-entropy and the rates of message generation by the source in the case where at the instant of message restoration after its transmission through the channel is known either prior to the time instant coinciding with the current instant or to an instant preceding to it. Such problem formulation may arise both in the theory of information transmission and in the problems related to control. It was shown in [62] that for the stationary sources the rate of message generation is always definite and in a wide class of cases is realized on the stationary pairs the input and output messages.

[3] As can be seen from (3.13), M defines the number of elements of the (integer) alphabet of message coding and, thereby, the number of the data bits R transmitted at each step.

"capture" the plant state. The second stage lies in zooming in. At this stage, the sensitivity parameter Δ is decreased, which allows one to drive asymptotically the plant to the zero state. The passage between the increase and decrease in the parameter Δ is defined by the value of the "*scaling variable*" $z \in \{-1, 1\}$. As the result, [26] considers the following *hybrid* system including continuous linear plant and discrete procedure of varying the sensitivity Δ:

$$\dot{x}(t) = Ax(t) + Bu\left(t, q(x\left(t\right)), \Delta\left(t\right)\right), \tag{3.14}$$

$$\Delta(t) = G\left(z, \lfloor t/\tau \rfloor, q(x(\lfloor t/\tau \rfloor \tau)), \Delta(\lfloor t/\tau \rfloor \tau)\right), \tag{3.15}$$

where $\tau > 0$ is a fixed parameter, the time discreteness step. Obviously, Eq. (3.15) describing dynamics of the parameter Δ is representable as a difference equation with the discrete time $k = 0, 1, \ldots$ Since the right-hand sides of (3.14) and (3.15) are discontinuous, their solutions are understood in the sense of Filippov [66, 67].

Let us consider a control law like

$$u(t) = \begin{cases} 0 & \text{for} \quad t < k_0\tau, \\ -Kq(x(t)), & \text{otherwise}, \end{cases} \tag{3.16}$$

where q is a uniform quantizer of sensitivity $\Delta(t)$ and k_0 is some positive integer. The following theorem was proved in [26].

Theorem 1. *Let the matrix K be selected so that all eigenvalues of the matrix $A - BK$ have negative real parts. Then, there exists a control strategy* (3.15), (3.16) *such that under an arbitrary $x(0)$ and $\Delta(0) = 0$ the trajectories of the closed-loop system* (3.14)–(3.16) *tend to zero for $t \to \infty$.*

The main idea underlying asymptotic stability of system (3.14)–(3.16) lies in the iterative reduction of the parameter of sensitivity $\Delta = \Delta(t)$ by multiplying it by the selected scaling factor $0 < \Omega < 1$. Thus, at the zooming-in stage Eq. (3.15) leads to $\Delta(t) = \Omega^{\nu}\Delta(k_0\tau)$, where $(k_0 + \nu)\tau < t \leqslant (k_0 + \nu + 1)\tau$, $\nu = 1, 2, \ldots$ The factor Ω and the quantization interval τ are calculated depending on the saturation level M and the maximal and minimal eigenvalues of the matrix Q of the solution of the Lyapunov equation $(A - BK)^{\mathrm{T}}Q + Q(A - BK) = -D$ with some matrix $D = D^{\mathrm{T}} > 0$. As follows from the proof of the theorem, convergence $x(t) \to 0$ is of *exponential* nature. At the beginning of the process (the zoom-out stage), the parameter Δ should grow with a rate exceeding that of divergence of the plant proper motions caused by instability of the matrix A. One can use the law $\Delta(t) = e^{2\|A\|\lfloor t/\tau \rfloor \tau}$, for example. The aforementioned method of generating $\Delta(t)$ as a sectionally constant (over intervals of duration τ) time function realizes the so-called dwell-time switching logic [68]. Another approach to the discrete-time systems where $\Delta(t)$ is changed each time as $\|q(x)\|$ becomes equal to or smaller than some given level is discussed in [26].

The described technique of stabilization requires a sufficiently great (and fixed) level of saturation M. It was also shown in [26] that the global asymptotic stabilization of plant (3.14) can be provided using a substantially smaller M by choosing a sufficiently small time-discreteness step τ. At that, the following procedure is used: if it is known that at the current time instant the plant state belongs to some hyperparallelepiped (*rectilinear box*) and the sensitivity Δ is selected so that the switching hyperplanes divide it into smaller hyperparallelepipeds, then from the corresponding quantized measurements one can determine a smaller parallelepiped comprising the system state, that is, specify the state estimate.

Similar results were obtained in [26] also for the problems of control of the *discrete* linear plants

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \ldots \tag{3.17}$$

It deserves noting that an additional condition arises for discrete time. It requires that plant (3.17) needs not to be "too unstable," namely, the inequality $\max\limits_{1\leqslant i\leqslant n}\left(\sum\limits_{j=1}^{n}|A_{ij}|\right)\leqslant 2$ must be satisfied. It was also noted in [26] that if the sensitivity Δ can have only a finite set of possible values, then only the *practical stability* can be reached and not the global asymptotic stability.

The problems of *output stabilization* also were considered in [26]. It is assumed that at generation of control not the entire state vector x can be measured and used, but only the output of the plant $y = Cx$, where C is a $p \times n$ matrix, $p < n$. The initial zoom-out stage cannot be used for such systems; therefore, only the *local* asymptotic stability is ensured for them. The paper demonstrated that the global stabilization of the output can be attained using more complicated, namely, dynamic control laws. For that purpose, the feedback in the *estimate* $\bar{x}$ of the plant state was used in this work.

The continuous systems with a quantizer having the saturation level $M = 1$ are discussed separately in [26]. For such systems, design of the sliding-mode control law providing the local asymptotic stabilization was developed, and the case of systems with scalar input and output was considered in detail. The possibility of extending the results obtained to the nonlinear systems also was discussed in [26].

An optimal static (*memoryless*) control law with quantization for stabilization of the linear first-order discrete systems is proposed in [69]. The unstable scalar plant

$$x_{k+1} = ax_k + u_{k+1}, \quad |a| > 1, \quad k = 0, 1, \ldots \tag{3.18}$$

is considered along the lines of [70–72]. The feedback is described by the static nonlinearity (with one-cycle delay):

$$u_{k+1} = \gamma(x_k). \tag{3.19}$$

Here, $\gamma(\cdot)$ is the *quantization function* because $\gamma(\cdot)$ is piecewise-constant and has a finite number N of the discontinuity points. It is required to design controller (3.19) such that almost for all $x_0 \in [-1, 1]$, that is, with the probability one, the process x_k hits in a finite time the interval $[-1/C, 1/C]$ for some $C > 1$ called the *contraction rate* and stays there. A decomposition of the system state interval $[-1, 1]$ into N subsets called the *quantization intervals*, which are not intervals of necessity, was proposed in [69]. The time instant of the first hit into the given interval averaged over the set of the initial conditions is denoted by T. Optimality is understood in the sense that if two of the parameters of T, N, and C are given, then the third parameter cannot be improved (reduced). Since it is not required in [69] that the quantizer be an interval quantizer, the proposed control strategy ensures better system performance and lower channel load that that of [70–72]. The necessary and sufficient condition for existence of the stabilizing static feedback with coder was obtained in [69], and the *Maxwell's demon* was discussed by way of example.

Some interesting results were established in [73] where, in particular, it was demonstrated that the coarsest (or *minimal-density*) quantizer stabilizing quadratically the discrete scalar-control linear system is *logarithmic* and can be obtained by solving a special problem of designing the linear-quadratic optimal control. The result was extended to the discrete control of the continuous systems with constant discretization step. The optimal quantization interval was shown to be dependent only on the value of the sum of unstable eigenvalues of the continuous system, and the related optimal quantizer was proved to be *logarithmic* with the base represented by a universal system-independent constant. Such scheme of time discretization and level quantization corresponds to the concept of the minimal attention control introduced in [47]. Finally, it was shown in [73] how to design with departure from the requirement of quadratic stabilization the logarithmic quantizer with only a *finite* number of the quantization levels reaching the *practical stability* of the closed-loop system.

The stabilizable unstable linear discrete systems like (3.17) with the scalar control $\{x\} = \mathcal{X} = \mathbb{R}^n$ are considered in [73]. It has been known that for them there exists a linear static state feedback ensuring existence for the closed-loop system of the quadratic Lyapunov function $V(x) = x^{\mathrm{T}}Px$ with some positive definite matrix $P = P^{\mathrm{T}} > 0$ (*quadratic stabilizability* is understood in the sense of existence of such function in other problems as well). Needed is to determine for the given matrix P a set $\mathcal{U} = \{u_i \in \mathbb{R} \colon i \in \mathbb{Z}\}$ of the values of control and an odd function $f\colon \mathbb{R}^n \to \mathcal{U}$ ("quantizer") such that $\Delta V(x) = V\big(Ax + Bf(x)\big) - V(x) < 0$ is satisfied for all nonzero $x \in \mathbb{R}^n$. The number of the quantization levels is assumed to be unlimited, and a notion of *coarseness* of the quantizer $f(x)$ is introduced which characterizes the density of the intervals of decomposition of the space $\mathcal{X}$.[4] It was established that the logarithmic quantizer for which the quantization levels, except for the zero level, obey the recurrent relation $u_{i+1} = \rho u_i$, $i \in \mathbb{Z}$, where the parameter $0 \leqslant \rho < 1$ is calculated in terms of the matrices $A, B,$ and P, is optimal in this sense. For a stable plant, there will be, obviously, a quadratic Lyapunov function for which $\rho = 0$. Additionally, the parameter ρ is invariant to the transformation of the basis of the plant equations. Disregarding quantization, the control is determined from the antigradient of the function $V(x)$ which leads to $u = K_{GD}x$, where the $(1 \times n)$ matrix of the controller coefficients is equal to $K_{GD} = -\dfrac{B^{\mathrm{T}}PA}{B^{\mathrm{T}}PB}$. The so-calculated action is quantized, and the signal $f(u)$ is fed into the control plant. The authors of [73] noticed that this method of control corresponds to the intuitive idea that the farther the system state from the desired one, the less precise information about it is required. Therefore, it suffices to make use of an imprecise control to send system motion in the desired direction. As was also noticed in [73], the property of scaling which underlies the logarithmic quantizer is intrinsic not only to the control with quadratic Lyapunov functions, but also to any other control defining seminorms, and, consequently, the findings of this paper extend to the control by more general Lyapunov functions.

The result obtained was extended in [73] to provision of stability with the *given decay index* $0 < \alpha < 1$ which manifests itself in the existence in the closed-loop system of a quadratic Lyapunov function $V(x)$ satisfying $V(x_{k+1}) < \alpha^2 V(x_k)$. It was demonstrated in [73] that the solution obtained is related with the well-known result on the design of linear quadratic-optimal controllers according to which the poles of the closed-loop optimal system coincide with the poles of the control plant if in the case of instability [74] the latter are stable and inverse to the poles of the control plant.

Consideration was also given to the problem of state estimation of plant (3.17) with the scalar output $y_k = Cx_k$. Without regard for quantization, the observer is described by the equation

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(C\hat{x}_k - y_k), \tag{3.20}$$

where $\hat{x}_k \in \mathbb{R}^n$ is the state estimate of the plant x_k and L is an $(n \times 1)$ matrix of the observer parameters.

It was shown that if one makes use of the logarithmic quantizer to which the residual signal of the estimate $\varepsilon_k = C\hat{x}_k - y_k$ is fed instead of the measurement signal y_k, then the results established for the problem of stabilization are also applicable to this case. We recall that a similar structure of observer was recommended in [24, 55] for the problems of control with communication constraints. The corresponding equations of the observer are as follows:

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + Lh_E(\varepsilon_k), \tag{3.21}$$

where $h_E(\cdot)$ is the function of the logarithmic quantizer.

[4] This definition corresponds to the *infinitely great* density for the uniform quantizer and to the *zero* density for the quantizer with a finite number of levels.

The equations of the quantizer in the state feedback and the observer with residual quantization lead naturally to the law of *output* stabilization.

The paper [73] also considered the problem of discrete control of the linear continuous plants

$$\dot{x}(t) = Fx(t) + Gu(t), \tag{3.22}$$

where F and G are, respectively, the $(n \times n)$ and $(n \times 1)$ matrices and $x(t) \in \mathbb{R}^n$ is the state vector. The sampling (time quantization) interval T is assumed to be constant, the control $u(t)$ is regarded as piecewise-constant within the intervals bounded by the time instants $t_k = kT$, $k = 0, 1, \ldots$ Obviously, (3.17) is a discrete model of plant (3.22) for $A = e^{FT}$, $G = \int_0^T e^{A(T-\tau)} B d\tau$. The problem of determining an optimal combination of the sampling interval and the quantizer ensuring stabilization of an unstable plant under least dense time sample and measurement quantization was formulated in [73]. The base ρ of the optimal quantizer for the so-determined system (3.17) depends on the sampling interval T. The following result was established in [73]: the optimal sampling interval T^* satisfies the equation

$$T^* \sum_{\mathrm{Re}\lambda_i(F) > 0} \lambda_i(F) = \ln(1 + \sqrt{2}), \tag{3.23}$$

where $\lambda_i(F)$ are the eigenvalues of the matrix F and the corresponding optimal logarithmic quantizer has the base

$$\rho^*(T^*) = \sqrt{2} - 1. \tag{3.24}$$

Expressions (3.23) and (3.24) establish availability of "universal"—within the framework of the discussed approach—plant-independent constants: the products of the optimal sampling interval by the sum of unstable plant poles and the base of the logarithmic quantizer.

The problem of using the quantizer with a *finite* number of levels ("*finite quantizer*" was also discussed in [73]. It was shown to bring about the *practical stability*. A *finite ρ-logarithmic quantizer* of the *order* N with the set $\mathcal{U}_N = \{-u_0, -\rho u_0, \ldots, -\rho^{N-1}u_0, 0, \rho^{N-1}u_0, \ldots, \rho u_0, u_0\}$, where $0 < \rho < 1$, $u_0 \in \mathbb{R}$, was introduced. Following [75–77], the notion of quadratic stabilizability is relaxed to the *practical quadratic stabilizability* according to which it is admitted that the trajectories may converge to a limit cycle or a chaotic attractor in some equilibrium neighborhood. In more detail: (3.17) is called the semiglobally practically quadratically stabilizable system if there exists a quadratic Lyapunov function $V(x) = x^{\mathrm{T}}Px$, $P = P^{\mathrm{T}} > 0$, such that for any compact set $\mathcal{C}$ including the origin and any β_s there exists a static controller in the state feedback $f(x)$ depending on $\mathcal{C}$ such that $V(x_{k+1}) < V(x_k)$ for all $x_k \in \mathcal{C} \backslash \Omega_s$ and $x_{k+1} \in \Omega_s$ each time as $x_k \in \Omega_s$, $\Omega_s \subset \mathcal{C}$, where $\Omega_s = \{x \in \mathcal{X} \colon V(x) \leqslant \beta_s\}$. It was shown in [73] that the stabilizable system (3.17) is practically quadratically semiglobally stabilizable by the finite ρ-logarithmic quantizer with an arbitrary $\rho < \rho^*$ and sufficiently great value of N. Relations for selection of the quantizer parameters ρ and N are given in [73].

The lower boundary of the density of the coarsest quantizer stabilizing quadratically the two-input linear discrete system was determined in [78, 79]. This result demonstrates to what extent the situation may be improved (in terms of reducing the quantization density) by using two inputs instead of one for quadratic stabilization of a system. It was also shown that the quantizer which is optimal in this sense is obtained as a generalization of the logarithmic optimal quantizer of [73] and is radial-logarithmic one.

The general equivalence between the system stabilization by the analog feedback controller, on the one hand, and the data transmission through a channel with feedback based on the method of [81, 82], on the other hand, was shown in [80]. It was also shown that the attainable rate of data transmission which is expressed by the *Bode's sensitivity integral* indicates to the basic limitations of the feedback control systems. This enables application of the control-theoretical methods at designing the data transmission systems.

By generalizing the findings of [73,84], a method of stabilization of the continuous linear systems with constant parameters via a limited-capacity communication channel was suggested in [83]. Availability of such a channel leads in the closed loop to signal sampling in time and level—the data having a limited number of bits are transmitted at discrete time instants. A method of determination of the upper boundary of the data transmission rate for stabilization of the system was proposed in [83]. In particular, solved is the problem of designing a discrete controller for quadratic stabilization of the system. Consideration was given to the *static* (*memoryless*) coders. The authors of [83] stress that quadratic stabilization is made possible also for the continuous-time processes, which guarantees the desired system behavior within the quantization intervals as well. In their opinion, the static coders may just make the trajectories to hit a limited neighborhood of the system origin, rather than to provide asymptotic stabilization.

4. FUNDAMENTAL BOUNDARIES OF THE DATA EXCHANGE RATES IN THE CONTROL AND OBSERVATION SYSTEMS

4.1. Mathematical Formulation of the Problem

Let us consider a plant following the mathematical model

$$x(t+1) = Ax(t) + Bu(t) + \xi(t), \quad x(0) = x_0; \tag{4.1}$$

$$y(t) = Cx(t) + \chi(t), \tag{4.2}$$

where $t = 0, 1, \ldots$ is time, $x(t) \in \mathbb{R}^n$ is the plant state vector, $u(t) \in \mathbb{R}^m$ is the control, $y(t) \in \mathbb{R}^k$ is the vector of sensor readings, $\xi(t) \in \mathbb{R}^n$ is the external action, $\chi(t) \in \mathbb{R}^k$ is the sensor noise vector, and A, B, and C are the constant real matrices of corresponding dimensions. The initial state x_0 is unknown. The uncontrollable plant (4.1) is unstable (the matrix A has the eigenvalue λ with $|\lambda| \geqslant 1$) and needs stabilization. We also consider the problem of constructing a reliable estimate $\widehat{x}(t)$ of the current state $x(t)$ which is allied in terms of the methods of studying and the results. The estimate $\widehat{x}(t)$ is constructed in real time, that is, at the instant t.

The sensor readings $y(t)$ are transmitted to the stabilization/observation system through the memoryless *discrete stationary communication channel* (DSC) [85–88] which accepts for transmission the *discrete signals* (*messages*) e from the given finite set $\mathcal{E}$, the channel's *input alphabet*. Therefore, the sensor readings are pretransformed by the *coder* in a form suitable for transmission:

$$e(t) = \mathfrak{E}[t, y(0), \ldots, y(t)] \in \mathcal{E}. \tag{4.3}$$

This leads inevitably to loss of data because the full description of the vector $y(t)$ contains an infinite number of data bits. At transmission the message is distorted $e \mapsto s$ and, possibly, lost. Here, s is an element of (finite) *output alphabet* $\mathcal{S}$ of the communication channel. (The special element $\circledast \in \mathcal{S}$ denotes loss of message.) On the basis of the data s received by the current instant, the *decoder-controller* generates the control

$$u(t) = \mathfrak{U}\left[t, s(0), s(1), \ldots, s(t)\right], \tag{4.4}$$

and the *decoder-observer* estimates the current state

$$\widehat{x}(t) = \mathfrak{X}\left[t, s(0), s(1), \ldots, s(t)\right]. \tag{4.5}$$

Explanation 4.1. In the problem of estimation, controller (4.4) is given; for the uncontrollable plant, $\mathfrak{U}[\cdot] \equiv 0$.

Both the decoder and coder are subject to construction. The *stabilization system* is made up of coder (4.3) and decoder-controller (4.4), and the *observation system*, of coder (4.3) and decoder-observer (4.5). Such system is called *consistent* if it enables attainment of the desired aim, that is, stabilizes the unstable plant (4.1) or estimates it reliably. The problem lies in determining the—desirably, necessary and sufficient— conditions under which this system can be constructed.

These conditions depend on the characteristics of the communication channel. The aforementioned popular model of such channel—the discrete memoryless channel—assumes that available are the statistical data about its operation in the form of the matrix of conditional probabilities

$$W(s|e) = \boldsymbol{P}\left[s(t) = s|e(t) = e\right]. \tag{4.6}$$

Stated differently, $W(s|e)$ is the probability of receiving the discrete signal s at the channel output, provided that sent was the signal e.[5] The sought-for conditions actually lie in describing the domain of stabilizability or observability in the space of parameters of both plant (4.1), (4.2) and channel:

$$\left[A, B, C, \{W(s|e)\}_{e\in\mathcal{E}, s\in\mathcal{S}}\right]. \tag{4.7}$$

Explanation 4.2. The discrete *noiseless channel*, that is, a channel $\mathcal{E} = \mathcal{S}$, $s(t) = e(t)$; $W(s|e) = 1$ for $s = e$ and $W(s|e) = 0$ for $s \neq e$ not losing and distorting the messages, is a special case of DSC.[5] Since the "physical nature" of the elements of e is unessential for the problems at hand, the part of collection (4.7) related with the channel comes to the number of elements $|\mathcal{E}|$ of the channel alphabet.

4.2. Comments to the Mathematical Formulation of the Problem

1. For the stabilization problem, the above formulation implies the perfect control channel.[7] In other words, it is assumed that the control channel has an unlimited capacity and immediately transmits messages without losses and distortions. This enables one to abstract from the details of transmission through this channel. Such situation may occur if the controller and actuator are combined in space.

2. In a more general formulation, the control channel is modeled explicitly as a memoryless DSC with the input $\mathcal{E}_u = \{e_u\}$ and output $\mathcal{S}_u = \{s_u\}$ alphabets and the matrix of transient probabilities $\{W_u(\cdot|\cdot)\}$. If the coder of this channel is considered as part of the decoder-controller, the latter's equation may be set down as

$$e_u(t) = \mathfrak{E}_u\left[t, s(0), s(1), \ldots, s(t)\right]. \tag{4.8}$$

The actuator generates control following an equation like

$$u(t) = \mathfrak{U}_a\left[t, s_u(0), s_u(1), \ldots, s_u(t)\right]. \tag{4.9}$$

The stabilization system is made up of the coder of sensor (4.3), decoder-controller (4.8), and decoder of actuator (4.9). The stabilizability domain is specified in the space of collections (4.7)

[5] It is assumed that under the condition $e(t) = e$ the channel output s is independent of noise in plant (4.1), (4.2), initial state x_0, and signal sent at other time instants $\tau < t$.

[5] Of the imperfections related with the channel, this model takes into consideration only the effect of data quantization, other traditionally accounted for effects being losses, distortions, and transmission delays.

[7] That is to say, the communication channel between the controller and actuator in distinction to the *observation channel* transmitting the sensor readings to the controller.

complemented by the matrix $\{W_u(\cdot|\cdot)\}$ of control channel parameters. This more general formulation sometimes comes to the preceding formulation (see comment 6 in Sec. 4.4).

3. A number of other extensions of the problem formulation such as the infinite-alphabet channels, delay channels, and nonstationary channels will be discussed in what follows. In particular, we will touch upon a situation where the sensor coder is informed about the result of the current transmission $s(t)$ through the communication channel.

4. Unless the contrary is stated, (4.3)–(4.5), (4.8), (4.9) consider any nonanticipative transformations, which is useful for specifying the limits attainable as the result of unlimited increment of the computing resources. The real accessible resources can be sometimes reflected by constraining the choice of the right-hand side in (4.3)–(4.5), (4.8), (4.9) to the membership in a certain class. For example, it is only natural to consider sometimes only the stationary static coders of the sensor and decoders of the actuator: $e(t) = \mathfrak{E}[y(t)]$ in (4.3) and $u(t) = \mathfrak{U}_a[s_u(t)]$ in (4.9).

5. In some cases, stabilizability/observability in the classes of all nonanticipative and more realistic coders and decoders are equivalent (see comment 5 from Sec. 4.4).

6. The mathematical description of the aim of control/observation and some other details of the problem formulation are variable and discussed below.

7. In general terms, the control/observation system is *consistent* if the error of stabilization $\|x(t)\|$/observation $\|x(t) - \widehat{x}(t)\|$ with the passage of time remains bounded in a reasonable sense to be discussed below. For the noiseless model $\xi(t) \equiv 0$, $\chi(t) \equiv 0$, convergence to zero for $t \to \infty$ is usually required.

8. Within the framework of the present discussion, in many cases the models obeying the ordinary differential equations are rearranged by standard methods [89] in a model based on the difference equation (4.1), (4.2).

4.3. General Comments to the Problem Formulation

The traditional approach to the problem lies in division of its "control" and "informational" aspects. At first, the researchers abstract away from the properties of the channel which is regarded as capable of transmitting an unlimited volume of information in time unit. As applied to the conditions for stabilizability, this leads, for example, to the well-known answer that the pair (A, B) should be stabilizable and (A, C), detectable. In practice, it is not the sensor output $y(\theta)$ but the result of its transmission through the communication channel that is fed into the input of the stabilization system designed within the framework of this abstraction. Transmission requires quantization of the signal $y(\theta)$, its coding, transmission proper, and decoding. The problem is treated here as the general communication problem solved using the general-purpose methods oriented to the abstract performance criteria for transmission. The factors dependent on discreteness of some components of the control loop are modeled typically as the components of the classical model, for example, the quantization errors as an additional sensor noise (see Sec. 2).

As was already noted, for a long time this, on the whole, traditional approach gave no ground for seeking an alternative. In the most general terms, this approach is quite appropriate if the communication and computation resources are "very extensive" as compared with the anticipated need or if there are no essential constraints on their choice and experimentation is possible, whereas the requirement on effective use of the resources is not topical.

The potential weak points of the discussed approach and the need for theoretical consideration of its applicability boundaries were formally highlighted in a number of theoretical studies. Even simple introduction of a quantizer in the feedback loop can chaotize the stable dynamics [12,43,44] and, in a certain context, make inapplicable some classical definitions of stability [12,43,44,48,65].[8] This approach may lead to an essential loss of control performance: even if the optimal (for the classical problem formulation) control law is used and then the optimal method of quantization and coding is used for data transmission, the control performance may turn out to be much worse than that attainable in the class of the control systems like (4.3), (4.4) [91].[9]

The idea of treating the "informational" part of the problem as a general problem of information transmission that must be solved using the "general-purpose" methods developed in the information theory was subjected to critical analysis in some publications ([4,48,65,92–100], and others). One of the reasons had to do with the adequacy to the problem of control of the traditional practice and theory of information coding which rely mostly on the concept of block coding. With the block code of length r, the information selected at the instant t is coded as the chain $(e_1,\dots,e_r)$ of messages e_i. These messages are transmitted one after another during the time interval $[t\colon t+r-1]$ of duration r, and transmission of new data is impossible until the instant $t+r$. If used in the feedback loop, this approach implies it is broken for the time r, which in the presence of perturbations may result in unacceptable stabilization errors for longer r. Reduction in the code length r may provide the same result because it is associated with higher probability of erroneous decoding [85,87]. A need for compromise arises here. One of the possible approaches lies in modifying the classical formulations of the information transmission problem with regard for the ideas suggested by the control problems. Specimens of this kind can be found, for example, in [92,99] and their reviews where consideration was given to the problem of successive optimal real-time quantization of the Markov stationary source and a "compromise" performance criterion was used. Another approach can be found in Sec. 4.1 where the "informational" part of the problem is not at all separated as an individual problem, and the resulting control performance is the single criterion.

With regard for the aforementioned circumstances and some theoretical facts [98,101,102], it is only natural to ask in what relation and to what extent the traditional methods of information coding are adequate to or optimal as applied to the problems of control. Is there need for new approaches or the problem yields to correct application of the well-known "general-purpose" methods? Projection of this question into the theoretical domain concerns, in particular, such basic channel characteristic as the Shannon capacity [85–88] which is equal to the maximal mean rate (in bits per time unit) with which information can be transmitted with an arbitrarily small error probability. (The error probability may be made arbitrarily small by an appropriate use of the block code of an adequate length.) At the same time, this capacity is equal to the maximal mutual information between the input and output of the communication channel:

$$\mathfrak{c} = \max_{\boldsymbol{P}(e)} \boldsymbol{I}, \qquad \text{where} \qquad \boldsymbol{I} := \sum_{e,s} \boldsymbol{P}(e\wedge s)\log_2 \frac{\boldsymbol{P}(e\wedge s)}{\boldsymbol{P}(e)\boldsymbol{P}(s)} \tag{4.10}$$

and $\boldsymbol{P}(e\wedge s) := W(s|e)\boldsymbol{P}(e), \boldsymbol{P}(s) := \sum_e \boldsymbol{P}(e\wedge s)$, the maximum being taken over all probabilistic distributions on the channel input alphabet. For many standard problems considered by the classical information theory, the relevant properties of the channel are totally defined by its capacity $\mathfrak{c}$. Does this situation exist for the problems of control? If no, then what new characteristics are required, and for which purposes and possibilities the Shannon capacity is responsible in this area?

[8] It was shown in particular in [12] that even for an ideal measurement $y(t)=x(t)$ and no noise both in the plant $\xi(t)\equiv 0$, $\chi(t)\equiv 0$, and the communication channel $\mathcal{E}=\mathcal{S}$, $s(t)=e(t)$ the stabilization system (4.3), (4.4) with the dynamic decoder (4.4) and static stationary coder (4.3) $e(t)=\mathfrak{E}[x(t)]$ is incapable of ensuring the convergence $x(t)\to 0$ for $t\to\infty$ from the standard definition of the asymptotic stability: there is no convergence for all initial states, except for the points of the set of zero Lebesgue norm.

[9] The paper [91] considered an analog to the classical stochastic linear-quadratic problem of optimization with Gaussian noise in the class of the control systems like (4.3), (4.4) with static stationary decoders $u(t)=\mathfrak{U}[s(t)]$ in (4.4) and noiseless communication channel $\mathcal{E}=\mathcal{S}$, $s(t)=e(t)$.

4.4. Basic Result in the Case of the Discrete Noiseless Communication Channel

This result finally took shape as a summary of some works that used formally different formulations and assumptions. At the same time, the results are essentially the same. We present and discuss a formulation of the basic result proceeding from the case of a noiseless channel and omitting some details that are considered in the following subsection where a review of publications is also given.

We consider the noiseless communication channel $\mathcal{E} = \mathcal{S}$, $s(t) = e(t)$ whose capacity is equal to the logarithm of the number of elements of the channel alphabet:

$$\mathfrak{c} = \log_2 |\mathcal{E}| \quad \text{(bits/time unit)}. \tag{4.11}$$

In what follows, the control system (4.1), (4.2) is assumed to be stabilizable and detectable (in the traditional sense of the word).

The following consolidated formulation of the stabilizability/observability criterion indicates to the fundamental boundary of the data transmission rate for which the desired aim is attainable and omits some nuances explained in the following subsection.

Theorem 2. *Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ be the eigenvalues of the matrix A from* (4.1) *listed with regard for their algebraic multiplicity and*

$$\eta(A) := \sum_{j:|\lambda_j| \geqslant 1} \log_2 |\lambda_j|. \tag{4.12}$$

The following assertions are true:

(i) *for existence of coder* (4.3) *and decoder-controller* (4.4) *stabilizing the system, it is necessary that*

$$\eta(A) \leqslant \mathfrak{c}, \tag{4.13}$$

and sufficient that

$$\eta(A) < \mathfrak{c}; \tag{4.14}$$

(ii) *let the decoder-controller* (4.4) *be given. For existence of coder* (4.3) *and decoder-observer* (4.5) *generating a reliable estimate $\widehat{x}(t)$ of the state $x(t)$, inequality* (4.13) *is necessary, and inequality* (4.14) *is sufficient.*

As was already noted in comment 7 (Sec. 4.2), in most general terms the unstable system (4.1), (4.2) is regarded as stabilized and its state estimate, as reliable if the error of stabilization $\mathrm{err}(t) := \|x(t)\|$ (correspondingly, the error of estimation $\mathrm{err}(t) := \|x(t) - \widehat{x}(t)\|$) remains with time bounded and in the absence of noise $\xi(t) \equiv 0$, $\chi(t) \equiv 0$ tends to zero for $t \to \infty$. The definitions met in the literature are variable and context-dependent; they are discussed in what follows.

Comments.

1. Condition (4.14) is obtained from (4.13) by replacing the nonstrict inequality by a strict one. In this sense, these conditions are allied and, therefore, necessary and almost sufficient for stabilizability and observability. We refer below to them as the *tight conditions*.

2. The strict inequality (4.14) is not only sufficient, but also necessary for plant stabilizability in stochastic noise in (4.1), (4.2) [97] and for exponential [103,104] and asymptotic [105] stabilizability of a noiseless plant.

3. In (4.12) one can confine oneself to summation only over the strongly unstable eigenvalues $\lambda_j : |\lambda_j| > 1$.[10] We also notice that

$$\eta(A) = \log_2 \left|\det A|_{L_+}\right|,$$

where L_+ is an unstable invariant subspace of the matrix A, that is, an invariant subspace corresponding to the spectral set $\sigma_+ := \{\lambda_j : |\lambda_j| \geqslant 1\}$, and $A|_{L_+}$ is a contraction of the operator A to L_+.

4. Theorem 2 is true not only for the noiseless channel, but also for a diversity of its generalizations. Although for the generalized models of the channel the right-hand side $\mathfrak{c}$ of inequalities (4.13) and (4.14) is not necessarily defined according to (4.11), the general sense of $\mathfrak{c}$—the mean number of information bits that can be transmitted through the channel in a time unit—remains the same.

5. By Theorem 2, inequality (4.13) is necessary for stabilizability and observability in the class of all nonanticipative coding and decoding algorithms. This class contains algorithms of computational complexity—defined by the memory volume, number of operations per time unit, and so on—growing unlimitedly in time. The majority of works substantiate the sufficiency of the strict inequality (4.14) by explicit construction of the stabilization/observation system of limited (algebraic) computational complexity.

6. Theorem 2 assumes formally that the nonidealities of the control channel are negligible (it is assumed, in particular, that the channel has an unlimited capacity) and in this connection it clearly is not modeled. Simple reasoning [96] enables one to extend this theorem to the case of the stabilization problem where both channels (of observation and control) are noiseless and have limited capacity. (In this case, the stabilization system obeys (4.3), (4.8) and (4.9).) Assertion (i) of Theorem 2 remains valid if the minimal capacity of the above two channels, $\mathfrak{c} = \min\{\log_2 |\mathcal{E}|; \log_2 |\mathcal{E}_u|\}$, is taken as $\mathfrak{c}$ in (4.13), (4.14).

Discussion of Theorem 2 and the value $\boldsymbol{\eta(A)}$ from (4.12). This value has the sense of the **entropy growth rate** (state uncertainty) of the **open-loop system** $x(t+1) = Ax(t)$ measured in bits per time unit. More precisely, it is a question of the mean rate of increase of the information volume necessarily included in the description of the current system state with the given constant precision. Against this background Theorem 2 becomes very transparent: it is required for stabilizability/observability that the channel be capable of transmitting information at a rate enabling compensation of the growth in uncertainty about the current state of the system.

In more detail: if the necessary condition of Theorem 2 is violated, that is, $\eta(A) > \mathfrak{c}$, the communication channel cannot transmit information at a rate required to maintain any given precision of describing the system state. Therefore, with time the decoder's idea of the plant's current state is doomed to inevitable unlimited impairment, and, therefore, it is quite obvious that the decoder cannot stabilize the plant and construct a consistent estimate of the current state. If the sufficient condition $\eta(A) < \mathfrak{c}$ is satisfied, then the channel in principle can keep the decoder's idea of the plant position within the limits of a bounded error. Additionally, in view of certain redundancy due to the strict inequality, there are grounds to figure on the possibility of reducing the error. Therefore, the conclusion that the plant can be stabilized and a consistent estimate constructed is not something unexpected.

Interpretation of $\eta(A)$ may be justified proceeding from the notion of differential entropy of the random vector $X \in \mathbb{R}^n$ with the probability density $p_X(x)$:

$$h(X) := -\boldsymbol{E} \log_2 p_X(X) = -\int\limits_{\mathbb{R}^n} p_X(x) \log_2 p_X(x)\, dx. \tag{4.15}$$

[10] We assume throughout the paper that the sum over an empty set is equal to zero.

This entropy may be regarded as a measure of information required to describe the vector X with a given precision (against the background of a certain density). Description with the precision 2^{-b} requires approximately $h(X) + nb + \log_2 \mathbf{Vol}\, B_0^1$ bits [106]. Here, $\mathbf{Vol}\, B_0^1$ is the volume (Lebesgue measure) of the unit sphere B_0^1 in the space $\mathbb{R}^n$.

Let the initial state x_0 be a random vector having probability distribution density and finite differential entropy. We assume for simplicity that the matrix A has no stable eigenvalues $|\lambda_j| \geqslant 1\ \forall j$. Elementary calculations (see, for example, [97, 107]) provide the following formulas justifying the above interpretation of $\eta(A)$:

$$h[x(t+1)] = h[x(t)] + \eta(A), \qquad \text{where} \quad x(t+1) = Ax(t), \quad t = 0, 1, \ldots \tag{4.16}$$

The above argument implies a probabilistic model of the plant and relies, if there are both unstable and stable eigenvalues, on the agreement to take into account only the unstable part of the system.[11] An alternative approach appealing to the notion of entropy of the deterministic dynamic system that was formulated by Kolmogorov [108, 109] and later developed along several lines (see, for example, the reviews in [110–112]) is less onerous in this respect. In particular, the metric Kolmogorov entropy was transformed in [113] into a more general notion of the topological entropy. In [114], this notion was adapted to the context of the problems of control through a limited-capacity communication channel, and the conditions for local stabilizability of the nonlinear plant were formulated in terms of this entropy. In [115], a similar program was realized on the basis of the metric entropy in the form given to this notion in [116], and the necessary and sufficient conditions for stabilizability (robust including) and observability of the nonlinear plant were established. The contribution of [114, 115] is discussed in Sec. 4.7 devoted to the nonlinear systems, and now we confine consideration to a fragment from [115] illustrating the discussed value $\eta(A)$.

Let us consider the nonlinear (uncontrollable) dynamic system

$$x(t+1) = F[x(t), \omega(t)], \quad x(t) \in \mathfrak{X}, \quad t = 0, 1, \ldots, \quad x(0) \in \mathfrak{X}_0, \tag{4.17}$$

where $x(t) \in \mathbb{R}^n$ is the state, $\omega(t) \in \Omega$ is the vector reflecting the dynamic system uncertainty, and $\mathfrak{X}_0 \subset \mathfrak{X} \subset \mathbb{R}^n$ and Ω are the given sets. These sets and the function $F(\cdot)$ are known; the initial state x_0 and the vectors $\omega(t)$ are unknown; the set $\mathfrak{X}_0$ expresses the a priori information about the initial state. For $k = 1, 2, \ldots$, we denote by $\mathfrak{X}_k$ the set of every possible[12] processes $[x(0), \ldots, x(k)]$ in system (4.17). This set expresses the a priori knowledge of the process. The set $Q \subset \mathfrak{X}_k$ is called the (k, ϵ)-network ($\epsilon > 0$) if any process $x_a(\cdot) \in \mathfrak{X}_k$ may be ϵ-approximated by the process $x_b(\cdot) \in Q$, that is, $\|x_a(t) - x_b(t)\| < \epsilon\ \forall t = 0, \ldots, k$. The least cardinality (number of elements) of this set is denoted by $q(k, \epsilon)$. If it is required to bring the precision of process description to ϵ against the background of the a priori knowledge, then the question may be reduced to the description of the nearest element of the a priori computable set Q for what purpose $\log_2 q(k, \epsilon)$ information bits are required. If the description itself is considered as a process over the time interval $[0 : k]$, then $(k+1)^{-1} \log_2 q(k, \epsilon)$ bits will be required on the average for a time step. For $k \to \infty$ and $\epsilon \to +0$, the limit of this value was called in [115] the *topological entropy* of system (4.17):

$$H[F(\cdot, \cdot), \mathfrak{X}_0, \mathfrak{X}, \Omega] := \lim_{\epsilon \to 0} \limsup_{k \to \infty} \frac{1}{k+1} \log_2 [q(k, \epsilon)]. \tag{4.18}$$

This limit indicates to the step-averaged increase in the volume of information necessarily contained in the description of the current process with the given precision. The required interpretation of

[11] In the general case, $h[x(t+1)] = h[x(t)] + \log_2 |\det A|$ and $\log_2 |\det A| \neq \log_2 |\det A|_{L_+}| = \eta(A)$ (we assume that $\det A \neq 0$). However, by considering the spectral projector π_+ on L_+ and the "unstable part" $x^+(t) := \pi_+ x(t)$ of the state vector, we obtain $x^+(t+1) = A|_{L_+} x^+(t) \Rightarrow h[x^+(t+1)] = h[x^+(t)] + \eta(A)$.

[12] That is, corresponding to every possible $x(0) \in \mathfrak{X}_0$ and $\omega(t) \in \Omega$ and satisfying the condition $x(t) \in \mathfrak{X}\ \forall t \in [0 : k]$.

$\eta(A)$ relies on the following fact [115]: for an arbitrary matrix A, compact set $\mathcal{X}_0$ containing 0, and $\mathcal{X} = \mathbb{R}^n$, the topological entropy of the linear system $x_{t+1} = Ax_t$ is equal to $\eta(A)$:

$$H(A, \mathcal{X}_0, \mathbb{R}^n) = \eta(A). \tag{4.19}$$

4.5. Details, Modifications, and Review of the Literature

The publications [48, 65, 117, 118] were the first to give a clear-cut definition of the fundamental boundaries of the data transmission rates in the problems of control and exercised an important influence on the further development of this area. We are going to characterize in more detail [65] which is interest, in particular, because the problem formulation studied there complements the formulations of many subsequent works. The problem lay in stabilization of the continuous-time unstable linear noiseless plant:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \in [0, \infty), \quad x(0) = x_0, \quad y(t) = Cx(t), \tag{4.20}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control, and $y(t) \in \mathbb{R}^p$ is the measurement. The channels of observation and measurement transmit messages e selected from alphabets of identical size D without distortions and at the same rate: one message per time $\delta > 0$. At the discrete time instants r_i, the static coder $c_i = h[y(r_i)]$ transforms the current measurement into the sequence $c_i = [e_1, \ldots, e_{l_i}]$ of messages of the length $l_i = l(c_i)$ selected by the coder. At the instant $r_i + \delta l(c_i)$, the full sequence c_i reaches the static decoder-controller which transforms its $d_i = g(c_i)$ into the sequence d_i (possibly, of another length $l(d_i)$) sent through the control channel. At the instant $r_{i+1} := r_i + \delta l(c_i) + \delta l(d_i)$, the transformed sequence reaches the static actuator and is used to change the control $u(r_{i+1} + 0) := k(d_i)$. The instants r_i are not given in advance, and the prefix coding is used to recognize the end of transmission. By the stabilizability is meant the possibility of constructing a control system $h(\cdot), g(\cdot), k(\cdot)$ maintaining the system trajectories within any predefined sphere centered at zero, provided that these trajectories begin quite near zero.[13]

As was established in [65], the following system of inequalities must be satisfied for such system to exist:

$$\sum_{i=0}^{\infty} D^{-m_i} \leqslant 1, \quad \sum_{i=0}^{\infty} D^{-n_i} \leqslant 1, \quad 1 \leqslant \sum_{i=0}^{\infty} \tau^{-m_i - n_i},$$

where $m_i := l(c_i)$, $n_i := l(d_i)$ and $\tau := \left|\det e^{\delta A}\right|$. (The two first conditions are related with the prefix coding and represent the well-known Kraft inequality [87].) If the sequences $\{m_i\}$ and $\{n_i\}$ have identical sets of elements, then from the above conditions it follows ([65, Corollary 1]) that

$$\log_2 \left|\det e^{2\delta A}\right| \leqslant \log_2 D. \tag{4.21}$$

With regard for the fact that the right-hand side is equal to the volume of information that can be transmitted from the sensor to the actuator through the available communication channels in time 2δ and the left-hand side shows to what extent the topological entropy of the open-loop system increases during this time,[14] this inequality is similar to condition (4.13). The sufficient conditions of [48, 65] refer to a more specific situation and, generally speaking, are far from (4.21).

It was shown in [117, 118]—and in more detail in [15]—that for the scalar $x = y$, $u \in \mathbb{R}$ plant (4.1), (4.2) without noise $\xi(t) \equiv 0$, $\chi(t) \equiv 0$ with a discrete time model, the nonstrict inequality

[13] This property is referred to as *containability.*

[14] More precisely, the entropy of the associated discrete system $x(t_{i+1}) = e^{2\delta A} x(t_i)$, $t_i := 2\delta i$, in the case where A has no stable eigenvalues.

(4.13) is necessary and sufficient for existence of the static coder (4.3) $e(t) = \mathfrak{E}[x(t)]$ and decoder-controller $u(t) = \mathfrak{U}[e(t)]$ ensuring local stability of the closed-loop system in the following—weaker as compared with [48, 65]—sense: there exists in the system phase space a bounded set S and its nonempty open subset $M \subset S$ such that the trajectories beginning in M do not leave S.[15] For the case of multidimensional linear plant with a real diagonizable matrix A, scalar control $u \in \mathbb{R}$, and binary channel $\mathfrak{c} = 1$, the sufficient conditions have been established as the strengthening $n \max_j \log_2 |\lambda_j| < 1$ of inequality (4.14).

In the general case of the multidimensional linear plant, the precise conditions for stabilizability/observability through the noiseless bounded-capacity communication channels were established in various formulations in [28, 51, 96, 97, 99, 104, 119–122]. The landmark achievements of the series of works [28, 51, 96, 97, 104, 120, 121] in studying the problem of stabilization are presented in an extended form in [96,97,104]. For the problem of estimation, allied results are given in [51,120,121]. In what follows, we confine ourselves to the characteristic of the works on stabilization which are distinguished for the stochastic formulation of the problem: the initial state x_0 from (4.1) is a random (not necessarily bounded) vector with a finite probabilistic moment; consideration is given to the moment stability. The scalar process of autoregressive moving average in the absence of noise and under special initial conditions (process zeroing at the instants preceding the experiment) was considered in [96]. The distribution of the initial state over a special one-dimensional space corresponds in model (4.1) to the last fact. Although having a similar sense, the established precise conditions for asymptotic stabilizability $\boldsymbol{E}\|x(t)\|^r \to 0$ for $t \to \infty$ are not identical to (4.14). It is only natural to attribute this divergence to the fact that the constraint of the initial state by a one-dimensional subspace in fact makes the dynamics of the noiseless open-loop system one-dimensional and, therefore, (4.16) and (4.19) are subject to correction. The exponential stabilizability in the sense of reaching the exponential moment stability $\sigma^{-rt}\boldsymbol{E}\|x(t)\|^r \to 0$ for $t \to \infty$ for the general multidimensional noiseless plant (4.1), (4.2) $\xi(t) \equiv 0$, $\chi(t) \equiv 0$ was considered in [104]. Here, $r > 0$ and $\sigma > 0$ are the given constants. The necessary and sufficient stabilizability conditions were established as

$$\sum_{j:|\lambda_j|>\sigma} \log_2 \left|\frac{\lambda_j}{\sigma}\right| < \mathfrak{c}.$$

For $\sigma = 1$, this inequality goes over to condition (4.14), and in the absolute value the changes of variables $x(t) := \sigma^{-t}x(t)$, $u(t) := \sigma^{-t}u(t)$, $y(t) := \sigma^{-t}y(t)$ and parameters $A := \sigma^{-1}A$, $B := \sigma^{-1}B$ are equivalent to this condition. The paper [97] established a fundamental theoretical fact that, under some natural technical assumptions, the quadratic stabilizability $\sup_{t=0,1,2,\ldots} \boldsymbol{E}\|x(t)\|^2 < \infty$ of the general multidimensional plant (4.1), (4.2) with stochastic noise $\xi(t)$, $\chi(t)$ is equivalent to satisfying the strict in equality (4.14).

Deterministic formulations of the problem were considered in [99, 122]. In particular, the initial state x_0 is an unknown vector from the given open set Λ_0. In the no-noise case $\xi(t) \equiv 0$, $\chi(t) \equiv 0$, the stabilization/estimation system is regarded as consistent if the corresponding error $\mathrm{err}(t) := \|x(t)\|$ (correspondingly, $\mathrm{err}(t) := \|x(t) - \widehat{x}(t)\|$) is (i) uniformly small for small initial states $\lim_{x_0\to 0} \sup_t \mathrm{err}(t) = 0$ and (ii) tends with time to zero $\lim_{t\to\infty} \sup_{x_0:\|x_0\|\leqslant r} \mathrm{err}(t) = 0\ \forall r > 0$. Consideration was given also to the case of a plant with bounded perturbation $\sup_t \|\xi(t)\| < \infty$ and precise measurement $\chi(t) \equiv 0$. In this case, it is required that the error be constrained by $\sup_t \sup_{x_0:\,\|x_0\|\leqslant r} \mathrm{err}(t) < \infty$. The nonstrict inequality (4.13) was shown to be necessary both for stabilizability and observability of both the perturbed and nonperturbed plants. For the case of full measurement $y(t) = x(t)$ and bounded set Λ_0 (knowledge of the boundary or its estimate is unnecessary), sufficiency of the strict inequality (4.14) for stabilizability and observability was es-

[15] In [15, 117, 118] this property is referred to as *boundability*.

tablished for the plants of both types. It was shown that in the case of uncontrollable ($u(t) \equiv 0$) unstable ($\exists j\colon |\lambda_j| > 1$) noiseless plant (4.1) ($\xi(t) \equiv 0$) the observation system with static stationary coder $e(t) = \mathfrak{E}[y(t)]$ fails to provide observation error boundedness for $|\mathcal{E}| < \infty$. The situation is different for the problem of stabilization. In the case of full and precise observation $y(t) = x(t)$ and bounded uncertainty of the initial state $\sup_{x_0 \in \Lambda_0} \|x_0\| < \infty$, the noiseless plant can be stabilized using a static stationary coder, provided that the channel capacity is sufficiently high [122].

Similar formulations were examined in [119] where consideration was given, however, to the linear continuous-time plant (4.20) without stable modes ($\Re\lambda_j \geqslant 0\ \forall j$). The control varies at the instants t_i, $0 = t_0 < t_1 < t_2 < \ldots, t_i \xrightarrow{i\to\infty} \infty$; the step $t_{i+1} - t_i$ is not constant and bounded by any requirements. During any interval $[t_i, t_{i+1}]$, one data bit is transmitted without distortions.[16] As in [99, 122], the initial state x_0 is unknown and belongs to the bounded set Λ_0 of a nonzero measure. It was shown that for existence of a dynamic coder and a decoder ensuring the uniform boundedness of the stabilization error, the following inequality should be satisfied:

$$\frac{1}{\ln 2} \sum_j \lambda_j \leqslant \varliminf_{i\to\infty} \frac{i}{t_i}. \tag{4.22}$$

In the case of diagonal matrix A, this inequality is sufficient as well, and the change of $\leqslant$ by $<$ provides the sufficient condition for uniform asymptotic stabilizability $\lim_{t\to\infty} \sup_{x_0\in\Lambda_0} \mathrm{err}(t) = 0$. Its sufficiency was demonstrated also for a bounded-noise plant.

Condition (4.22) is similar in its sense to condition (4.13). Indeed, it is natural to interpret its left-hand part

$$\frac{1}{\ln 2} \sum_j \lambda_j = \sum_j \log_2 \left| e^{\lambda_j} \right| \overset{(4.13),(4.19)}{=\!=\!=\!=} H(e^A, \Lambda_0, \mathbb{R}^n)$$

as the rate of growth of the entropy of the open-loop system $\dot{x} = Ax$, whereas the right-hand part represents the mean capacity of the communication channel.

The case of nonstationary capacity available at the given step was also considered in [97,105,122]: at a discrete instant t the channel can transmit $R(t)$ data bits immediately and without distortions. Then, in (4.13), (4.14) one must take the average channel capacity $\mathfrak{c} := \lim_{T\to\infty} T^{-1} \sum_{t=0}^{T-1} R(t)$ as $\mathfrak{c}$. In more detail: [105] established the precise conditions for uniform asymptotic stabilizability $\lim_{t\to\infty} \sup_{x_0 : \|x_0\| \leqslant r} \mathrm{err}(t) = 0$ in the form (4.13), (4.14) for the noiseless plant $\xi(t) \equiv 0$, $\chi(t) \equiv 0$ with real diagonizable matrix A. The precise conditions for quadratic stabilizability $\sup_t \boldsymbol{E}\|x(t)\|^2 < \infty$ of the plant in noise and arbitrary matrix A were established in [97], and [122] established sufficiency of the strict inequality (4.14) for observability of the noiseless plant in the case of precise and full measurement $y(t) = x(t)$.[17]

In [103] even a more general deterministic model of the communication channel was considered where not only possible oscillations of the channel capacity were taken into consideration, but the data transmission delays, losses, and distortions as well.[18] The communication channel is regarded

[16] By passing to the difference model describing the change of states $x(t_i)$ at the instants t_i, we obtain in distinction to [99,122] a model like (4.1), (4.2) with the nonstationary matrix A and the channel capacity $\mathfrak{c} = \mathfrak{c}_i = (t_{i+1} - t_i)^{-1}$.

[17] The channel properties were constrained in [105,122] by the existence of the limit $\lim\limits_{T\to\infty} T^{-1} \sum\limits_{t=0}^{T-1} R(t)$. In the opinion of the present authors, the proposed proofs of the sufficient conditions used in fact an implicit insignificant strengthening of this property. The necessary stabilizability conditions were given in [97] in terms of $\liminf\limits_{T\to\infty} T^{-1} \sum\limits_{t=0}^{T-1} R(t)$. For substantiation of the sufficient conditions, the channel alphabet was taken time periodic.

[18] The last of the factors is allowed for in a limited volume: it is assumed that the measures to compensate it (for example, as the error correcting code) have already been taken and manifested themselves as a reduction in the actual mean channel capacity, that is, as a result the messages are either lost or transmitted without distortions.

as a device transmitting messages with delays and losses. Let $b^+(t_0, t_1)$ be the maximal possible amount of information (in bits) carried by the transmissions initiated and completed during the time interval $[t_0 : t_1]$. Let in turn $b^-(t_0, t_1)$ be the amount of information that can be transmitted with assurance over this interval independently of the forming situation. The main requirement on the channel in [103] lies in stabilization of the mean values of these two values toward the common limit $\exists \lim_{t_1-t_0\to\infty}(t_1-t_0)^{-1}b^+(t_0,t_1) = \lim_{t_1-t_0\to\infty}(t_1-t_0)^{-1}b^-(t_0,t_1) = \mathfrak{c}$. The precise conditions for the exponential stabilizability of the noiseless plant were obtained in terms of this limit which has the sense of the mean channel capacity as (4.14). A number of examples of channels satisfying this requirement was presented in [103,123], and their mean capacity $\mathfrak{c}$ was calculated. In particular, it was proved that the limited (including the nonstationary) time delays in data transmission do not affect the value of $\mathfrak{c}$ and, therefore, the stabilizability conditions (4.13), (4.14).[19]

A stochastic modification of the scalar plant (4.1) $x, u \in \mathbb{R}$, $A = a$, $B = b \in \mathbb{R}$ without noise $\xi(t) \equiv 0$ with precise measurement $y(t) = x(t)$ was considered in [124, 125]: the coefficients of the equation from (4.1) are defined as $a = a[z(t)]$, $b = b[z(t)]$ by the current state $z(t) \in \{z_1, \ldots, z_k\}$ of the irreducible stationary Markov chain with a finite phase space and the transition probability matrix $T = (\boldsymbol{P}[z(t+1) = i|z(t) = j])_{i,j=1}^k$. The initial state x_0 is a random variable; the current state of the chain $z(t)$ is known to the coder and decoder. It was established that under certain technical assumptions such plant is asymptotically stabilizable in the ηth moment $\boldsymbol{E}|x(t)|^\eta \to 0$ for $t \to \infty$ through the noiseless communication channel of capacity $\mathfrak{c}$ if and only if $\mathfrak{c} > \eta^{-1} \log_2 \rho(\mathcal{A}^\eta T)$. Here, $\mathcal{A}$ is the diagonal matrix with the diagonal elements $|a(z_1)|, \ldots, |a(z_k)|$ and $\rho(\cdot)$ is the greatest nonnegative eigenvalue of the matrix, its existence being guaranteed by the Frobenius–Perron theorem [126, Ch. XIII, § 2].

Although not focusing on the main topic of this section, that is, the establishment of the precise boundary of the stabilizability/observability domain, the works [12, 15, 26, 61, 127–130] throw on it additional light illuminating the situation up to or within the nearest neighborhood of one of the sides. Some of them (for example, [26,127,130]) also made a recognized methodological contribution to this subject. On the whole, many of the aforementioned publications go outside the scope of the present section, and, therefore, we confine ourselves only to their corresponding part and consider some of them in what follows from another standpoint.

The sufficient conditions for global asymptotic stabilizability and observability were established in [26,127] for the noiseless plant $\xi(t) \equiv 0$, $\chi(t) \equiv 0$ and the noiseless channel. In certain cases, these conditions enter the boundary defined by (4.13), (4.14). For the plant with external perturbation, similar studies as applied to the strengthened stability allowing for the requirement of robustness in this perturbation[20] were carried out in [128]. In [130], the sufficient stabilizability conditions were obtained, and an "asynchronous" stabilization scheme was proposed according to which message transmission through a communication channel and generation of controls are done with different frequencies.

Explanations pertinent to the case of scalar plant and static stationary coder and decoder $e(t) = \mathfrak{E}[y(t)]$, $u(t) = \mathfrak{U}[e(t)]$ and concerning the attainable forms of stabilization of the scalar noiseless

[19] This does not contradict the well-known possibility of losing stability owing to the time delay introduced (for the given controller) in the feedback loop. Since it is a matter of stabilizability and not stability, changes in the properties of the channel (occurrence of delays, for example) implies changes in the problem conditions, which implies the possibility of changing the dynamic controller. The arguments of [103] show that if the strict inequality (4.14) is satisfied, then it is possible to design a control system stabilizing the plant in the presence of arbitrary and a priori unknown data transmission delays bounded from above by a certain constant. We also notice that the delays may have an adverse effect on the attainable stabilization performance.

[20] The closed-loop system is regarded as stable if there are continuous, increasing, and unlimited functions $\gamma_i(\cdot)\colon [0,\infty) \to [0,\infty)$, $i = 1, 2, 3$, such that $\gamma_i(0) = 0$, $\|x(t)\| \leqslant \gamma_1(\|x_0\|) + \gamma_2(\sup_t \|\xi(t)\|)$ and $\limsup_{t\to\infty} \|x(t)\| \leqslant \gamma_3(\limsup_{t\to\infty} \|d(t)\|)$. These inequalities must be satisfied for any trajectory, any time instant, and independently of the initial conditions. This definition is discussed in [131].

plant, including the approach of the system parameters to the boundary of the stabilizability domain [15], can be found in [12, 15, 117, 118, 129]. We omit the details and just notice that in this case the classical asymptotic stability $\mathrm{err}(t) \to 0$ is not achievable because of the finiteness of the set of the controls used. Achievable is only its weakened counterpart: the trajectories starting in some bounded set S tend to a smaller invariant subset $L \subset S$ characterized by chaotic dynamics. The findings of [15] indicate to the fact that at approaching the boundary of the stabilizability domain the sets S and L inevitably approach each other and chaotization extends to the entire "stabilized" part of the phase space S. More precisely, [15] analyzed situations on the boundary of the stabilizability domain in the case of precise measurement $y(t) = x(t)$, noiseless channel, and static stationary coder and decoder $e(t) = \mathfrak{E}[y(t)]$, $u(t) = \mathfrak{U}[e(t)]$ with connected sets of the level $\{y \colon \mathfrak{E}[y] = e\}$, $e \in \mathcal{E}$. The aforementioned boundary situation means that inequality (4.13) is satisfied as equality. It was shown that the stabilizing system is defined uniquely (to within trivialities) by the maximal and minimal values of the controls used u: $\mathbf{Vol}\,\{y \colon \mathfrak{U}[\mathfrak{E}(y)] = u\} \neq 0$, employs the uniform state quantization, and ensures invariance of the interval with the ends defined by and proportional to the aforementioned values, the trajectories being unbounded outside the interval. This interval is a minimal invariant set within which the dynamics of the closed-loop system is ergodic. We notice, that in less straitened circumstances, that is, far from the boundary, the uniform quantization is not generally an optimal choice [73, 92, 99]. For example, from a certain point of view preferable is the logarithmic scale [73].

For the linear stationary plant in Gaussian noise and a discrete noiseless communication channel, as well as for the stationary continuous channel with additive Gaussian noise, the constraints imposed by the finite channel capacity not only on the possibility of stabilization itself, but also on the attainable stabilization quality were considered in [61] for the case where it is possible. Simultaneously, consideration was given to the question of attainable quality of observation. The standard root-mean-square performance criteria were used. In the technical part, consideration was given to the previously raised question of the rate of growth in time of the volume of information required for permanent maintenance of the given rms precision D of the description of the state of an open-loop linear system. At that, in contrast to the previously characterized asymptotic (in D) formulations of the question, the answer was obtained for the given D.

In the case of noise in plant (4.1), sufficiency of the established boundary of the stabilizability domain was substantiated in [97, 119, 122] for the case of certain characteristics of noise (uniform estimates [119, 122] or estimates of the probabilistic moments [97]). As was demonstrated in [132], this constraint is unessential because within the limits of the indicated boundary it is always possible to construct a "noise-universal" stabilization system which is independent of the noise parameters and provides a bounded error for any uniformly bounded noise. At that, the precision of stabilization depends on the level of noise. For a certain subdomain within the indicated boundary, a similar system providing stabilization robust to the external perturbation was proposed and studied [128].

4.6. Stochastic Communication Channel

4.6.1. Introductory remarks. The last subsection considered the case where the communication channel model allows only for the effects of quantization and delay at data transmission, the third traditionally accounted for factor, distortion at transmission being regarded as unessential. It is only natural to use such approach at the initial stage of theory development, but at the same time it contrasts with classical theory of information and communication where some fundamental results concern distortion at data transmission. At formulation of the control/observation problem, account for them seems an inevitable step in the development of the theory.

The classical information theory traditionally treats the communication channel as a stochastic system described in the simplest memoryless case by the conditional probability of distribution of the channel output under the given input[21] Determination of the channel capacity, that is, the amount of information that can be transmitted in a time unit, has the status of a problem to which the classical theory has more than one answer. The basic answer is represented by the Shannon channel capacity (4.10) which is equal to the maximal mean rate at which the information may be transmitted with an arbitrarily small error probability. Another characteristic, the zero error capacity $\mathfrak{c}_0$, is equal to the maximal mean rate at which the information may be transmitted with the zero error probability [133, 134]. There are other notions of channel capacity (see, for example, [135, 136]) intended for special cases and needs. Are these notions sufficient to enable one to judge the possibility of plant stabilization and reliable estimation of its state? If yes, then for what possibilities is responsible one or another notion? If no, then what new characteristics of the communication channel are required?

For the communication channel in noise, consideration of the possibilities following from the availability of the auxiliary feedback channel used to notify the coder about the results of transmission through the main channel is of traditional interest [85, 137, 138]. In the ideal case, the feedback channel transmits the exhaustive notification without distortions and delays, and then the coder equation (4.3) becomes as follows:

$$e(t) = \mathfrak{E}[t, y(0), \dots, y(t), s(0), \dots, s(t-1)] \in \mathcal{E}. \tag{4.23}$$

Availability of a perfect channel of information feedback enables the coder to be aware of the current events and the decoder's concepts, for example, by modeling it according to (4.4), (4.5). This affords ground to expect that the channel errors may be compensated. At the same time, the feedback under discussion does not allow one to increase the rate of information transmission through the main memoryless channel with an arbitrarily small error probability, that is, its Shannon capacity [49, 139].[22] At the same time, this feedback is capable of increasing the rate of error-free transmission [134], raising the so-called reliability function [142], and simplifying the information coding procedure [138]. This subject is discussed in detail in [138]. General discussion of the role of information feedback in the communication channels of control can be found in [24, 99, 122, 143, 144].

The perfect "backward" channel can model bilateral communication between the coder and decoder of the main channel, the power of the decoder signal being much higher than that of the coder, which enables one to disregard the errors of transmission through the backward channel. Communication between a satellite and the ground-based stationary autonomous underwater sensors and the base station exemplify such cases exemplify such situations. For the controlled systems, information transmission from the decoder to coder requires no special facilities (individual communication channel) because an arbitrary volume of information may be transmitted by the use of control actions on the plant [101, 102, 107, 143, 145–147].

As for the model of the stochastic communication channel, the works devoted to determination of the fundamental boundary of the data transmission rate in the stabilization/observation systems may be divided into two groups. The general model of the stochastic channel borrowed from the classical information theory was considered in [98, 101, 102, 107, 132, 143, 146–151]. The works [93, 143, 152–159] considered special cases of this model: [143, 152–154] considered the erasure channel, [93, 155, 156], the binary symmetric channel, and [157–159], the truncation channel losing a random number of the end bits in the R-bit packet.[23] A model of the channel transmitting a random

[21] In the case of finite input and output channel alphabets, by the matrix $W(\cdot|\cdot)$ of transition probabilities.
[22] In the case of a channel with memory, such possibility, generally speaking, does exist [137, 140, 141].
[23] The truncation channel is a generalization of the classical erasure channel.

number of information bits without errors which is close to the last case was studied in [160]. For the reason alone that the channel model is stochastic, the process in the stabilization/observation system is random and application of the probabilistic criteria such as stability with the probability one (almost surely), moment stability, and stability in probability is appropriate. The results for these three cases have individual distinctions and are considered separately below.

In this subsection we assume for simplicity that the problem of observation examines the uncontrollable plant $\mathfrak{U}(\cdot) \equiv 0$ of (4.4). Otherwise, one needs to discuss the nuances concerned with presence or absence of the information feedback.

4.6.2. Stabilizability and Observability with the Probability One.

Noiseless plant. In (4.1), (4.2), $\xi(t) \equiv 0$, $\chi(t) \equiv 0$, and the initial state x_0 is a random vector independent of the communication channel. It is required to provide convergence to zero $\mathrm{err}(t) \to 0$ for $t \to \infty$ of the error $\mathrm{err}(t) := \|x(t)\|$, correspondingly $\mathrm{err}(t) := \|x(t) - \widehat{x}(t)\|$, of stabilization/estimation with the probability one (almost surely).

In this case, *Assertions* (i) *and* (ii) *of Theorem* 2 *retain their validity if* $\mathfrak{c}$ *in* (4.13) (4.14) *is the Shannon capacity of the communication channel* [107, 143, 147, 148].

The need for inequality (4.13) was established in [143] for the case of common channel with aftereffect, coders (4.3) without an information feedback, assumption of finiteness of the differential entropy of the initial state x_0. In this case, the channel is defined by the generally speaking infinite input and output alphabets $\mathcal{V}, \mathcal{W}$ and system of conditional probabilistic distributions $\boldsymbol{P}(dw_t|v_0, \dots, v_t, w_0, \dots, w_{t-1})$, $t = 0, 1, 2, \dots$ Here, $\boldsymbol{P}(dw_t|v_0, \dots, v_t, w_0, \dots, w_{t-1})$ defines the probability of getting the channel output w_t at the instant t, provided that the signals $v_0, \dots, v_t$ were sent at the instants $\theta = 0, 1, \dots, t$ and the signals $w_0, \dots, w_{t-1}$ were previously received at the instants $\theta = 0, 1, \dots, t-1$, respectively. In (4.13), (4.14),

$$\mathfrak{c} := \lim_{T\to\infty} \frac{1}{T} \mathfrak{C}_T, \quad \mathfrak{C}_T := \max_{\boldsymbol{P}(V_0^T)} I\left(V_0^T; W_0^T\right),$$

where $V_0^T = (v_0, \dots, v_T)$, $W_0^T = (w_0, \dots, w_T)$, $I(\cdot;\cdot)$ is the mutual information and maximization is carried out over all permissible probabilistic distributions on the set $\mathcal{V}^{T+1} = \{V_0^T\}$.[24] Sufficiency of inequality (4.14) was proved in [143] only for a special stationary memoryless channel, that with erasure and information feedback. This channel transmits the R-bit message without distortions with the probability $1 - p$ and loses it with the complementary probability p, the losses being time-independent and the coder notified about them. The Shannon capacity of this channel is $\mathfrak{c} = (1 - p)R$. A similar result was established in [154] for a stronger assumption of convergence to the limit of the mean frequency in the erasure channel. This assumption makes the model of [154] a special case of that considered in [103].

Both the necessary and sufficient parts of conditions (4.13), (4.14) were substantiated in [107, 147, 148] for the general case of the discrete stationary memoryless communication channel with finite input and output alphabets. (The Shannon capacity of such channel follows (4.10).) The need for inequality (4.13) was substantiated for an arbitrary initial state having the distribution density and coders both with, (4.23), and without, (4.3), the information feedback. As the result, it was shown that this relation exerts no appreciable effect on the domains of stabilizability and observability of the noiseless plant. The need for the discussed inequality was substantiated for weaker forms of stabilizability/observability. For example, inequality (4.13) is satisfied each time as the time-mean error of stabilization/observation may be bounded with a nonzero probability

[24] All distributions are permissible in the case of the finite alphabet $\mathcal{V}$. In the case of the Gaussian channel with additive white noise ($\mathcal{V} = \mathcal{W} = \mathbb{R}^s$, $\boldsymbol{P}(w_t|V_0^t, W_0^{t-1})$ is the normal distribution with the mean v_t and the given covariance matrix K), permissible are the distributions satisfying the constraint on the input power $\boldsymbol{E}\|v_\theta\|^2 \leqslant \rho\ \forall \theta = 0, \dots, t$, where the constant ρ is given. Other cases are explained in [143].

$\boldsymbol{P}\left[\sup_T (T+1)^{-1}\sum_{t=0}^{T}\mathrm{err}(t)<\infty\right]>0$. The need for inequality $\mathfrak{c}\leqslant\eta(A)$ highlights the fact that if it is violated, $\mathfrak{c}>\eta(A)$, any algorithm of stabilization/observation diverges exponentially with the probability one, that is, $\exists\,\alpha>1$: $\overline{\lim}_{t\to\infty}\alpha^{-t}\mathrm{err}(t)=\infty$ almost surely and the probability of a substantial error is bounded from below by values independent of the choice of the stabilization/observation system:

$$\varliminf_{t\to\infty}\boldsymbol{P}\Big[\mathrm{err}(t)\geqslant b(t)\Big]\geqslant 1-\frac{\mathfrak{c}}{\eta(A)},$$

where $b(t)>0$, $t=0,1,\ldots$, is an arbitrary sequence for which $\frac{\log_2 b(t)}{t}\to 0$ for $t\to\infty$. Additionally,

$$\boldsymbol{P}\Big[\mathrm{err}(t)\geqslant b\Big]\geqslant 1-\frac{\mathfrak{c}}{\eta(A)}-\frac{1}{t}\times\frac{1-h(x_0)+\mathfrak{c}+\frac{n}{2}\log_2\left(2\pi e\max\{b^2,b_0^2\}\right)}{\eta(A)}\quad\forall b>0,\ t\geqslant 1.$$

The last estimate is valid for plant (4.1) without stable modes $|\lambda_j|\geqslant 1\ \forall j$ if the initial state has finite differential entropy $h(x_0)$ and is almost surely bounded by $\|x_0\|\leqslant b_0$. A similar inequality may be derived from Lemma 3.2 of [143].

Although sufficiency of the inequality $\mathfrak{c}>\eta(A)$ for stabilizability/observability was substantiated in [107, 147, 148] for lack of an information feedback channel, some nuances of the problem of observation are related with such channel. In the absence of such feedback, it is possible to secure convergence to zero of the estimation error with an arbitrarily high probability. However, this is achieved at the expense of successive use of the block codes of variable and infinitely growing length, which requires unlimitedly growing memory resources and operating speed.[25] This disadvantage is overcome by a perfect information feedback, that is, in the class of coders (4.23). In this case, if the inequality $\mathfrak{c}>\eta(A)$ is satisfied, then it is possible to construct an observation system of bounded (algebraic) complexity based on a fixed-length block code and ensuring convergence to zero of the estimation error with the probability one. Such a system was described explicitly, except for the block code. The point is that suitable is any code transmitting data at a rate lower than the Shannon channel capacity with the error probability not greater than the desired one. The classical information theory guarantees existence of such code. Construction of codes with characteristics close to the theoretically attainable boundary is a standard problem of the coding theory (see, for example, [161, 162]). Therefore, in the presence of the information feedback, the asymptotically precise estimation can be ensured with the probability one by a realistic observer of uniformly bounded algebraic complexity using the standard general-purpose algorithms of block coding of information for transmission through the communication channel.

This conclusion remains in force also for the problem of stabilization even in the absence of the information feedback channel. It was shown in [107, 147] as a preliminary result that a feedback channel with an arbitrarily small mean rate of information transmission suffices for construction of a stabilization system of bounded algebraic complexity based on the fixed-length block code and providing asymptotic stabilization of the plant $x(t)\to 0$ for $t\to\infty$ with the probability one.[26] Then it was shown that this transmission requires no special facilities (communication channels) because the feedback can be established already in virtue of the fact that the decoder-controller affects the plant motion and the sensor observes this motion. As a result, the decoder-controller is capable of coding a message by imparting a certain distinction to the plant motion, and the sensor's coder can receive the message and detect and decode this feature. Apart from [107, 147], various schemes of information transmission[26] were suggested in [101, 102, 143, 146].[27] In particular, it was

[25] It deserves noting that the observation system generates an asymptotically precise estimate $\lim_{t\to\infty}\|\widehat{x}(t)-x(t)\|=0$ in real time, that is, the estimate $\widehat{x}(t)$ of the state $x(t)$ at the instant t is generated precisely at this instant.

[26] It suffices that the channel transmits without errors one bit in an arbitrarily long time.

[26] Also in the presence of bounded noise in the plant.

[27] We mean the schemes intended for servicing the stabilization/observation systems through a bounded-capacity communication channel. Outside this context, the idea of information transmission by means of the control actions

shown in [146] that in this way an arbitrary volume of information can be transmitted in unit time if the plant noise is bounded and its estimates are known.

Obviously, caution must be exercised in using the discussed schemes of information transmission which are in potential contradiction with the main aim of control. For example, the best result of stabilization lies in maintaining the plant precisely in the desired position, whereas the transmission of information using the discussed method requires deviation from this position. The need for transmission of an arbitrarily small amount of information in a time unit that was noticed in [107, 147] indicates a path to a possible trade-off.

These results are in agreement with Sec. 4.4 because formula (4.11) used there gives the Shannon capacity of the noiseless communication channel. On the whole, these results imply that under the condition of negligible uncertainty either in the communication channel or the plant the majority of channels that are of practical importance are in principle capable of ensuring stability in the strong sense, that is, with the probability one.[29]

Plant in noise. The last conclusion is replaced by its opposite in a more realistic situation where both the plant and communication channel are noisy. Against the background of the channel errors, arbitrarily (uniformly) small additive plant perturbations inevitably accumulate and sooner or later lead with the probability one to an arbitrarily great stabilization/observation error. This is true of any nonanticipative stabilization/observation system including that with the information feedback (4.23) and unlimited memory, as well as for the majority of channels that are of practical interest. However, there are channels enabling stability with the probability one. Ability of a noisy communication channel to ensure such stability is identical to its ability to transmit information with the zero error probability [151]. The capacity in the absence of noise serves as a numerical characteristic of this ability [134]. Unfortunately, it is equal to zero for many channels [133].

We specify the situation before going into detail. In (4.1) and (4.2), the noise sequences $\{\xi(t)\}$ and $\{\chi(t)\}$ are independent of each other; each is made up of mutually independent and identically distributed vectors, the noise distribution in the plant $\xi(t)$ has density, the initial state x_0 is a random noise-independent vector, the stochastic communication channel is independent of the control plant, that is, noise and the initial state. Noise is uniformly bounded $\boldsymbol{P}[\sup_t \|\xi(t)\| \leqslant D] = 1$, $\boldsymbol{P}[\sup_t \|\chi(t)\| \leqslant D_\chi] = 1$, where $0 < D < \infty$ and $0 \leqslant D_\chi < \infty$. It is required to make the stabilization/estimation error bounded with the probability one: $\boldsymbol{P}[\sup_t \mathrm{err}(t) < \infty] = 1$. In this case *Theorem* 2 *retains validity if* $\mathfrak{c} = \mathfrak{c}_0$ *in* (4.13), (4.14) *is the communication channel capacity in the absence of noise*[30] [151].

Moreover, if inequality (4.13) is violated, $\eta(A) > \mathfrak{c}_0$, then for any nonanticipative stabilization/estimation system $\overline{\lim}_{t\to\infty} \mathrm{err}(t) = \infty$ with the probability one [151]. This is true for arbitrarily uniformly small perturbations of the plant $D \approx 0$ even if there is no sensor noise, $D_\chi = 0$. Although there are nonzero-capacity communication channels, for many practically significant channels $\mathfrak{c}_0 = 0$ [133, 138].[31] An unstable linear plant ($\exists j\colon |\lambda_j| > 1$) cannot be stabilized or observed through such channel with a bounded error and the probability one. If, on the contrary, inequality (4.14) is satisfied for $\mathfrak{c} = \mathfrak{c}_0$), then it is possible to reach uniform boundedness of the error $\exists D_\infty < \infty\colon \boldsymbol{P}[\sup_t \|\mathrm{err}(t)\| \leqslant D_\infty] = 1$ (for bounded initial state) [166, §§ 7.7, 7.8].

For some channels, the information feedback enhances capacity in the absence of errors—$\mathfrak{c}_{0F} > \mathfrak{c}_0$ [134], where the left and right sides represent the channel capacities with and without such

on the plant can be found in earlier works devoted, for example, to the problem of decentralized control of the linear plant [145, 163–165].

[29] Because this majority has a nonzero Shannon capacity [138].

[30] Brief information concerning this notion is given in Appendix 1.

[31] Some examples can be found in Appendix 1.

feedback.[32] The capacity $\mathfrak{c} := \mathfrak{c}_{0F}$ of a feedbacked channel always appears in the stabilizability conditions (4.13), (4.14). The causes of this fact were discussed above: it is always possible to transmit information from the decoder-controller to the sensor's coder by means of the control actions on the plant. For the problem of observation of the state of an uncontrollable plant $\mathfrak{U}(\cdot) \equiv 0$ in (4.4)), the situation is even simpler: the capacity without, $\mathfrak{c}_0$, or with, $\mathfrak{c}_{0F}$, the feedback channel appears in (4.13), (4.14) depending on whether it is indeed, that is, what relation—(4.3) or (4.23)—describes the coder. As the result, for the channels with $\mathfrak{c}_{0F} > \mathfrak{c}_0$ there exist linear plants which may be stabilized in the absence of a special information feedback channel, but cannot be observed in the open state with a bounded error.

The main result considered here was first established in [152, 153] for the erasure channel with both finite [153] and infinite [152] alphabets. This channel transmits message without distortion with the probability $1 - p$ and loses it with the probability $p \in (0, 1)$, the losses being time-independent. In the absence of channel errors, its capacity is zero. Correspondingly, [152, 153] demonstrated that an unstable linear plant cannot be stabilized through such channel with the probability one or observed with a bounded error. In the case at hand, the roots of this fact are transparent: since the channel failures are time-independent, according to the law of large numbers an arbitrarily long sequence of continuous failures arises sooner or later with the probability one. During a corresponding time interval the control/observation channel is open, which in the environment of unpredictable external perturbations of the plant inevitably leads to a great stabilization/observation error. More complicated argumentation is required in the general case of a channel with an arbitrary form of data distortion at transmission [151].

A fact consonant to that discussed here was established in [157, 158] for a dedicated communication channel transmitting binary packages of a fixed nominal length and losing at that a random number of the end characters (truncation channel). Yet, in distinction to the plant examined here, in [157,158] consideration was given to a linear noiseless plant $\xi(t) \equiv 0$, $\chi(t) \equiv 0$ and a strengthened uniform stability $\sup_t \sup_{x_0} \|\mathrm{err}(t)\| < \infty$, where the second sup is taken over the initial states from the unit sphere. It was shown that for stabilizability of the plant in this sense it is necessary that $r_{\min} > \eta(A)$. Here, $r_{\min}$ is the maximal number of bits that are not lost in any situation, that is, with the probability one. Since $r_{\min}$ is the capacity of the error-free transmission of the channel under consideration, this condition coincides with the stabilizability condition considered in this subsection. At the same time, there exists an essential difference due to the differences in the plant model and the concept of stability. We notice for illustration that the loss of uniform stability does not rule out the possibility of very low probability of realization of great stabilization errors. Moreover, namely this happens if the necessary condition $\eta(A) < r_{\min}$ from [157, 158] is violated , but $\eta(A)$ is still smaller than the Shannon capacity of the channel. Then, $\lim_{t\to\infty} \mathrm{err}(t) = 0$ with the probability one for an appropriate stabilization system [107]. As the result, the probability of uniform stabilization $\boldsymbol{P}[\sup_t \|\mathrm{err}(t)\| < d]$ in the sense of [157, 158] may be made arbitrarily close to one by selecting a sufficiently large d. The result of [151] states that in this case for any nonanticipative stabilization system $\overline{\lim}_{t\to\infty} \|\mathrm{err}(t)\| = \infty$ with the probability one if arbitrarily small perturbations $\sup_t \|\xi(t)\| \leqslant \varepsilon \approx 0$ act (almost surely) on the plant.[33]

Since it is impossible to ensure a bounded error of stabilization/estimate, the critical analysis may be focused on (i) the channel model or (ii) stability criterion. The first line of research to a certain extent is associated with a departure from the popular models of the classical information theory. Some examples can be found in [167–169]. This subsection focuses further on reviewing the second line of research and considers the plant in noise.

[32] That is to say, the result of transmission through the communication channel $s(t)$ becomes known to the coder by the instant $t + 1$ at starting the next transmission.

[33] At that the noise $\xi(t)$ is distributed identically according to a certain density.

4.6.3. Moment stabilizability and observability. These notions imply that there exists a stabilization/observation system ensuring boundedness of the probabilistic moment of the error $\sup_t \boldsymbol{E}\|\text{err}(t)\|^\eta < \infty$. In the classical "linear" theory, the choice of η often is of no primary importance and is defined by the assumptions about noise and the initial state, as well as by the considerations of convenience. For example, in the problem of stabilization of system (4.1), (4.2) by the linear stationary controller $u = Ky$ in the quadratic summable noise and with the initial state $\sup_t \boldsymbol{E}\|\xi(t)\|^2 < \infty$, $\sup_t \boldsymbol{E}\|\chi(t)\|^2 < \infty$, $\boldsymbol{E}\|x_0\|^2 < \infty$, it is only natural to require that $\sup_t \boldsymbol{E}\|\text{err}(t)\|^2 < \infty$. Yet, a formally weaker requirement of $\sup_t \boldsymbol{E}\|\text{err}(t)\|^\eta < \infty$, $\eta < 2$, may be advanced, but this will not change the set of solutions $\{K\}$.

In the problem of stabilization[34] through the stochastic communication channel the index η acquires the features of a numerical estimate of the attainable degree of stability, which manifests itself, for example, in that the linear plant with uniformly bounded—and, therefore, having bounded moments of any order—noise and initial state is stabilizable for some discrete channels in the ηth moment only to some limiting value $\eta = \eta_*$ [98, 101, 102]. If channel or plant is changed, then η_* changes as well. An increase in η implies improved asymptotics of the decrease in probability of great deviations $\|\text{err}\| > d$ with the increase of the tolerance level $d \to \infty$. Against the background of the law of large numbers, this allows one to expect that the inevitable—in virtue of the results of Sec. 4.6.2—large systematic stabilization errors occur rarely and in this sense the system is better stabilized. Therefore, the limiting value of η_* may be interpreted as the characteristic of the attainable quality of stabilization. The domain of moment stabilizability in the space of plant and channel parameters depends on the moment index η [98, 101, 102]. (Appendix 2 explains the facts presented in this paragraph by way of a simple example.)

In the case of a common discrete memoryless communication channel, precise conditions for the moment stabilizability/observability of the scalar plant were established in [98, 101, 102, 170] in terms of a new parametric notion of the channel capacity christened the *anytime capacity* or AT-capacity.

AT-**capacity.** A special case of the problem of the unstable scalar plant ($|\lambda| > 1$)

$$x(t+1) = \lambda x(t) + u(t) + \xi(t) \in \mathbb{R}; \quad x(0) = 0; \quad |\xi(t)| \leqslant D; \quad y(t) = x(t) \tag{4.24}$$

with a certain noise level D (formulated in Sec. 4.1) was considered in [98,101,102,170]. The packets are transmitted from the coder to the decoder through a discrete stationary memoryless communication channel with the input, $\mathcal{E}$, and output, $\mathcal{S}$, alphabets and the matrix of transition probabilities $W(s|e)$, $s \in \mathcal{S}$, $e \in \mathcal{E}$. The model allows only for the randomness due to the channel, the perturbations $\xi(t)$ being regarded as deterministic. The plant is called the *η-stabilizable/observable* if there exists a stabilization/observation system for which $\sup_t \boldsymbol{E}|\text{err}(t)|^\eta \leqslant \Phi < \infty$ for any perturbation $|\xi(\theta)| \leqslant D$.

According to [98, 101, 102, 170], the η-stabilizability/observability is (almost) equivalent to the existence of a system of information transmission with certain features. This system and the properties are nontraditional. At the same time, such information system is "extractable" from any system η-stabilizing the plant. And vice versa, the plant can be η-stabilized with such information system.

The systems of information transmission through the communication channel that are considered in [98, 101, 102] usually consist of coder and decoder and are designed as follows:

- An R-bit packet $P(t)$ arrives to the coder input at each time instant $t = 0, 1, 2, \dots$ from an external information source.

[34] A similar situation exists in the problem of observation.

- The coder sends a message $e(t)$ through the channel at each time instant t. The message is generated on the basis of the entire information accessible to the coder.[35]
- At each instant $t = 0, 1, 2, \ldots$ the decoder constructs the estimates $\widehat{P}(\theta|t)$ of all previously sent packets $P(\theta)$, $\theta = 0, \ldots, t$, on the basis of the messages $s(0), \ldots, s(t)$ arriving through the channel.

Therefore, operation with this packet $P(\theta)$ continues infinitely $t \geqslant \theta$, and its current version is allowed to be reconsidered infinitely many times. This is the distinction of the given scheme from, for example, the block coding where operation with a packet is bounded by the time of processing the block which codes the packet and the resulting version is final.

The considered system *attains* the (*uniform*) *level of reliability* $\alpha > 0$ if the error of incorrect decoding of a fixed packet decreases α-exponentially with time:

$$\boldsymbol{P}\Big[\exists\, \theta \leqslant \tau \colon \widehat{P}(\theta|t) \neq P(\theta)\Big] \leqslant K 2^{-\alpha(t-\tau)} \qquad \forall t, \tau \colon \tau \leqslant t. \tag{4.25}$$

Here, the constant K is independent of t, τ and $P(\theta)$. The supremum $C_{any}(\alpha)$ of the rates R[36] at which the information may be transmitted with the reliability level α, was called in [98, 101, 102] the αAT-capacity (α-*anytime capacity*) of the channel. (If such transmission is impossible, then $C_{any}(\alpha) := 0$.)

Let us assume that there exists an information feedback. If plant (4.24) *is η-stabilizable* ($\eta > 0$), *then* $C_{any}(\eta \log_2 |\lambda|) \geqslant \log_2 |\lambda|$. *Inversely, if this inequality is satisfied in the strengthened form* $\exists \varepsilon > 0 : C_{any}(\eta \log_2 |\lambda| + \varepsilon) > \log_2 |\lambda|$, *then plant* (4.24) *is η-stabilizable* [98, 101, 102].

Similar results were established in [98, 170] for the problem of observation. It was pointed out in [98, 101, 102] that the assumption of information feedback may be relaxed.

An information system reaching (in the limit) the necessary level of reliability $\eta \log_2 |\lambda|$ at the desired rate $\geqslant \log_2 |\lambda|$ is "contained" in a sense in any system η-stabilizing the plant, and the proofs given in [98, 101, 102] rely on its explicit "extraction." By contrast, the proposed system η-stabilizing the plant makes direct use of the discussed nontraditional scheme of information transmission. Although construction of such scheme represents a nontraditional problem, its formulation is closer to the classical problems of the information transmission theory than the original problem of stabilization of the unstable plant. This gives promise that the methodological potentialities of the classical information theory could be adapted and used. Examples can be found in [98] where, for instance, an adapted classical method of random coding [171, 172] was used.

It is common knowledge [98, 170] that $\mathfrak{c}_0 \leqslant C_{any}(\alpha) \leqslant \mathfrak{c}$ and $C_{any}(\alpha) \to \mathfrak{c}$ for $\alpha \to +0$, where $\mathfrak{c}$ is the Shannon channel capacity and $\mathfrak{c}_0$ is its capacity in the absence of errors. The AT-capacity was calculated for the erasure channel[37] transmitting R-bit packets, $C_{any}(\alpha) = \dfrac{R\alpha}{\alpha + \log_2[(1-p)(1-2^{\alpha}p)^{-1}]}$, and the channel with additive white Gaussian noise and constraint on the power of input for which the AT-capacity is independent of the reliability level α and is equal to the Shannon capacity. The AT-capacity was also determined for some channels having combined features of the aforementioned two channels [173, 174] and the simplest channel with memory, the channel with additive white Gaussian noise, constraint on the input power, and Markov transition from one of the two possible states into the other (Gilbert–Elliott model) under information feedback embracing the

---

[35] It includes $P(0), \ldots, P(t)$, and, in the case of information feedback, also $s(0), \ldots, s(t-1)$.

[36] We recall that each packet $P(t)$ has R bits. The extended definition of AT-capacity "operates" not only at the integer, but at the fractional rates R as well. The so-defined capacity can take on any real values $c \geqslant 0$, but for brevity the corresponding details are omitted here.

[37] It is described in Appendix 2. In the case of feedback, the expression for AT-capacity follows from the basic fact set forth in Appendix 2 and the main result of the present subsection.

channel state [175]. In the case of common discrete memoryless communication channel, the AT-capacity is related to the exponent of the error probability for the block codes $E(R)$[38] by the formula $C_{any}[E(R)\log_2 e] \geqslant R\log_2 e\ \forall R < \mathfrak{c}$ [98] which can be used for the estimate from below $C_{any}(\cdot)$ if the function $E(\cdot)$ is known.

In the general case, there is no formula[39] for $C_{any}(\cdot)$, and the ideas of practical realization are bounded to the use of the classical method of random coding and imply that the coder and decoder have access to the common source of randomness. An increased complexity inherent to the random codes in the context under consideration is aggravated by the need for operating with infinite-length data arrays. By its definition, the proposed system of information transmission has computational complexity that grows infinitely in time: growing memory and the number of operations executed at time unit that are required to process an infinitely growing file of the estimates of the previously received packets. The possibilities for restricting complexity were not studied explicitly for the general case.[40]

Constructive moment stabilization and estimation. The problem of estimating the state $x(t) \in \mathbb{R}$ of the unstable, $|\lambda| > 1$, and uncontrollable, $u(t) \equiv 0$, scalar linear plant (4.24) through a symmetrical binary communication channel was considered in [93,155,156]. This channel transmits without distortions with the probability $p \in (0,1)$ the one-bit packets $e(t) = 0,1$ and inverts them $s(t) = e(t) + 1 \mod 2$ with the complementary probability $1 - p$. The inversions are time-independent; the coder is notified about them. The proposed estimation algorithms are described explicitly in depth and make use of an auxiliary file whose length varies in time and is formally unconstrained. The conditions for the plant and channel parameters under which the algorithm provides bounded anticipated values of the error $\boldsymbol{E}|x(t) - \widehat{x}(t)|$ and length of the auxiliary file were established in [93].

Precise constructive conditions for moment stabilizability were obtained in [157–159] for the scalar plant and the truncation communication channel. The channel model was substantiated in [159] by the mobile radiocommunication applications. The $\overline{r}$-bit packets $e = (b_1, \dots, b_{\overline{r}})$, $b_i = 0,1$ are sent through the channel, a random number $r(t) \in [r_{\min}, \overline{r}]$ of the first bits $s = (b_1, \dots, b_{r(t)})$ reaches the addressee, the following bits being lost. The channel is stationary and memoryless: the values of $r(t)$ are independent and identically distributed. In (4.24), $\lambda = \lambda(t)[1 + z_a(t)]$ and $\xi(t) = d(t) + z_f(t)$. Here $z_a(t)$ and $d(t)$ are bounded random perturbations $|z_a(t)| \leqslant \overline{z}_a < 1$, $|d(t)| \leqslant \overline{d}$, the values of $\lambda(t)$ are mutually independent and identically distributed, and $\{z_f(t)\}$ is the output of the causal nonlinear operator with the input $\{x(t)\}$ which meets the condition $|z_f(t)| \leqslant \overline{z}_f \sup_{\theta=t-\rho,\dots,t} |x(\theta)|\ \forall t$ with some $\overline{z}_f < 1$ and $0 \leqslant \rho \leqslant \infty$. The estimates $\overline{z}_a, \overline{d}, \overline{z}_f$ and ρ are known; at the instant t the coder learns $x(t), \lambda(t)$ and $r(t-1)$, which means that there exists an information feedback. Consideration is given to the moment robust stability meaning that $\sup_t \boldsymbol{E} \sup_{x(0)\in[-\frac{1}{2},\frac{1}{2}]} |x(t)|^\eta < \infty$, at that $\boldsymbol{E} \sup_{x(0)\in[-\frac{1}{2},\frac{1}{2}]} |x(t)|^\eta \to 0$ for $t \to \infty$ if $\overline{d} = \overline{z}_f = 0$, that is, there is no additive noise in plant (4.24). The stabilizability conditions were established under certain technical assumptions. In the case of $\overline{z}_a = \overline{d} = \overline{z}_f = 0$, that is, for the plant

---

[38] We recall that $E(R) = \overline{\lim}_{N\to\infty} -\dfrac{\ln p_{\text{err}}(N)}{N}$, where $p_{\text{err}}(N)$ is the best, that is, the least, probability of incorrect decoding at transmission of information at the rate R that is reachable using the block codes of length N.

[39] Similar, for example, to (4.10), that is, reducing calculation of the AT-capacity to the solution of a finite-dimensional extremal problem.

[40] If (4.25) is satisfied, the packets eventually are decoded correctly, that is, $\forall \theta\ \exists\ t_\theta \geqslant \theta$: $P(\theta) = \widehat{P}(\theta|t)\ \forall t \geqslant t_\theta$ with the probability one, and the anticipated length $\boldsymbol{E}d_t$ of the part of the estimate file that is liable to further corrections $d_t := \min\{d\colon \widehat{P}(\theta|t) = P(\theta)\ \forall \theta \leqslant t - d\}$ is bounded in t. Some considerations concerning recognition of d_t by the decoder are set forth in [98] for the erasure channel defined in Appendix 2.

$x(t+1) = \lambda(t)x(t) + u(t)$), these conditions are necessary and sufficient and come to the inequality

$$\boldsymbol{E}r(t) > \underbrace{\boldsymbol{E}\log_2|\lambda(t)|}_{\mathfrak{c}} + \frac{1}{\eta}\log_2\left[\boldsymbol{E}2^{\eta[\boldsymbol{E}r(t)-r(t)]}\right] + \frac{1}{\eta}\log_2\left[\boldsymbol{E}2^{\eta[\log_2|\lambda(t)|-\mathfrak{c}]}\right]. \tag{4.26}$$

Sufficiency of the strengthened and augmented condition (4.26) was established for the general case. The robust stabilization system including the code for data transmission through the channel was described explicitly and has a uniformly bounded algebraic complexity. The necessary stabilizability conditions were extended to some multidimensional stochastic systems (4.1) without additive noise ($A = A(t)$ in (4.1) is a random matrix with special properties).

4.6.4. Stabilizability and observability in probability. These properties imply that it is possible to ensure boundedness of the error of stabilization/estimation in probability: for any $p \in (0,1)$, there is a tolerance level $B = B(p) > 0$ observed at any time instant with the probability not less than p: $\boldsymbol{P}\big[\|\mathrm{err}(t)\| < B\big] \geqslant p\ \forall t$. The definition is in keeping with the standard notion of convergence in probability and describes the weakest of the considered forms of stability.[41] At the same time, this definition correctly separates the stable and unstable systems under the classical linear formulation of the problem [132, 149].[42]

Assertions (i) and (ii) of Theorem 2 as applied to stabilizability and observability in probability retain their validity if $\mathfrak{c}$ *in* (4.13), (4.14) *is the Shannon capacity of the communication channel* [98, 101, 102, 132, 149].

For the scalar linear plants (4.24) in noise that is uniformly bounded by a certain boundary, this fact was established in [98, 101, 102] within the framework of the aforementioned theory of AT-capacity. It was also shown that, given a perfect information feedback in the channel, for existence of a stabilization system featuring $\boldsymbol{P}\big[\|\mathrm{err}(t)\| > B\big] \leqslant f(B)\ \forall t, B \geqslant 0$, where $f(\cdot)$ is the given function defining the desired rate of decrease of the probability of great error, it is necessary to satisfy for some $K > 0$ the inequality $C_{g-any}(g) \geqslant \log_2|\lambda|$, where $g(d) := f(K|\lambda|^d)$. The value $C_{g-any}(g)$ is defined similarly as $C_{any}(\alpha)$ with replacement of the right-hand side of (4.25) by $g(t-\tau)$. Inversely, if the above inequality is satisfied, then it is possible to construct a stabilization system for which $\boldsymbol{P}\big[\|\mathrm{err}(t)\| > B\big] \leqslant g(K + \log_{|\lambda|} B)\ \forall t, B > 0$ for some K. The above results rely on using the "*anytime*" coders and decoders defined as systems of indefinitely increasing computational complexity.[43]

The general case of multidimensional noisy linear plant (4.1), (4.2) was considered in [132, 149]. The necessary stabilizability/observability condition (4.13) was established in the class of all nonanticipative coders and decoders. It was also shown that if the sufficient condition (4.14) is satisfied, then stabilization and reliable observation can be attained with a system of uniformly bounded algebraic computational complexity using the traditional procedures of block coding of information for transmission through the channel. Except for the code of data transmission through the channel, the corresponding systems were described explicitly. The point is that any code transmitting data at a rate not lower than the Shannon channel capacity with an error probability not higher than the desired one is suitable. The classical information theory guarantees that it exists; construction of codes with characteristics close to the theoretically attainable ones is the standard problem of the coding theory [161, 162]. In the case of perfect information feedback and unbounded noise, the proposed stabilization/observation system is universal in the sense that it provides probability-bounded error under any noise having a finite and time-bounded probabilistic moment of a certain order, knowledge of the noise level estimate being unnecessary. It was shown

[41] Error boundedness almost surely or in the sense of the probabilistic moment entails its boundedness in probability.

[42] For instance, under natural technical assumptions the linear feedback $u = Kx$ stabilizes plant (4.1) in probability if and only if the eigenvalues of the dynamic matrix $A + BK$ of the closed-loop system lie on an open unit disk [132].

[43] See the discussion in Sec. 4.6.3.

that in the case of a uniformly bounded noise with certain upper bound one can do without a special channel of the information feedback. In this case, stabilization is guaranteed only if the aforementioned boundary is observed.

4.7. Extension of Conditions (4.13) *and* (4.14) *to the Nonlinear Systems*

This generalization requires an analog of $\eta(A)$ from (4.12). Its definition from (4.12) is fit only for the linear systems. At the same time, as was noticed in Sec. 4.4 when discussing Theorem 2, $\eta(A)$ has the sense of the rate of growth of the entropy of the open-loop linear system. By departing from the previously developed concepts of the entropy of the nonlinear dynamic system[44] and adapting them to the context of the problems of control and observation through the limited-capacity communication channel, counterparts of Theorem 2 were established in [114, 115] for the general case of the nonlinear dynamic system.

The problem of estimating the state of the uncertain nonlinear system (4.17) was considered in [115]. Its topological entropy follows (4.18). It is assumed that the channel of data transmission from the sensor to the estimation system transmits immediately and without errors the b-bit packets $h(jT)$ at the instants that are multiples of the given $T > 0$; the symbol $\mathfrak{C}_R$ denoting the class of all such channels with the data transmission rate $b/T \leqslant R$. At the instants jT that are multiples of T the coder generates a packet $h(jT)$ on the basis of observations $x(0), x(1), \dots, x(jT)$. On the basis of the received packets $h(0 \cdot T), h(1 \cdot T), \dots, h(j \cdot T)$, the decoder-observer constructs the state estimates $\widehat{x}(t)$ for $t = (j-1)T + 1, \dots, jT$. System (4.17) is referred to as *observable by channels of the class* $\mathfrak{C}_R$ if for any $\epsilon > 0$ there is a channel (defined by b and T) of this class, as well as the coder and decoder-observer supporting estimation with a uniform precision ϵ:

$$\|x(t) - \hat{x}(t)\|_\infty < \epsilon \quad \forall t = 1, 2, 3, \dots \tag{4.27}$$

This relation must be satisfied for any solution of system (4.17). For observability in the above sense, satisfaction of the inequality $H := H\big[F(\cdot,\cdot), \mathcal{X}_0, \mathcal{X}, \Omega\big] \leqslant R$ is necessary, and satisfaction of the inequality $H < R$ [115] is sufficient. Here, $H\big[F(\cdot,\cdot), \mathcal{X}_0, \mathcal{X}, \Omega\big]$ is defined in (4.18). It was established in [115] that $H = 0$ for the robust stable systems (4.17), $H = \eta(A)$ for the linear systems, and also conditions were indicated under which inevitably $H = \infty$. It was shown by making a start from this result that under natural technical assumptions the uncertain systems like

$$x(t+1) = [A + B\omega(t)]\, x(t) + b, \quad x(1) \in \mathfrak{X}_1 \tag{4.28}$$

with the unstable matrix A (there is an eigenvalue λ with $|\lambda| > 1$) for $b = 0$ and with an arbitrary matrix A for $b \neq 0$ are nonobservable. Here, $\omega(t) \in \Omega$ is the matrix function of uncertain parameters, the sets $\mathfrak{X}_1$ and Ω are defined, at that the former is compact and the latter is a neighborhood of zero. Additionally, the error of any estimation system is asymptotically infinite independently of the rate of data transmission through the channel R: the supremum in t of the left-hand side of (4.27) is $+\infty$. In the case of $b = 0$, stable matrix A, and $\Omega = \{\omega\colon \|\omega\| \leqslant c\}$, the uncertain system (4.28) is observable using a channel of an arbitrarily small capacity R at satisfaction of the discrete analog of the circular criterion

$$\max_{z \in \mathbb{C}: |z| = 1} \|(zI - A)^{-1}B\| < \frac{1}{c}.$$

Along the lines of the discussed approach, [176] established the sufficient conditions for robust observability/stabilizability of the system obeying the differential equations with certain linear part

[44] For more detail see the discussion of Theorem 2 in Sec. 4.4.

nonlinearity satisfying the Lipschitz inequality with a certain constant.[45] Robustness means that the system must immediately ensure observability/stabilization for all Lipschitzian nonlinearities with this constant. The sufficient conditions make reference to solvability of the pair of algebraic Lur'e–Riccati equations in the class of positive definite matrices and includes a requirement similar by implication to the condition $H < R$. Generalizations to the case of monotone nonlinearities were proposed in [178]. The results of [176] were complemented and developed for the problem of estimation in [179] where, in particular, more general classes of nonlinearities were studies.

The problem of stabilization of the nonlinear control plant

$$x(t+1) = F[x(t), u(t)], \quad t = 0, 1, 2, \ldots \tag{4.29}$$

was studied in [114]. Here, the state $x(t)$ and control $u(t)$ are the elements of the topological space $\mathfrak{X}$ and set U, respectively; the function $F[\cdot, u]\colon \mathfrak{X} \to \mathfrak{X}$ is continuous for all u. The dependence of the right-hand side of Eq. (4.29) on the free parameter $u(t)$ prevents direct application of the standard definition of the topological entropy of the continuous map on the compact invariant set $K \subset \mathfrak{X}$ [110–113]. This problem is eliminated by feedbacking the system with subsequent infimization of the entropy of the closed-loop system with respect to the feasible feedbacks. The corresponding infimum which characterizes the original open-loop system was called the *topological feedback entropy*.

Two such allied constructions were proposed in [114]. Passing to the details, we dwell on the simpler one and the results based on it. We consider system (4.29) on the compact subset $K \subset \mathfrak{X}$ of the phase space $\mathfrak{X}$; the inside K is nonempty. The subset is "potentially" invariant in the following sense: there exists a compact subset $K' \subset \operatorname{int} K$ such that from any state $x_0 \in K$ in a single step one can get inside this subset $\exists u \in U\colon x_1 = F(x_0, u) \in \operatorname{int} K'$. Consideration is given to the pairs $p = [\alpha, G]$, where α is the open covering of the set K, that is, the family of open subsets $A \subset \mathfrak{X}$ whose union includes K, and G assigns to each element $A \in \alpha$ a sequence of controls $u_A(0), \ldots, u_A(\tau_p - 1) \in U$ of length τ_p which is different for different pairs p. We confine our field of vision only to those pairs p for which the trajectory of system (4.29) generated by this sequence in one step gets inside some compact subset $K' \subset \operatorname{int} K$ and is held there: $x(1), \ldots, x(\tau_p) \in \operatorname{int} K'$. This must be satisfied for any initial state $x(0) \in A$ and any element A of the covering α of which the set K' is independent. (The described invariance of K guarantees availability of such pairs.) For the given pair p, any sequence of the covering elements $\mathcal{A} = \{A_0, \ldots, A_k\}$ generates the sequence of controls $\{u(t)\}_{t=0}^{k\tau_p - 1}$

$$\{u(t)\}_{t=i\tau_p}^{(i+1)\tau_p - 1} := G(A_i), \quad i = 0, \ldots, k-1,$$

which in turn converts each initial state $x(0) = \overline{x}$ into the trajectory $x_{\overline{x},\mathcal{A},p}(0), \ldots, x_{\overline{x},\mathcal{A},p}(k\tau_p)$. Let $B_{\mathcal{A},p}$ be the set of all initial states $\overline{x}$ for which $x_{\overline{x},\mathcal{A},p}(i\tau_p) \in A_i$, $i = 0, \ldots, k$. The totality $\mathcal{B}_{p,k}$ of all such sets corresponding to every possible sequence $\mathcal{A}$ of the given length k makes up an open covering K [114] from which a finite subcovering can be extracted in view of compactness of K. We select a subcovering with the least number of elements and denote it by $\beta_{p,k}$. The *topological entropy of system* (4.29) *on* K *with regard for feedback* is the value

$$h(F, K, U) := \inf_p \lim_{k \to \infty} \frac{\log_2 \beta_{p,k}}{k\tau_p},$$

where the limit exists [114]. It was shown that the inequality

$$R \geqslant h(F, K, U)$$

[45] This condition is a special case of the sector constraint in the theory of absolute stability [177].

must be satisfied for existence of a control system providing invariance of the set K and using a limited-capacity noiseless communication channel R for data transmission from the sensor to the controller. The corresponding strict inequality suffices for existence of such system.

The characterized notion was used to analyze the local asymptotic system stabilizability in the neighborhood of equilibrium x_*. (In more exact terms, it is assumed that $x_* = F[x_*, u_*]$ for some control $u_* \in U$.) Under additional technical assumptions including metrizability $\mathfrak{X}$ and availability of a metric in U, introduced was the notion of the *local topological entropy of system* (4.29) *at the point* (x_*, u_*) *with regard for the feedback*:

$$h(F, x_*, u_*) := \lim_{\delta \to +0} \overline{\lim_{\varepsilon \to +0}}\, h\left(F, B^{\varepsilon}_{x_*}, B^{\delta}_{u_*}\right). \tag{4.30}$$

Here B^r_z is a sphere of radius r centered at z. It was shown that for the local asymptotic stabilizability the inequality $R \geqslant h(F, x_*, u_*)$ is necessary and $R > h(F, x_*, u_*)$, sufficient. In the case where $\mathfrak{X}$ and U are Euclidean spaces, the function F is continuously differentiable, and the pair $[F'_x(x_*, u_*), F'_u(x_*, u_*)]$, where F'_x and F'_u are the matrices of the corresponding partial derivatives, is completely controllable, the local entropy (4.30) is equal to $\eta\,[F'_x(x_*, u_*)]$ from (4.12) [114].

5. STABILIZATION AND OBSERVATION THROUGH LIMITED-CAPACITY COMMUNICATION NETWORKS

Many of the existing control systems are realized as complexes of multiple spatially distributed sensors, controllers, and actuators exchanging data through a digital limited-capacity network. The last section considered the question of the fundamental boundaries of the data transmission rates for the simplest network consisting of one or two communication channels. In what follows, we consider a generalization to more complicated networks. The majority of the known results refer to the problem of stabilization.

The problem of centralized exponential stabilization of the multisensor noiseless linear system was considered in [103]. An individual limited-capacity communication channel is assigned to each sensor for transmission of data to the unique central controller; there are no channels for back transmission and data exchange between the sensors. The system state is, generally speaking, nondetectable in the classical sense from the readings of an individual sensor, but detectable from their totality. In fact, one faces the problem of merging the potentialities of individual sensors with the aim of plant stabilization in the conditions of constrained data transmission rate. The necessary and sufficient stabilizability condition were established, and the reachable degree of exponential stability was determined for the case where these conditions are satisfied.

More specifically, consideration was given to the linear controllable systems of the form

$$x(t+1) = Ax(t) + Bu(t); \quad x(0) = x_0; \quad y_j(t) = C_j x(t), \quad j = 1, \ldots, k, \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^{n_u}$ is the control, k is the number of sensors, $y_j(t) \in \mathbb{R}^{n_{y,j}}$ is the reading of the jth sensor, and the matrix A has the eigenvalue λ with $|\lambda| \geqslant 1$. A model of the communication channel allowing for the variations in its instantaneous capacity, delays, losses, and data distortions at transmission was discussed in Sec. 4.5. The main requirements is as follows: transmissions through the channel from the jth sensor initiated and completed during the time interval $[t_0 : t_1]$ carry at most $b^+_j(t_0, t_1)$ information bits, independently of the situation they enable one to transmit at least $b^-_j(t_0, t_1)$ bits, the mean values of these estimates stabilize about the common limit $\exists \lim_{t_1 - t_0 \to \infty} (t_1 - t_0)^{-1} b^{\pm}_j(t_0, t_1) =: \mathfrak{c}_j$.

At each time instant $t = 0, 1, \ldots$, the coder of the jth sensor generates a packet $e_j(t) \in \mathcal{E}_j$ sent to the controller through the communication channel:

$$e_j(t) = \mathfrak{E}_j[t, y_j(0), \ldots, y_j(t)]. \tag{5.2}$$

The decoder constructs the control $u(t) = \mathcal{U}[t, \overline{e}(t)]$ on the basis of the totality of the packets $\overline{e}(t)$ received through all channels by the instant t. The stabilization system made up by k coders and the decoder *stabilizes exponentially* the plant with the index $\mu \in (0,1)$ if $\|x(t)\| \leqslant K_x(d)\mu^t$, $\|u(t)\| \leqslant K_u(d)\mu^t \ \forall t \geqslant 0$ for $\|x_0\| \leqslant d$ and any $d > 0$.

The pair (A, B) is regarded as stabilizable in the classical terms. Introduced are an unstable subspace $M_{unst}(A)$ of the matrix A, as well as the unobservable and nondetectable subspaces of the $j$the sensor:

$$L_j^{-o} := \{x \in \mathbb{R}^n \colon C_j A^\nu x = 0 \ \forall \nu \in [0\colon n-1]\}, \quad L_j^- := M_{unst}(A) \cap L_j^{-o}.$$

The stabilizability conditions have the form of a system of inequalities which in principle are numerable in terms of the sensor groups $J \subset [1:k]$. The inequality depends on the group by means of the subspace nondetectable by this group $L(J) := \bigcap_{j \in J} L_j^-$. (For uniformity, we denote $L(\emptyset) := M_{unst}(A)$, where $\emptyset$ is an empty set.) Since different groups J may generate a common subspace $L(J)$, it is advisable to numerate the inequalities directly by the elements of the set $\mathfrak{L} = \{L = L(J) \colon J \subset [1:k]\}$.

It was shown under an additional technical assumption that the exponential stabilizability of a plant (with some $\mu \in (0,1)$) amounts to satisfying for all $L \in \mathfrak{L}$ the inequality

$$\log_2 |\det A|_L| < \sum_{j \notin J(L)} \mathfrak{c}_j, \quad \text{where} \quad J(L) := \{j \in [1:k] \colon C_j x = 0 \quad \forall x \in L\}, \tag{5.3}$$

where $A|_L$ is the contraction of the operator A to the invariant subspace L and the sum is taken over the sensors that do not completely ignore L. If these inequalities are satisfied and the matrix A has no stable eigenvalues, then the reachable degree μ^0 of exponential stability (the infimum of the reachable μ) is as follows:

$$\log_2 \mu^0 = \max_{L \in \mathfrak{L}} \frac{1}{\dim L} \left(\log_2 |\det A|_L| - \sum_{j \notin J(L)} \mathfrak{c}_j \right). \tag{5.4}$$

An explicit formula for μ^0 was determined also in the general case of a matrix with stable eigenvalues.

Centralized stabilization of the multisensor systems was also examined in [99] under the assumption that each sensor transmits its message to the controller immediately and without noise through the assigned limited-capacity communication channel and that the current control is known to the coders of all sensors. The argumentation implies that the system is reducible to the real-diagonal form so that the corresponding "modes" have simple relations with the sensors. This means that the "mode" either does not effect the sensor output or is uniquely defined from it.[46] A stabilization system suggested by the analogies with the classical problem of coding data from dependent sources [180] was proposed in [99] where the conditions were established under which the system ensures stability. These conditions come to solvability of the system of linear inequalities in natural numbers having the sense of the number of the levels of quantization of individual modes. The

[46] Stated differently, a situation like the two-dimensional system $x_1(t+1) = \lambda x_1(t)$, $x_2(t+1) = \lambda x_2(t)$ observed by three sensors $y_1 = x_1 + x_2$, $y_2 = x_2$, and $y_3 = x_1$ is inadmissible. Here, the second and third sensors are in simple relations with both "modes" x_1 and x_2, and the first is not: the "mode" x_i effects the sensor reading y_1, but it is impossible to determine it from the readings of this sensor. At least one sensor has such "defect" for any linear change of the phase variables. We notice that for the argumentation of [99] it is of principal importance that the system be representable as an ensemble of independent linear subsystems whose states are simply related with the sensors. Reducibility to the real-diagonal form is less essential. A case where the assumption of simple relations of the sensors and modes is violated was considered in [103]. In this case, the design of the stabilizing controller becomes fundamentally more difficult.

discussed result was extended in [181] to the case where the messages of the sensors are transmitted to the controller through a fixed-topology noiseless communication channel.

The problem of decentralized stabilization of the linear real-diagonal unstable noiseless system with multiple sensors and actuators was considered in [182]:

$$x(t+1) = \mathbf{diag}(\eta_1, \ldots, \eta_n)x(t) + \sum_{i=1}^{l} B_i u_i(t),\ y_j(t) = C_j x(t),\ j \in [1:k],\ t = 0, 1, \ldots.$$

Here, u_i is the control action of the ith actuator and y_j is the reading of the jth sensor. The sensors are connected to the actuators by direct allocated communication channels of time-variable capacity: at the instant t the sensor j can transmit to the ith actuator a message selected from the given finite alphabet $S_{j\to i}(t)$ immediately and without distortions. ($|S_{j\to i}(t)| \equiv 1\ \forall t$ corresponds to the lack of channel.) The averaged capacity of such channel is characterized by

$$\mathfrak{c}_{j\to i} := \varliminf_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \log_2 |S_{j\to i}(t)|. \tag{5.5}$$

At the time instant t coder (5.2) of the jth sensor generates the collection $e_j(t) = [e_{j\to 1}(t), \ldots, e_{j\to l}(t)]$ of the messages $e_{j\to i}(t) \in S_{j\to i}(t)$ to all actuators. The decoder of the ith actuator generates control relying on all received messages $u_i(t) = \mathcal{U}_i[t, e_{\to i}(0), \ldots, e_{\to i}(t)]$, where $e_{\to i}(\theta) := [e_{1\to i}(\theta), \ldots, e_{k\to i}(\theta)]$. The *decentralized stabilization system* made up by all coders and decoders $\mathcal{E}_1[\cdot], \ldots, \mathcal{E}_k[\cdot], \mathcal{U}_1[\cdot], \ldots, \mathcal{U}_l[\cdot]$ ensures *uniform asymptotic stability* if $\sup_{\|x(0)\|\leqslant d} \|x(t)\| \to 0$ for $t \to \infty$ for any $d > 0$. The necessary and sufficient (distant from them) conditions for uniform asymptotic stabilizability were determined with the assumption of $\eta_\alpha \neq \eta_\beta\ \forall \alpha \neq \beta$, as well as observability and controllability of the original system by the entire totality of sensors and actuators. The necessary conditions lie in satisfying the inequality

$$\sum_{h\in H} \log_2 |\eta_h| < \sum_{(i,j):i\text{th actuator affects at least one variable } x_h \text{ c } h\in H} \mathfrak{c}_{j\to i} \tag{5.6}$$

for any nonempty subset $H \subset \{h \in [1:n] \colon |\eta_h| \geqslant 1\}$ of the "unstable part of the spectrum" of the matrix A. In the right-hand side the sum is taken over all (i, j) for which the hth row of the matrix B_i is other than zero for some $h \in H$. This sum has the sense of the summary capacity of all communication channels through which the data reach the actuators affecting the dynamics of the considered part of the state vector $x_H := \{x_h\}_{h\in H}$. (Therefore, the channels that are knowingly useless for control of the vector x_H are disregarded.) The left-hand side of (5.6) contains the topological entropy (4.19) of the open-loop system describing the dynamics of x_H. In the light of these observations, the need for (5.6) follows from (4.13) where the nonstrict inequality is replaced by the strict one on the basis of Comment 2 to Theorem 2.

The idea of neglecting in the right-hand side of (5.6) not only the channels to the actuators i that do not affect x_H, but also the channels from the sensors j insensitive to x_H, that is, such that the hth column of the matrix C_j is zero for all $h \in H$, might seem natural. Namely this was done in condition (5.3) concerning the centralized stabilization. However, this idea is inapplicable to decentralized control. The point is that the sensor j that is insensitive to x_H may, nevertheless, receive indirectly the information about x_H by analyzing the impact on the accessible measurements of the actuators $i \in I_{observing\ x_H}$ having "direct"' access to x_H, that is, connected by the communication channels to the sensors sensitive to x_H. If the "social circle" of the jth sensor $I_{j\to} := \{i\colon$ is the communication channel between j and $i\}$ includes the actuator $i_* \notin I_{observing\ x_H}$ which is knowingly impossible in the case of centralized control, then transmission of the "indirect" information from the sensor j to the actuator i_* may make sense, especially if the actuators

Fig. 1. System with two sensors and two actuators.

$i \in I_{observing\, x_H}$, although observing x_H, cannot affect it, whereas the actuator i_*, being deprived of the possibility of "direct observation" of x_H, can do it. This aforementioned effect is iterable because by using the above method i_* can transmit the data to other actuators i_{**}.

The above considerations come to the already discussed phenomenon of information transmission by the controls, which means that, apart from the information transmission channels that are indicated explicitly in the problem formulation, the control system has "hidden" channels. Generally speaking, all communication channels may be important for decentralized stabilization and, therefore, the communication topology may turn out to be more complicated that it seems at first glance. A simple illustrative example is depicted in Fig. 1.

A system with two unstable "modes" x_1 and x_2 is observed by two sensors $S1andS2$ and controlled by two actuators $A1$ and $A2$. The relations of observability and controllability are reflected by the dashed lines. The solid lines show the "evident" communication channels. Therefore, the actuator Ai controls the mode x_i, but has no "direct" information about it. At the same time, if the dashed arrows are interpreted as data transmission channels, then $A2$ may receive data about x_2 along the route $a \to b \to c \to d \to e$ where both "explicit" channels are used. Similarly, $A1$ can get information about x_1. As a result, if the "explicit" channels have capacity sufficient for simultaneous transmission of the necessary data flows $x_1 \to A1$ and $x_2 \to A2$, then the system under study can be stabilized [166, Ch. 9].

Constructive schemes of using controls for information transmission in the problems of decentralized network stabilization of the linear systems under bounded communication channel capacity were proposed in [166, Ch. 9].

The authors of [182] consciously disregarded the information transfer by controls. As the result, the sufficient stabilizability conditions established in [182] are far from the necessary ones. These sufficient conditions lie in solvability of the following system of linear inequalities with respect to $p_{h,i,j} \geqslant 0$:

$$\begin{aligned} &\sum_{(i,j):\begin{cases} i\text{th actuator affects } x_h \\ j\text{th sensor is sensitive to } x_h \end{cases}} p_{h,i,j} > \log_2 |\eta_h| \quad \forall h\colon |\eta_h| \geqslant 1, \\ &\sum_{h:\begin{cases} i\text{th actuator affects } x_h \\ j\text{th sensor is sensitive to } x_h \end{cases}} p_{h,i,j} < \underline{\mathfrak{c}}_{j\to i} \quad \forall (i,j). \end{aligned} \tag{5.7}$$

Here, $\underline{\mathfrak{c}}_{j\to i}$ is defined according to (5.5), and $p_{h,i,j}$ has the sense of the mean information transmission rate x_h through the communication channel $j \to i$.

As was already noted, in the general case the necessary (5.6) and sufficient (5.7) conditions are not allied. Nevertheless, in a special case where each actuator affects all unstable "modes" $x_h\colon |\eta_h| \geqslant 1$ and each sensor is sensitive to all such "modes" [182], these conditions are close to each other.

The general case of the decentralized network control system of a linear process was considered in [166, Ch. 9]. It was assumed that the network consists of the *elements* exchanging data and endowed by the computational facilities used for transformation of the input data flows into the outgoing flows. Some of the elements (sensors) measure directly the parameters of the external process, and the other elements (actuators) directly affect the process. The rest of the elements operate as repeaters and intermediate processors (controllers) participating in the decentralized manner in the distributed transformation of the sensor signals in the control actions on the unstable process. By an appropriate design of the data processing algorithms for all elements, it is required to achieve stability of the closed-loop system consisting of an external (for the network) process and elements interacting through the communication network.

The network is assumed to be defined, that is, it is indicated from what elements, at what rates[47] and how the information may be transmitted. The message transmitted through the network may be delayed, lost, and distorted, in particular, because of mutual collisions.[48] The communication topology is of arbitrary nature and may be varied dynamically by the authorized elements. The situation where the communication channel of two network elements (channel subscribers) is switched to service another pair of subscribers offers a simplest example. The decision about switching is made by a special element (channel supervisor) on the basis of information available. Part of the considered problem of stabilization is represented by construction of the corresponding algorithm (protocol).

The system of algorithms for data processing by the controllers is selected from the class defined a priori in the problem formulation. It allows one, for example, to reflect the possibilities of the digital processors servicing the controllers: constraints on memory, rate, nomenclature of operations, and so on. In turn, this class is not constrained by any requirements and is arbitrary in this sense. In particular, a situation may occur where for the given controller this class offers a single data processing algorithm, that is, it is not improbable that in the network some controllers are defined a priori. An example is offered by a network having an element in the form of a memory unit serviced according to a predefined protocol. Another example is given by the network with switchable channel and given switching protocol. At the same time, such a restriction is not imposed on the selection of the data processing algorithms by the sensors and actuators.

An external (to the network) unstable process obeys a linear difference equation; there is bounded noise in the plant and sensor. Conditions under which the process can be stabilized, that is, for the network elements one can select some admissible data processing algorithms such that the closed-loop system is stable, were established in [166, Ch. 9]. A constructive description of these stabilizing algorithms was given for the case where these condition are satisfied.

In more precise terms, the problem of stabilization was reduced to the traditional problem of data transmission in the multiterminal information networks as formulated in the following standard terms. At the given points of the network there are N sources of information for each of which the information destination is indicated. Since consideration is given to the deterministic model of the network, a nonrandomized criterion for quality of transmission is used: after decoding at the destination point, the received packet should agree with the sent one within the probability one. The conditions for stabilizability are given in terms of the domain $\mathcal{R}$ of all collections $(r_1, r_2, \dots, r_N)$ of the rates r_i at which the data may be transmitted from the ith source to the corresponding destination simultaneously for all $i = 1, \dots, N$ (under the existing constraints on the admissible algorithms of data processing at the network elements). The aforementioned conditions are precise: for process stabilizability, it is necessary and sufficient that a certain vector characterizing the degree of instability of the external process belong to the domain $\mathcal{R}$ and its interior, respectively. Design of the stabilization system was reduced to construction of a system for data transmission at the rates $(r_1, r_2, \dots, r_N)$ corresponding to the components of this vector.

It is important to emphasize that the mentioned above domain $\mathcal{R}$ is not associated with the network from the formulation of the stabilization problem, but with its modification which lies in introducing into the network additional vertices (elements) and unlimited-capacity communication channels. The additional elements and channels are "dummies," and the modification itself is a

[47] In the network there may be channels of both finite and infinite capacity.
[48] The deterministic model was used to allow for these effects.

computational trick. Channels of three types are added. Those of the first type transmit data from each actuator to all sensors capable of detecting its actions. These channels express explicitly the aforementioned possibility of transmitting information by means of the control actions on the plant. (In the situation of Fig. 1, there are two such channels represented by the upper and lower pairs of the dashed arrows.) The channels of the second type transmit data from the "dummy" sources of information associated with the unstable "modes" of the external process; an additional "dummy" vertex is introduced into the network to "allocate" each source. The channel outgoing from this vertex is broadcasting data simultaneously to all sensors capable to observe the corresponding "mode." Finally, the channels of the third type have multiple users, real alphabet, and additive interference of the transmitted signals. Each of these channels transmits data to an additional "dummy" vertex, the data reception point in the modified network. These vertices are also related one-to-one to the unstable "modes" of the external process (and at that they are distinct from the similar vertices associated with the second-type channels). The "users" of the channel leading to the given "mode" are represented by all actuators capable to exert influence on it. The discussed domain $\mathfrak{R}$ answers the question of the rates at which it is possible to transmit data simultaneously along the modified network from each of the "dummy" sources of information to the data reception point associated with the same unstable "mode" as the source. At that, the modified network is considered isolated from the external process that must be stabilized.

6. ALLOWANCE FOR INFORMATION CONSTRAINTS—NONLINEAR SYSTEMS

The problem of stabilization of the nonlinear affine systems with scalar control and feedback quantization was examined in [183]. Consideration was given to the following control plants:

$$\dot{x} = f(x) + d(x)u, \quad f(0) = 0, \tag{6.1}$$

where the functions f and g belong to the class $\mathcal{C}^l$, $x \in X \subseteq \mathbb{R}^n$, $u \in U \subseteq \mathbb{R}$.

The stabilizing and robust stabilizing quantizers are designed for system (6.1) using the method of Lyapunov functions. The problem is solved with the use of the logarithmic quantizer. Application of the proposed schemes of quantization to the control of motion trajectories of the one-wheeled transportation vehicles was described.

The problem of asymptotic output-stabilization of the nonlinear systems represented in the Lur'e form with nondecreasing nonlinearities and bounded communication channel capacity was considered in [178]. The channel errors and delays, external disturbances, and measurement imprecisions are disregarded. The observer of [184] was used to estimate the plant state. The coder and decoder use the vector of corrections of the state estimate that is transmitted at the discrete time instants through the communication channel. The methods of the theory of absolute stability [66,177] (frequency theorem, Popov criterion) and the apparatus of linear matrix inequalities are used to substantiate the result. The coder scaling coefficient is calculated from recurrent relations. It was shown that for a sufficiently high communication channel capacity it is possible to select the parameters of the coder and decoder that ensure global asymptotic stability of the closed-loop system.

The problems of stabilization of the nonlinear systems under information constraints about the given point were discussed also in [114, 127, 185–189], but the result of [114] is of local nature, and [127,185–189] consider only the problem of stabilization of the system equilibrium state. They cannot be applied directly to the problems of *synchronization* where its is required to ensure convergence of the trajectories to some set (manifold) in the system state space and not to a given point. The first findings on synchronization of the nonlinear systems under constraints on the data transmission rate were made in [190, 191] where the so-called *observer-based synchronization* was considered (see [192–199]). Consideration was given to the class of first-order coders with time-exponential scaling. The method of synchronization used in [190, 191] leads to an inversely proportional dependence of the limiting synchronization error on the rate of data transmission through the communication channel.

The paper [200] is devoted to the problem of *controlled* synchronization through a limited-capacity communication channel. In distinction to [190,191], it considers controlled synchronization of the nonlinear systems with output feedback where the control signal is calculated on the basis of measurements of the error signal between the master and slave systems transmitted through the communication channel. The control law is designed using the *passification theorem* [196, 197, 201, 202]. For the ideal case of no perturbations, noise, and distortions in the communication channel, it was shown that the synchronization error can be made to tend asymptotically to zero if the rate of data transmission through the channel exceeds some threshold value. Comparison of the considered hybrid (continuous-discrete) system with an auxiliary continuous system (the so-called *continuous model*) featuring the desired properties of stability and passivity is the key method used in [200] to study the controlled synchronization. This approach was methodically developed in the 1970's under the name of the "*method of continuous models*" [203, 204].

The problem of synchronization for the class of the nonlinear systems under parameter uncertainty and bounded rate of information transmission through the communication channel between the master and slave systems was investigated in [190,191,196,205–211]. Proposed was a method of adaptive synchronization on the basis of adaptive observers and nonstationary coders with memory. For the adaptive synchronization systems, Chao obtained numerical characteristics of the synchronization process under different loadings of the communication channel and demonstrated the possibility of using the results obtained for information transmission by modulation of the chaotic signal. The results were extended in [211, 212] to the problem of synchronization of the nonlinear systems through the communication networks with the "tree" topological structure.

A hybrid control system (equilibrium stabilization system) comprising a continuous (affine in control) nonlinear plant and a discrete linear dynamic controller was examined in [213]. The controller output (control action) is quantized. The control plant is defined by the state equations:

$$\begin{aligned} \dot{x}(t) &= f\big(x(t)\big) + K\bar{u}(t), \\ \bar{u}(t) &= \bar{u}_k, \quad t \in [k, k+1), \quad k = 0, 1, \ldots, \end{aligned} \tag{6.2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $\bar{u}(t) \in \mathbb{R}^m$ is the control signal received by fixing over the interval of discreteness $T = 1$ the output of the quantizer to whose input the control u_k generated by the discrete controller arrives. The function $f(x) \in \mathbb{R}^n$ in (6.2) is assumed to be continuously differentiable, $f(0) = 0$. The results obtained are similar to those of the method of continuous models [203, 204]. Namely, as was established in [213], the uniform asymptotic stability of the equilibrium of the discrete system follows from the exponential stability of its idealized linearized model. It was shown that if the equilibrium (*trivial solution*) of the linearized system without quantization ("idealized system") is asymptotically stable, then in the case of quantization the solutions are uniformly ultimately bounded, the size of the limit set may be made arbitrarily small by reducing the quantization step. It was also shown that by reducing the quantization step one can reach asymptotically an arbitrarily small deviation between the trajectories of the idealized system and that with quantization. It was also noted that the methodology used in the proof is applicable to more general cases such as the systems with nonlinear discrete controllers, systems with nonlinear measurements of the plant output, and so on.

7. SOME OTHER FORMULATIONS OF AND APPROACHES TO THE PROBLEMS

As was noticed in the paper [214] devoted to the control of microengines through the limited-capacity communication networks, the main obstacle to enhancing the control performance in the distributed systems lies not in the limitedness of the computing resources, but rather in the time available for exchange of data between the sensors, control computer, and the actuators through the common communication network. A daisy-chain network configuration where the signal propagates successively between the network nodes was considered in [214]. In such network, only one operation of data transmission can be executed at each time instant. Therefore, it is important to determine what variables are most essential in terms of system performance in order to allocate them the greatest transmission time. The paper considered the linear discrete models with different discretization frequencies for different control loops, but the data arriving and transmitted through the network were assumed to have an unlimited number of positions.

The problem of minimization of the mean-root square error at transmission of the vector signal through an analog communication channel under constraints both on the total power of the signal in the channel and on the powers of individual components of the transmitted signal was considered in [215]. The dynamics of the signal source and the peculiarities arising in the feedback problems were not considered. The optimal coder and decoder are sought in the form of static linear operators.

Limitedness of the signal/noise ratio as the cause of limited information transmission rate in the control systems was considered also in [216–218]. Stabilizability of the unstable linear plant (whether continuous or discrete) by a linear stationary feedback controller through a noisy communication channel under the bounded signal/noise ratio was investigated in [216, 217]. They follow the classical works [49, 85] in using the notion of available signal power P, noise power N, and channel bandwidth W to determine the channel capacity C channel in bits per second from the well-known formula $C = \log_2(1 + P/N)$. Thereby, the constraints on the data transmission rate are expressed in terms of the signal/noise ratio and the channel bandwidth. The necessary and sufficient conditions for plant stabilizability through the communication channel at both the state and output feedbacks were determined. In particular, it was shown that the nonminimal-phase plant gives rise to additional constraints. An interesting result of [217] is the comparison of the established boundaries with inequality (4.13). For that, the communication channel capacity at the given signal/noise ratio for the Gaussian noisy channel with an unlimited bandwidth was determined in [217]. It was shown that the communication channel capacity required for stabilization of a plant with the state feedback by a linear controller with constant parameters is equivalent (in the bits per second) to the minimal rate of data transmission obeying by inequality (4.13). If only the output of the nonminimal-phase plant is measurable, then the least required ratio is greater that at stabilization by state and exceeds the boundary of Theorem 2. An approach enabling one to treat the results of [216,217] from the standpoint of the general concept of solution of the linear-quadratic optimization problem is expounded in [218].

When speaking about the relation between the theories of control and information, one must mention the works where the information-theoretical terms such as entropy, relative entropy, anisotropy, mutual information, and so on are used to express the aim of control, performance criterion, or constraints [219–225]. Such approaches enable one to see in a new light the optimal and adaptive systems, extend the class of the designed systems, and endow them with new characteristics. However, the problems formulated in the mentioned above works do without communication channels, that is, information processes as such are disregarded.

We notice one more approach to taking into account the computational constraints by representing the computing device as an information transformer [226] equivalent to the ideal calculator connected in series to some communication channel. At that, the limitedness of the computer performance corresponds to the limited capacity of the channel. The capacity R_c of the equivalent channel may be evaluated as the informational capacity of the computer: the maximal amount of information generated by the computer in unit time which in the simplest case is equal to the product of the computer capacity ν by the number of elementary operations ω executed also at the same time unit:

$$R_c = \nu\omega.$$

A reasoning similar to [96] (see Comment 6 to Theorem 2) shows that a result similar to Theorem 2 is valid for $\mathfrak{c} = \min\{R, R_c\}$ in the case of channels in the no-noise measurement and control at the capacity R and allowance for limited controller capacity R_c. Similar estimates are true also for the problems of synchronization of the nonlinear tree-like systems and networks [209, 211]: the equivalent network capacity is regarded as equal to the minimum of the capacities of all its channels and the information capacities of all computers.

8. APPLIED PROBLEMS

As was noticed in [65, 227], in the applied problems it was usually accepted to separate the communication aspect of the control system from its dynamic characteristics because this simplifies the study and usually is used to advantage in the classical control problems. The situation, however, changes dramatically when one controller (decision maker) controls many subsystems through a limited-capacity communication channel. Such systems are represented by the mobile sensor networks, self-organizing mobile cooperative groups of mobile robots, in particular, groups of pilotless aircraft or orbital clusters, wireless sensor media used for surface monitoring, operation control systems, signal power control systems in the mobile network telephony, control of the arrays of microactuators [214] and groups of autonomous robots, control systems of flexible power networks, control systems of production and traffic, and control systems of nanorobot arrays. In such and similar systems, successive interaction with all subsystems may prove to be impossible because of physical constraints. For example, the security conditions for the pilotless aircraft may hinder continuous communication with the central control point. Remoteness from the Earth of the robotic space buggies and limitedness of their on-board power resources entail sparse communication patterns. In addition, if, as in the case of microactuator control, the number of subsystems exceeds some threshold, then communication at a rate required for the real-time control may become impossible.

Use of the aforementioned methods of control and estimation to the applied problems is described in the scientific literature. The main attention is paid to the problems of observation of the moving objects.

The problem of control of sensors and data transmission at tracking a maneuvering target was considered in [228]. The following two typical situations were examined in particular. In the first case, an antenna with a controlled beam is used to track more than one target (the target motions are assumed to be mutually independent). The task is to determine what target should be selected at each time instant by the sighting unit in order to optimize the given objective functional. Another situation arises if more than one sensor is connected via a multiplexor to a limited-capacity communication channel for transmission of the data about the target motion to the computer tracking it on the basis of data from various sensors. In this case, one should select the best way of switching the flows of data transmitted from the sensors through the communication channel such that the "optimal" target tracking is ensured at the given limited rate of data transmission through the communication channel. Target motion is represented in [228] by the jump Markov linear system. Consideration is given to the quadratic loss function; the problem of its optimization is formulated as that of stochastic dynamic programming. The paper proposed suboptimal algorithms that are realizable from the computational point of view such as the sensor adaptive interacting multiple model algorithm (SAIMM-algorithm) and the sensor adaptive integrated Probabilistic data association algorithm (SAIPDA-algorithm).

The problem of minimizing the flows of information about the target motion at multiple-sensor tracking was considered in [229]. Minimization is attained owing to the correlation between the measurements of various sensors. The method of adaptive adjustment of the coder by automatic scaling on the basis of the current process information was demonstrated in [38] for data transmission through a limited-capacity channel at tracking a moving object. Its efficiency was verified experimentally on the "Vertolet" laboratory bench [37]. The problem of intercepting a moving target if its constrained information is acquired from various (moving and fixed) sources was examined in detail in [230].

9. CONCLUSIONS

The recent decades are notable for explosive interest in the domain lying at the crossing of the control and information theories. The above problems and results are far from exhausting all that is known by now. The issues associated with the theorem of data transmission rate (Sec. 4) was presented more fully. Importance of this theorem is due to the fact that it relates the central notions of the information and control theories and states that the aim of control is reached if the rate of information transmission through the communication channels exceeds that of information generation (uncertainty) by the control plant. We note that with this interpretation the theorem is nothing but the generalized differential form of the Ashby's *law of requisite variety* [231] that was very popular at the earlier stage of cybernetics. Mathematical detailing of different versions of this theorem reveals a number of nuances and problems arising at attempting to establish mathematically correct formulations in involved situations. In particular, the Shannon's definition of the channel capacity sometimes proves to be insufficient. The results obtained become the milestones along the difficult road toward approaching the theories of information and control symbolizing the return to the initial holistic understanding of cybernetics as the science of control and communication [232].

In the further studies the emphasis is made on the problems of control of the nonlinear and network systems and on allowance for the computational networks. Some approaches to allowance for the latter problem were represented in the review.

As in any other developing theory, in the presented field of science there are unclear questions and paradoxical situations. One of such paradoxes was analyzed at length: in the majority of practical cases stabilization in the presence of an arbitrarily low noise both in the plant and communication channel is impossible if it is the question of stabilization with the probability one (see Sec. 4.6.2), but stabilization "in probability" is possible if the conditions of the corresponding theorem of data transmission rate are satisfied (see Sec. 4.6.3). Another paradox is associated with the possibility of stabilizing the unstable plants with no communication channels, that is, zero capacity. Indeed, the results of the classical works of A. Stephenson [233,234], P.L. Kapitsa [235], N.N. Bogolyubov [236], and their followers [237–239] show that under certain conditions the unstable nonlinear plants such as the flipped pendulum controlled by moving the point of suspension can be stabilized by a periodic program action of a sufficiently hight frequency (vibrational stabilization). Therefore, the feedback channel is not the necessary condition for stabilizability which, at the first glace, comes into conflict with the theorem of data transmission rate. Analysis of this and other paradoxes is the subject matter of future research.

ACKNOWLEDGMENTS

The authors should like to thank Profs. A.V. Savkin (Sydney, Australia) and R.J. Ewans (Melbourne, Australia) for useful discussions.

APPENDIX 1

CAPACITY OF THE COMMUNICATION CHANNEL IN THE ABSENCE OF ERRORS

We consider the discrete memoryless channel with finite input, $\mathcal{E} = \{e\}$, and output, $\mathcal{S} = \{s\}$, alphabets and the transition probability matrix $W(s|e)$.

Channel without information feedback. The *block code* of length N is a collection of M input words of length N of the channel $E^1, \dots, E^M$, $E^\nu = (e_0^\nu, \dots, e_{N-1}^\nu)$. By sending the word E^ν during N time cycles the informant notifies about selection of the νth option from the M predefined ones. The code *rate* obeys the ratio $R := \dfrac{\log_2 M}{N}$. The *rule of decoding* $\mathfrak{D}$ assigns a certain ν to each output N-word: $\mathfrak{D}\colon \mathcal{S}^N \to [1 : M]$. This rule is *error-free* if $\mathfrak{D}(S) = \nu$ for any ν and any word $S = (s_0, \dots, s_{N-1})$ whose reception probability is other than zero $\boldsymbol{P}(S|E^\nu) = \prod_{i=0}^{N-1} W(s_i|e_i^\nu) > 0$, provided that E^ν was sent. The *capacity in the absence of errors* is $\mathfrak{c}_0 := \sup R$, where sup is taken over all arbitrary-length block codes admitting error-free decoding [134]. Simultaneously, $\mathfrak{c}_0 = \sup_N \dfrac{1}{N} M_{\max}(N) = \lim_{N\to\infty} \dfrac{1}{N} M_{\max}(N)$ [133], where $M_{\max}(N)$ is the maximal number of the distinguishable input words of length N. (The words E' and E'' are *distinguishable* if the probability of generating a common channel output word S, that is, $\{S : \boldsymbol{P}(S|E') > 0\} \cap \{S : \boldsymbol{P}(S|E'') > 0\} = \emptyset$), is zero.)

Channel with information feedback. In this case, the result $s(t)$ of transmission through the channel becomes known to the coder by the instant $t + 1$ of the next transmission. The *block-function* of length N is a recursive rule for generation of the input word of length N on the basis on the index $\nu = 1, \dots, M$:

$$e(t) = \mathfrak{E}_*[t, e(0), \dots, e(t-1), s(0), \dots, s(t-1), \nu], \qquad \text{(A1.1)}$$
$$t = 1, \dots, N-1, \quad e(0) = \mathfrak{E}_*[0, \nu].$$

This word is used as before for notification about the choice of the νth option of the predefined M. The rest of the definition details are the same; the corresponding capacity is denoted by $\mathfrak{c}_{0F}$.

Sometimes the information feedback increases the capacity in the absence of errors $\mathfrak{c}_{0F} > \mathfrak{c}_0$ [134]. There is no general formula for $\mathfrak{c}_0$ [133], but for $\mathfrak{c}_{0F}$ it is known as $2^{-\mathfrak{c}_{0F}} = \min \max_{s\in\mathcal{S}} \sum_{e\in\mathcal{E}_s} \boldsymbol{P}(e)$ [134]. Here, min is taken over all probabilistic distributions $\{\boldsymbol{P}(e)\}$ on the channel's input alphabet $\mathcal{E}$, and $\mathcal{E}_s$ is the set of all input symbols $e \in \mathcal{E}$ generating this output symbol s with a nonzero probability $W(s|e) > 0$. This formula for $2^{-\mathfrak{c}_{0F}}$ is true if there exists a pair e', e'' of distinguishable input symbols: $W(s|e')W(s|e'') = 0\ \forall s$; stated differently, $\mathfrak{c}_{0F} = \mathfrak{c}_0 = 0$. Additionally, $\mathfrak{c}_{0F} > 0 \Leftrightarrow \mathfrak{c}_0 > 0 \Leftrightarrow$ there exists a pair of distinguishable input symbols [134].

In virtue of the last fact, for many channels that are of practical interest $\mathfrak{c}_{0F} = \mathfrak{c}_0 = 0$ [133]. We present some simplistic examples.

Erasure channel with finite alphabet $\mathcal{E}$ (cardinality $K \geqslant 2$) transmits the message $e \in \mathcal{E}$ without distortion with the probability $1 - p$ and loses it with the probability $p \in (0, 1)$. The output alphabet $\mathcal{S} = \mathcal{E} \cup \{\circledast\}$, where $s(t) = \circledast \Leftrightarrow$ message $e(t)$ is lost. Since $W(\circledast|e) = p > 0\,\forall e$, any two input symbols are indistinguishable and, therefore, $\mathfrak{c}_{0F} = \mathfrak{c}_0 = 0$. The Shannon capacity of this channel is $\mathfrak{c} := (1 - p)\log_2 K > 0$ [85].

Binary symmetric channel transmits the symbol $e = 0, 1$ correctly $s = e$ with the probability $p \in (0, 1)$ and inverts it $s = e + 1 \mod 2$ with the probability $1 - p$. Since $W(1|0) = 1 - p > 0$ and $W(1|1) = p > 0$, the symbols 0 and 1 are indistinguishable and, therefore, $\mathfrak{c}_{0F} = \mathfrak{c}_0 = 0$. The Shannon capacity of the channel is $\mathfrak{c} = 1 + p\log_2 p + (1 - p)\log_2(1 - p) > 0$ for $p \neq 1/2$ [85].

Channel with nonzero transition probabilities $W(s|e) > 0\ \forall s, e$ also has a nonzero capacity in the absence of errors. Moreover, $\exists s\colon W(s|e) > 0 \forall e \Rightarrow \mathfrak{c}_{0F} = \mathfrak{c}_0 = 0$.

Any channel with the *characteristic graph* depicted in Fig. 2 serves as a counterexample of the noisy channel with nonzero capacity in the absence errors. The graph vertices are identified with the

Fig. 2. Characteristic graph.

input symbols of the channel $e \in \mathcal{E}$, and the (nonoriented) branches connect the indistinguishable symbols. In the absence of errors, the capacity is defined completely by the characteristic graph [134]. For the channels with the graph of Fig. 2, $\mathfrak{c}_0 = \dfrac{1}{2}\log_2 5 > 0$ [240].

For the channels with the input alphabet of power $\leqslant 4$, the capacity in the absence of errors was calculated in [134]. A detailed review of this field can be found in [133].

APPENDIX 2

MOMENT STABILIZABILITY OF THE SCALAR LINEAR PLANT THROUGH DISCRETE ERASURE CHANNEL WITH INFORMATION FEEDBACK[49]

The communication channel either immediately transmits an R-bit message without distortions or loses it. The losses are time-independent, the probability is constant $p \in (0,1)$, the coder is notified about a loss of packet with one-cycle delay. We consider the unstable $|\lambda| > 1$ scalar plant

$$x(t+1) = \lambda x(t) + u(t) + \xi(t); \quad x(0) = 0; \quad |\xi(t)| \leqslant D; \quad y(t) = x(t)$$

with a certain noise level D. At the instant t, the coder sends the packet $e(t) = \mathfrak{E}[t, y(0), \dots, y(t), I(0), \dots, I(t-1)]$, where $I(\theta) = 1$ if the packet $e(\theta)$ was lost $s(\theta) = \circledast$, otherwise, $I(\theta) = 0$, $s(\theta) = e(\theta)$. The decoder operates according to (4.4). The plant is referred to as *uniformly η-stabilizable* if there exist coder and decoder for which $\sup_t \boldsymbol{E} \sup |x(t)|^\eta < \infty$. The second sup is taken over entire noise $|\xi(\theta)| \leqslant D$ (for the given $I(\theta)$).

The *plant under consideration is uniformly η-stabilizable* ($\eta \in [1, \infty)$) *if and only if the following strict and nonstrict inequalities are satisfied* [98]:

$$\varphi(\eta) := |\lambda|^\eta \left[2^{-\eta R}(1-p) + p\right] < 1, \qquad \varphi(\eta) \leqslant 1. \tag{A2.1}$$

Simple analysis of the function $\varphi(\cdot)$ shows that it assumes values smaller than one for $\eta \geqslant 0$ only if $\log_2 |\lambda| < \mathfrak{c} := (1-p)R$. Here, $\mathfrak{c}$ is the Shannon capacity of the channel under consideration [85], that is, the inequality itself is the condition (4.14). We regard it as satisfied in what follows. Then the equation $\varphi(\eta) = 1$ has two nonnegative roots $\eta = 0$ and $\eta = \eta_* = \eta_*(|\lambda|, R, p) > 0$, at that $\varphi(\eta) < 1$ for $\eta \in (0, \eta_*)$ and $\varphi(\eta) > 1$ for $\eta > \eta_*$. As the result, the conditions (A2.1) may be rearranged in

$$\eta \leqslant \eta_*(|\lambda|, R, p), \quad \eta < \eta_*(|\lambda|, R, p).$$

One can also easily verify that the function $\eta_*(a, R, p)$ decreases strictly in $a > 1$, $p \in [0,1]$ and strictly increases in R. Whence it follows that the domain of η-stabilizability $\{(\lambda, R, p) \colon \eta_*(|\lambda|, R, p) < \eta\}$ depends on the index of moment η.

We also notice that $\lim_{R\to\infty} \eta_*(|\lambda|, R, p) = -\dfrac{\log_2 p}{\log_2 |\lambda|}$. (The passage to the limit corresponds to the transformation of the channel alphabet from finite to infinite.) As the result, $\eta_*(|\lambda|, R, p) \leqslant -\dfrac{\log_2 p}{\log_2 |\lambda|}$, that is, it is impossible to provide the moment stability with an index higher than

[49] This material is based on [98].

$-\dfrac{\log_2 p}{\log_2 |\lambda|}$ independently of the size R of the transmitted packets and, therefore, of the Shannon capacity of the channel.

An outline of the proof of conditions (A2.1).

Necessity. In the case of k losses (at fixed time instants) over the interval $[0 : t]$, in virtue of (4.4) the second sum in

$$x(t) = \sum_{\theta=0}^{t-1} \lambda^{t-1-\theta} d(\theta) - \sum_{\theta=0}^{t-1} -\lambda^{t-1-\theta} u(\theta)$$

takes on at most $M := 2^{R(t+1-k)}$ values q. When $\xi(\theta)$ runs the domain $|\xi(\theta)| \leqslant D$, the first sum deletes the interval $\left[-\dfrac{|\lambda|^t - 1}{|\lambda| - 1} D, \dfrac{|\lambda|^t - 1}{|\lambda| - 1} D\right]$. For any allocation of M points q, this interval includes the element z with $|z-q| \geqslant M^{-1} D \dfrac{|\lambda|^t - 1}{|\lambda| - 1} \geqslant M^{-1} D |\lambda|^{t-1}$. Therefore, $\sup_{\xi(\cdot)} |x(t)| \geqslant 2^{-R(t+1-k)} |\lambda|^t D_1$, where $D_1 := D|\lambda|^{-1}$, and

$$\begin{aligned}
\boldsymbol{E} \sup |x(t)|^\eta &= \sum_{k=0}^{t+1} \boldsymbol{E}\Big[\sup |x(t)|^\eta \Big| k \text{ losses}\Big] \boldsymbol{P}[k \text{ losses}] \geqslant \\
&\geqslant \sum_{k=0}^{t+1} 2^{-\eta R(t+1-k)} |\lambda|^{\eta t} D_1^\eta \boldsymbol{P}[k \text{ losses}] = \\
&= D_1^\eta 2^{-\eta R(t+1)} |\lambda|^{\eta t} \sum_{k=0}^{t+1} 2^{\eta R k} \boldsymbol{P}[k \text{ losses}] = \\
&= D_1^\eta 2^{-\eta R(t+1)} |\lambda|^{\eta t} \sum_{k=0}^{t+1} 2^{\eta R k} C_{t+1}^k p^k (1-p)^{t+1-k} = \\
&= D_1^\eta |\lambda|^{\eta t} 2^{-\eta R(t+1)} [1 - p + 2^{\eta R} p]^{t+1} = \\
&= D_1^\eta |\lambda|^{-\eta} \cdot \Big[|\lambda|^\eta \Big(2^{-\eta R}(1-p) + p\Big)\Big]^{t+1} \overset{\text{(A2.1)}}{=\!=} D_1^\eta |\lambda|^{-\eta} \cdot \Big[\varphi(\eta)\Big]^{t+1}.
\end{aligned}$$

Since $\boldsymbol{E} \sup |x(t)|^\eta$ is bounded in t, the second inequality of (A2.1) is true.

Sufficiency. The decoder constructs recursively $\delta(t+0) := \mathfrak{F}[\delta(t-0), s(t)]$, $\delta(0) = \delta_0 > 0$ the estimate $\delta(t) > 0$ of stabilization precision $|x(t)| \leqslant \delta(t-0)$. In view of the information feedback at the instant t, the coder learns $s(t-1)$ and can duplicate the computation of $\delta(t-0)$. At this instant it divides the interval $[-\delta(t-0), \delta(t-0)] \ni x(t)$ into 2^R subintervals of equal lengths $2^{-(R-1)}\delta(t-0)$ and sends through the channel the number of the subinterval including the state $x(t)$ by coding this number as an R-bit packet. If the packet reaches the decoder, then the latter regards the middle of the corresponding interval as the estimate $\widehat{x}(t)$ of the current state $|x(t) - \widehat{x}(t)| \leqslant 2^{-R}\delta(t-0)$. The control $u(t)$ is selected so that $\widehat{x}(t) \xrightarrow{u(t)} x(t+1) = 0$ for $\xi(t) = 0$, that is, $u(t) := -\lambda \widehat{x}(t)$. In fact,

$$|x(t+1)| = |\lambda x(t) + u(t) + \xi(t)| \leqslant |\lambda||x(t) - \widehat{x}(t)| + D \leqslant 2^{-R}|\lambda|\delta(t-0) + D$$

and therefore,

$$\delta(t+0) := 2^{-R}|\lambda|\delta(t-0) + D \quad \text{for} \quad I(t) = 0.$$

In the case of a lost package, $u(t) := 0$ and, therefore,

$$|x(t+1)| = |\lambda x(t) + \xi(t)| \leqslant |\lambda||x(t)| + D \leqslant |\lambda|\delta(t-0) + D,$$
$$\delta(t+0) := |\lambda|\delta(t-0) + D \quad \text{for} \quad I(t) = 1.$$

This construction guarantees that $|x(t)| \leqslant \delta(t-0)\ \forall t$.

By applying the simple inequality $(a+b)^\eta \leqslant (1+\varepsilon)a^\eta + \alpha(\varepsilon,\eta)b^\eta\ \forall a,b \geqslant 0$, where $\varepsilon > 0$ is a small parameter, to the recursion

$$\delta(t+0) = \underbrace{|\lambda| \left[2^{-R}[1-I(t)] + I(t)\right] \delta(t-0)}_{a} + \underbrace{D}_{b}$$

and taking into consideration that $I(t)$ is independent of $\delta(t-0)$, we obtain

$$\delta^\eta(t+0) \leqslant (1+\varepsilon)|\lambda|^\eta \left[2^{-R\eta}[1-I(t)] + I(t)\right] \delta^\eta(t-0) + \alpha(\varepsilon,\eta)D^\eta,$$
$$\boldsymbol{E}\delta^\eta(t+0) \leqslant (1+\varepsilon)\ \underbrace{|\lambda|^\eta \left[2^{-R\eta}[1-p] + p\right]}_{=\varphi(\eta)<1 \text{ in virtue of (A2.1)}} \boldsymbol{E}\delta^\eta(t-0) + \alpha(\varepsilon,\eta)D^\eta.$$

By taking ε sufficiently small, we obtain the inequality $(1+\varepsilon)\varphi(\eta) < 1$ and, therefore, $\boldsymbol{E}\delta(t-0)^\eta$ is bounded and, consequently, the stabilization error $\boldsymbol{E}|x(t)|^\eta \leqslant \boldsymbol{E}\delta(t-0)^\eta$ is bounded as well.

REFERENCES

1. Nair, G.N., Fagnani, F., Zampieri, S., and Evans, R., Feedback Control under Data Rate Constraints: An Overview, *Proc. IEEE*, 2007, vol. 95, no. 1, pp. 108–137.
2. Baillieul, J. and Antsaklis, P.J., Control and Communication Challenges in Networked Real-Time Systems, *Proc. IEEE Spec. Issue Technol. Networked Control Syst.*, 2007, vol. 95, no. 1,
3. Hespanha, J., Naghshtabrizi, P., and Xu, Y., A Survey of Recent Results in Networked Control Systems, *Proc. IEEE Spec. Issue Technol. Networked Control Syst.*, 2007, vol. 95, no. 1, pp. 138–162.
4. Murray, R., Ed., Control in an Information Rich World: Report of the Panel on Future Directions in Control, Dynamics, and Systems, Pasadena: Caltech, 2002. Manuscript at `http://www.cds.caltech.edu/~murray/cdspanel/`
5. Widrow, B., Statistical Analysis of Amplitude-Quantized Sampled-Data Systems, *Trans. AIEE.*, 1961, vol. 79, no. 2, pp. 555–567.
6. Liu, B. and Kaneko, T., Error analysis of digital filters realized with floating-point arithmetic, *Proc. IEEE*, 1969, vol. 57, no. 10, pp. 1735–1747.
7. Liu, B., Effect of finite word length on the accuracy of digital filters–a review, *IEEE Trans. Circuits Theory*, 1971, vol. 18, no. 6, pp. 670–677.
8. Bogner, R.E. and Constantinides, A.G., Eds., *Introduction to Digital Filtering*, London, New York: Wiley, 1975. Translated under the title *Vvedenie v tsifrovuyu fil'tratsiyu*, Moscow: Mir, 1976.
9. Arens, V.D., Fedorov, S.M., Khitrik, M.S., and Luchko, S.V.; Khitrik, M.S. and Fedorov, S.M., Eds., *Dinamika sistem upravleniya raket s bortovymi tsifrovymi vychislitel'nymi mashinami* (Dynamics of the Rocket Control Systems with On-board Computers), 2nd. ed., Moscow: "Mashinostroenie", 1976.
10. Gray, R.M. and Neuhoff, D.L., Quantization, *IEEE Trans. Inform. Theory*, 1998, vol. 44, pp. 2325–2383.
11. Delchamps, D.F., Extracting State Information from a Quantized Output Record, *Syst. Control Lett.*, 1989, vol. 13, pp. 365–372.
12. Delchamps, D.F., Stabilizing a Linear System with Quantized State Feedback, *IEEE Trans. Automat. Control*, 1990, vol. 35, no. 8, pp. 916–924.
13. Koplon, R. and Sontag, E.D., Linear Systems with Sign-Observations, *SIAM J. Control Optim.*, 1993, vol. 31, no. 5, pp. 1245–1266.
14. Feely, O., A Tutorial Introduction to Nonlinear Dynamics and Chaos and their Application to Sigma-Delta Modulators, *Int. J. Circuit Theory Appl.*, 1997, vol. 25, pp. 347–367.

15. Baillieul, J., Feedback Coding for Information-Based Control: Operating Near the Data Rate Limit, in: *Proc. 41st IEEE Conf. Decision Control*, vol. ThP02-6, Las Vegas, Nevada, USA, 2002, Dec., pp. 3229–3236.

16. Tsypkin, Ya.Z. and Popkov, Yu.S., *Teoriya nelineinykh impul'snykh sistem* (Theory of Nonlinear Impulsive Systems), Moscow: Nauka, 1973.

17. Tou, J.T., *Optimum Design of Digital Control Systems*, New York: Academic, 1963.

18. Lewis, J.B. and Tou, J.T., Optimum Sampled-Data Systems with Quantized Control Signals, *Trans. AIEE.*, 1965, vol. 82, no. 2, pp. 195–201.

19. Larson, R.E., Optimum Quantization in Dynamic Systems, *IEEE Trans. Automat. Control*, 1967, vol. 12, pp. 162–168.

20. Marleau, R.S. and Negro J.E., Comments on "Optimum Quantization in Dynamic Systems", *IEEE Trans. Automat. Control*, 1972, vol. 4, pp. 273–274.

21. Fischer, T.R., Optimal Quantized Control, *IEEE Trans. Automat. Control*, 1982, vol. 27, no. 4, pp. 996–998.

22. Larson, R.E. and Tse, E., Author's Reply, *IEEE Trans. Automat. Control*, 1972, vol. 4, pp. 274–275.

23. Curry, R.E., A Separation Theorem for Nonlinear Measurements, *IEEE Trans. Automat. Control*, 1969, vol. 14, pp. 561–569.

24. Tatikonda, S., Sahai, A., and Mitter, S., Control of LQG Systems under Communication Constraints, in: *Proc. 37th IEEE Conf. Decision Control*, vol. WP04, Tampa, Florida USA, IEEE, 1998, pp. 1165–1170.

25. Tatikonda, S., Sahai, A., and Mitter, S., Control of LQG Systems under Communication Constraints, in: *Proc. Amer. Control Conf.*, San Diego, California USA, AACC, 1999, pp. 2778–2782.

26. Brockett, R.W. and Liberzon, D., Quantized Feedback Stabilization of Linear Systems, *IEEE Trans. Automat. Control*, 2000, vol. 45, no. 7, pp. 1279–1289.

27. Matveev, A.S. and Savkin, A.V., Optimal State Estmation in Networked Systems with Asynchronous Communication Channels and Switched Sensors, in: *Proc. 40th IEEE Conf. Decision Control*, vol. TuM13, Orlando, Florida USA, IEEE, 2001, pp. 825–830.

28. Nair, G.N. and Evans, R.J., Mean Square Stabilizability of Stochastic Linear Systems with Data Rate Constraints, in: *Proc. 41st IEEE Conf. Decision Control*, vol. WeM02, Las Vegas, Nevada USA, IEEE, 2002, pp. 1632–1637.

29. Williamson, D., Finite Wordlength Design of Digital Kalman Filters for State Estimation, *IEEE Trans. Automat. Control*, 1985, vol. AC-30, no. 10, pp. 930–939.

30. Goodman, D.J. and Gersho, A., Theory of an Adaptive Quantizer, *IEEE Trans. Commun.*, 1974, vol. COM-22, no. 8, pp. 1037–1045.

31. Zhang, S.D. and Lockhart, G.B., Design and Simulation of an Efficient Adaptive Delta Modulation Embedded Coder, *IEE Proc. Vis. Image Signal Proc.*, 1995, vol. 142, no. 3, pp. 155–160.

32. Zierhofer, C.M., Adaptive Sigma-Delta Modulation with One-bit Quantization, *IEEE Trans. Circuits Syst. II.*, 2000, vol. 47, no. 5, pp. 408–415.

33. Aldajani, M.A. and Sayed, A.H., Stability and Performance Analysis of an Adaptive Sigma-Delta Modulator, *IEEE Trans. Circuits Syst. II.*, 2001, vol. 48, no. 3, pp. 233–244.

34. Venayagamoorthy, G.K. and Zha, W., Comparison of Nonuniform Optimal Quantizer Designs for Speech Coding with Adaptive Critics and Particle Swarm, *IEEE Trans. Industry Appl.*, 2007, vol. 43, no. 1, pp. 238–244.

35. Golding, L.S. and Schultheiss, P.M., Study of an Adaptive Quantizer, *Proc. IEEE*, 1967, vol. 55, pp. 293–297.

36. Gomez-Estern, F., Canudas de Wit, C., Rubio, F., and Fornés, J., Adaptive Delta-Modulation Coding for Networked Controlled Systems, in: *Proc. Amer. Control. Conf.*, New York, USA, Jul. 11–13, 2007, FrA20.6.

37. Andrievsky, B., Fradkov, A.L., and Peaucelle, D., State Estimation over the Limited-Band Communication Channel for Pitch Motion Control of LAAS Helicopter Benchmark, in: *Proc. 17th IFAC Symp. Automat. Contr. Aerospace (ACA'2007)*, Toulouse, France, 2007.

38. Andrievsky, B., Adaptive Coding for Transmission of Position Information over the Limited-Band Communication Channel, in: *Proc. 9th IFAC Workshop Adaptat. Learning Control and Signal Proc. (ALCOSP'2007)*, St. Petersburg: Russia, IFAC, Aug. 29–31, 2007.

39. Zheng, J., Duni, E.R., and Rao, B.D., Analysis of Multiple-Antenna Systems with Finite-Rate Feedback Using High-Resolution Quantization Theory, *IEEE Trans. Signal Proc.*, 2007, vol. 55, no. 4, pp. 1461–1475.

40. Rotea, M.A. and Williamson, D., Optimal Realizations of Finite Wordlength Digital Filters and Controllers, *IEEE Trans. Circuits Syst. I.*, 1995, vol. 42, no. 2, pp. 61–72.

41. Gol'denberg, L.M., Levchuk, Yu.P., Polyak, M.N., *Tsifrovye fil'try* (Digital Filters), Moscow: Svyaz', 1974.

42. Ushio, T. and Hirai, K., Chaos in Non-Linear Sampled-Data Control Systems, *Int. J. Control*, 1983, vol. 38, no. 5, pp. 1023–1033.

43. Ushio, T. and Hsu, C., Chaotic Rounding Error in Digital Control Systems, *IEEE Trans. Circuits Syst.*, 1987, vol. 34, pp. 133–139.

44. Ushio, T. and Hirai, K., Chaotic Behavior in Piecewise-Linear Sampled-Data Control Systems, *Int. J. Nonlinear Mech.*, 1985, vol. 20, no. 5/6, pp. 493–506.

45. Delchamps, D.F., Some Chaotic Consequences of Quantization in Digital Filters and Digital Control Systems, in: *Proc. ISCAS'89*, 1989, pp. 602–605.

46. Delchamps, D.F., The 'Stabilization' of Linear Systems with Quantized Feedback, in: *Proc. 27th IEEE Conf. Decision Control*, vol. WP1, Austin, Texas, USA, IEEE, 1988, pp. 405–410.

47. Wong, W.S. and Brockett, R.W., State Estimation with Finite Communication Bandwidth Constraints, in: *Proc. 34th IEEE Conf. Decision Control*, New Orleans, LA: 1995, pp. 1400–1401.

48. Wong, W.S. and Brockett, R.W., Systems with Finite Communication Bandwidth Constraints – Part I: State Estimation Problems, *IEEE Trans. Automat. Control*, 1997, vol. 42, no. 9, pp. 1294–1299.

49. Shannon, C.E., A Mathematical Theory of Communication , *Bell Syst. Tech. J.*, 1948, vol. 27, pp. 379–423, 623–656.

50. Li, X. and Wong, W.S., State Estimation with Communication Constraints, *Syst. & Control Lett.*, 1996, vol. 28, pp. 49–54.

51. Nair, G.N. and Evans, R.J., State Estimation via a Capacity-Limited Communication Channel, in: *Proc. 36th IEEE Conf. Decision Control*, vol. WM09, San Diego, Califomia USA, IEEE, 1997, pp. 866–871.

52. Zhang, Z. and Berger, T., Estimation via Compressed Information, *IEEE Trans. Inform. Theory*, 1989, vol. 34, no. 2, pp. 198–211.

53. Han, T.S. and Amari, S., Parameter Estimation with Multiterminal Data Compression, *IEEE Trans. Inform. Theory*, 1995, vol. 41, no. 6, pp. 1802–1833.

54. Han, T.S. and Amari, S., Statistical Inference under Multiterminal Data Compression, *IEEE Trans. Inform. Theory*, 1998, vol. 44, no. 6, pp. 2300–2324.

55. Borkar, V.S. and Mitter, S.K., LQG Control with Communication Constraints, *Research Report LIDS-P-2326*, Dec. 1995, p. 12.

56. Shoham, Y. and Gersho, A., Efficient Bit Allocation for an Arbitrary Set of Quantizers, *IEEE Trans. Acoust., Speech, Signal Proc.*, 1988, vol. 36, no. 9, pp. 1445–1453.

57. Gersho, A., Optimal Nonlinear Interpolative Vector Quantization, *IEEE Trans. Comput.*, 1990, vol. 38, no. 9, pp. 1285–1287.

58. Zeger, K., Vaisey, J., and Gersho, A., Globally Optimal Vector Quantizer Design by Stochastic Relaxation, *IEEE Trans. Signal Proc.*, 1992, vol. 40, no. 2, pp. 310–322.

59. Bansal, R. and Basar, T., Simultaneous Design of Measurement and Control Strategies for Stochastic Systems with Feedback, *Automatica*, 1989, vol. 25, no. 5, pp. 679–694.

60. Gabor, G. and Gyorfi, Z., *Recursive Source Coding*, New York: Springer, 1986.

61. Tatikonda, S., Sahai, A., and Mitter, S., Stochastic Linear Control over a Communication Channel, *IEEE Trans. Automat. Control*, 2004, vol. 49, no. 9, pp. 1549–1561.

62. Gorbunov, A.K. and Pinsker, M.Sh., Epsilon-entropy and Rate of Nonanticipative Message Generation with Forecast, *Probl. Peredachi Inf.*, 1973, vol. 9, no. 3, pp. 12–21.

63. Chou, J., Chen, S., and Horng, I., Robust Stability Bound on Linear Time-Varying Uncertainties for Linear Digital Control Systems under Finite Wordlength Effects, *JSME Int. J. Ser. C*, 1996, vol. 39, pp. 767–771.

64. Feng, X. and Loparo K., Active Probing for Information in Control Systems with Quantized State Measurements: A Minimum Entropy Approach, *IEEE Trans. Automat. Control*, 1997, vol. 42, pp. 216–238.

65. Wong, W.S. and Brockett, R.W., Systems with Finite Communication Bandwidth Constraints – II: Stabilization with Limited Information Feedback, *IEEE Trans. Automat. Control*, 1999, vol. 44, no. 5, pp. 1049–1053.

66. Gelig, A.Kh, Leonov, G.A., and Yakubovich, V.A., *Ustoichivost' nelineinykh sistem s needinstvennym sostoyaniem ravnovesiya* (Sability of Nonlinear Systems with a Nonunique Equilibrium), Moscow: Nauka, 1978.

67. Filippov, A.F., Differential Equation s with Discontinuous Right Side, *Mat. Sb.*, 1960, vol. 51, no. 1, pp. 99–128.

68. Morse, A.S., Supervisory Control of Families of Linear Set-Point Controllers – Part I: Exact Matching, *IEEE Trans. Automat. Control*, 1996, vol. 41, no. 10, pp. 1413–1431.

69. Delvenne, J.-C., An Optimal Quantized Feedback Strategy for Scalar Linear Systems, *IEEE Trans. Automat. Control*, 2006, vol. 51, no. 2, pp. 298–303. Manuscript at `http://www.inma.ucl.ac.be/ delvenne/delvenne.ps`

70. Fagnani, F., Chaotic Quantized Feedback Stabilizers: The Scalar Case, *Communicat. Inform. Syst.*, 2004, vol. 4, no. 1, pp. 53–72.

71. Fagnani, F. and Zampieri, S., Stability Analysis and Synthesis for Scalar Linear Systems with a Quantized Feedback, *IEEE Trans. Automat. Control*, 2003, vol. 48, no. 9, pp. 1569–1584.

72. Fagnani, F. and Zampieri, S., Quantized Stabilization of Linear Systems: Complexity Versus Performance, *IEEE Trans. Automat. Control*, 2004, vol. 49, no. 9, pp. 1534–1548.

73. Elia, N. and Mitter, S.K., Stabilization of Linear Systems with Limited Information, *IEEE Trans. Automat. Control*, 2001, vol. 46, no. 9, pp. 1384–1400.

74. Kwakernaak, H, and Sivan, R., *Linear Optimal Control Systems*, New York: Wiley, 1972. Translated under the title *Lineinye optimal'nye sistemy upravleniya*, Moscow: Mir, 1977.

75. Isidori, A., Semiglobal Robust Regulation of Nonlinear Systems, in: *Lecture Notes Control Inform. Sci.*, 1996, vol. 215, pp. 27.

76. Khalil, H.K. and Esfandiari, F., Semiglobal Stabilization of a Class of Nonlinear Systems Using Output Feedback, *IEEE Trans. Automat. Control*, 1993, vol. 38, pp. 1412–1415.

77. Lin, Z. and Saberi, A., Robust Semiglobal Stabilization of Minimumphase Input-Output Linearizable Systems via Partial State and Output Feedback, *IEEE Trans. Automat. Control*, 1995, vol. 40, pp. 1029–1041.

78. Elia, N., Coarsest Quantizer Density for Quadratic Stabilization of Two-Input Linear Systems, in: *Proc. 10th Mediterranean Conf. Control Automat. (MED2002)*, Lisbon, Portugal, 2002.

79. Elia, N. and Frazzoli, E., Quantized Stabilization of Two-Input Linear Systems: A Lower Bound on the Minimal Quantization Density, in: Tomlin, C.J. and Greenstreet, M.R., Eds., *HSCC 2002, LNCS 2289*, Berlin, Heidelberg: Springer, 2002, pp. 179–193.

80. Elia, N., Control-oriented Feedback Communication Schemes, in: *Proc. 42nd IEEE Conf. Decision Control*, vol. ThA03, Maui, Hawaii, USA, 2003, pp. 3161–3166.

81. Schalkwijk, J.P.M., A Coding Scheme for Additive Noise Channels with Feedback – II: Bandlimited Signals, *IEEE Trans. Inform. Theory*, 1966, vol. IT-12, no. 2, pp. 183–189.

82. Schalkwijk, J.P.M. and Kailath, T., A Coding Scheme for Addtive Noise Channels with Feedback – I: No Bandwidth Constraint, *IEEE Trans. Inform. Theory*, 1966, vol. IT-12, no. 2, pp. 172–182.

83. Ishii, H. and Francis, B.A., Quadratic Stabilization of Sampled-Data Systems with Quantization, *Automatica*, 2003, vol. 39, pp. 1793–1800.

84. Ishii, H. and Francis, B.A., Stabilizing a Linear System by Switching Control with Dwell Time, *IEEE Trans. Automat. Control*, 2002, vol. 47, no. 12, pp. 1962–1973.
85. Shannon, C.E. and Weaver, W., *The Mathematical Theory of Communication*, Urbana: Univ. Illinois Press, 1949.
86. Tartakovskii, G.P., *Teoriya informatsionnykh sistem* (Theory of Information Systems), Moscow: Fizmatkniga, 2005.
87. Fano, R.M., *Transmission of Information: A Statistical Theory of Communication.* MIT Press, 1961. Translated under the title *Peredacha informatsii. Statisticheskaya teoriya svyazi*, Moscow: Mir, 1965.
88. Kolesnik, V.D. and Poltyrev, G.Sh., *Kurs teorii informatsi* (A Course of the Information Theory), Moscow: Nauka, 1982.
89. Åström, K.J. and Wittenmark, B., *Computer Controlled Systems. Theory and Design*, Englewood Cliffs, New Jersey: Prentice Hall, 1984. Translated under the title *Sistemy upravleniya s EVM*, Moscow: Mir,
90. Curry, R.E., *Estimation and Control with Quantized Measurements*, Cambridge, MA: MIT Press, 1970.
91. Matveev, A.S. and Savkin, A.V., The Problem of LQG Optimal Control via a Limited Capacity Communication Channel, *Syst. & Control Lett.*, 2004, vol. 53, pp. 51–64.
92. Borkar, V., Mitter, S., and Tatikonda, S., Optimal Sequential Vector Quantization of Markov Sources, *SIAM J. Control Optim.*, 2001, vol. 1, no. 40, pp. 135–148.
93. Şimşek, T., Jain, R., and Varaiya, P., Scalar Estimation and Control with Noisy Binary Observations, *IEEE Trans. Automat. Control*, 2004, vol. 49, no. 9, pp. 1598–1603.
94. Mitter, S.K., Control with Limited Information: The Role of Systems Theory and Information Theory. ISIT 2000 Plenary Talk, in: *IEEE Inform. Theory Soc. Newslet.*, 2000, pp. 1–23.
95. Mitter, S.K., Control with Limited Information, *Eur. J. Control*, 2001, vol. 7, pp. 122–131.
96. Nair, G.N. and Evans, R.J., Stabilization with Data-Rate-Limited Feedback: Tightest Attainable Bounds, *Syst. & Control Lett.*, 2000, vol. 41, pp. 49–56.
97. Nair, G.N. and Evans, R.J., Stabilizability of Stochastic Linear Systems with Finite Feedback Data Rates, *SIAM J. Control Optim.*, 2004, vol. 43, no. 2, pp. 413–436.
98. Sahai, A., Anytime Information Theory, *PhD Thesis*, Cambridge, MA, USA: Massachusetts Institute of Technology, 2001, pp. 175. Manuscript at `http://dspace.mit.edu/handle/1721.1/8770`
99. Tatikonda, S., Control Under Communication Constraints, *PhD Thesis*, Cambridge, MA, USA: Massachusetts Institute of Technology, 2000, pp. 225. Manuscript at `http://hdl.handle.net/1721.1/16755`
100. Witsenhausen, H., On the Structure of Real-Time Source Coders, *Bell Syst. Tech. J.*, 1979, vol. 58, pp. 1437–1451.
101. Sahai, A., The Necessity and Sufficiency of Anytime Capacity for Control over a Noisy Communication Link, in: *Proc. 43rd IEEE Conf. Decision Control.*, vol. WeB02, Atlantis, Paradise Island, Bahamas: IEEE, 2004, pp. 1896–1901.
102. Sahai A., Mitter S., The Necessity and Sufficiency of Anytime Capacity for Stabilization of a Linear System over a Noisy Communication Link. Part I: Scalar Systems, *IEEE Trans. Inform. Theory*, 2006, vol. 52, no. 8, pp. 3369–3395, (preprint: arXiv: cs.IT/0601007).
103. Matveev, A.S. and Savkin, A.V., Multirate Stabilization of Linear Multiple Sensor Systems via Limited Capacity Communication Channels, *SIAM J. Control Optim.*, 2005, vol. 44, no. 2, pp. 584–618.
104. Nair, G.N. and Evans, R.J., Exponential Stabilisability of Finite-Dimensional Linear Systems with Limited Data Rates, *Automatica*, 2003, vol. 39, pp. 585–593.
105. Nair, G.N., Evans, R.J., and Caines, P.E., Stabilising Decentralised Linear Systems under Data Rate Constraints, in: *Proc. 43rd IEEE Conf. Decision Control*, vol. ThC10, Atlantis, Paradise Island, Bahamas: IEEE, 2004, pp. 3992–3997.
106. Dembo, A., Cover, T.M., and Thomas, J.A., Information Theoretic Inequalities, *IEEE Trans. Inform. Theory.*, 1001, vol. 37, no. 6, pp. 1501–1518.
107. Matveev, A.S. and Savkin, A.V., An Analogue of Shannon Information Theory for Detection and Stabilization via Noisy Discrete Communication Channels, *SIAM J. Control. Optim.*, 2007, vol. 46, no. 4, pp. 1323–1361.

108. Kolmogorov, A.N., New Metric Invariant of the Transitive Dynamic Systems and Automorphisms of the Lebesgue Space, *Dokl. Acad. Nauk SSSR*, 1958, vol. 119, no. 5, pp. 861–864.

109. Kolmogorov, A.N., On Entropy per Time Unit as the Metric Invariant of Automorphisms, *Dokl. Acad. Nauk SSSR*, 1959, vol. 124, no. 4, pp. 754–755.

110. Kornfel'd, I.P., Sinai, Ya.G., and Fomin, S.V., *Ergodicheskaya teoriya* (Ergodic Theory), Moscow: Nauka, 1980.

111. Martin, N.F.G. and England, J.W., Mathematical Theory of Entropy, in *Encyclopedia of Mathematics and its Applications*, vol. 12, Reading, Mass.: Addison-Wesley, 1981. Translated under the title *Matematicheskaya teoriya entropii*, Moscow: Mir, 1988.

112. Sinai, Ya.G., *Vvedenie v ergodicheskuyu teoriyu* (Introduction to the Ergodic Theory), Moscow: Fazis, 1996.

113. Adler, R.L., Konheim, A.G., and McAndrew, M.H., Topological Entropy, *Trans. Amer. Math. Soc.*, 1965, vol. 114, no. 2, pp. 309–319.

114. Nair, G.N., Evans, R.J., Mareels, I., and Moran, W., Topological Feedback Entropy and Nonlinear Stabilization, *IEEE Trans. Automat. Control*, 2004, vol. 49, no. 9, pp. 1585–1597.

115. Savkin, A.V., Analysis and Synthesis of Networked Control Systems: Topological Entropy, Observability, Robustness, and Optimal Control, *Automatica*, 2006, vol. 42, no. 1, pp. 51–62.

116. Bowen, R., *Equilibrium States and Ergodic Theory of Anosov Diffeomorphisms*, Berlin: Springer, 1975.

117. Baillieul, J., Feedback Designs for Controlling Device Arrays with Communication Channel Bandwidth Constraints, in: Pasik-Duncan, B., Ed., *Proc. 4th ARO Workshop on Smart Structures, Ser. Lecture Notes in Control and Informat. Sci.*, New York: Springer, 2002, pp. 35–57.

118. Baillieul, J., Feedback Designs in Information-Based Control, Pasik-Duncan, B., Ed., in: *Proc. Workshop Stochastic Theory and Control. Ser. Lecture Notes in Control and Informat. Sci.*, New York: Springer, 2002, pp. 35–57.

119. Hespanha, J.P., Ortega, A., and Vasudevan, L., Towards the Control of Linear Systems with Minimum Bit-rate, in: *Proc. Int. Simp. Mathemat. Theory of Networks and Syst.*, Notre Dame, USA, 2002, pp. 15.

120. Nair, G.N. and Evans, R.J., State Estimation under Bit-rate Constraints, in: *Proc. 37th IEEE Conf. Decision Control*, vol. WA09, Tampa, Florida USA, IEEE, 1998, pp. 251–256.

121. Nair, G.N. and Evans, R.J., A Finite-dimensional Coder-Estimator for Rate-constrained State Estimation, in: *Proc. 14th IFAC World Congr.*, Beijing, China, 1999, pp. 19–24.

122. Tatikonda, S. and Mitter, S., Control under Communication Constraints, *IEEE Trans. Automat. Control*, 2004, vol. 49, no. 7, pp. 1056–1068.

123. Matveev, A.S. and Savkin, A.V., Stabilization of Multisensor Networked Control Systems with Communication Constraints, in: *Proc. 5th Asian Control Conf.*, Melbourne, Australia, 2004, pp. 1905–1913.

124. Nair, G.N., Dey, S., and Evans, R.J., Communication-limited Stabilizability of Jump Linear Systems, in: *Proc. 15th Int. Symp. Mathemat. Networked Syst.*, USA, Univ. of Notre Dame, 2002.

125. Nair, G.N., Dey, S., and Evans, R.J., Infimum Data Rates for Stabilising Markov Jump Linear Systems, in: *Proc. 42nd IEEE Conf. Decision Control*, vol. TuP06, Maui, Hawaii, USA, IEEE, 2003, pp. 1176–1181.

126. Gantmakher, F. R., *Teoriya matrits* (Theory of Matrices), Moscow: Nauka, 1966. Translated into English under the title *Theory of Matrices*, Chelsea, New York, 1959.

127. Liberzon, D., Hybrid Feedback Stabilization of Systems with Quantized Signals, *Automatica*, 2003, vol. 39, pp. 1543–1554.

128. Liberzon D. and Nešić, D., Input-to-State Stabilization of Linear Systems with Quantized State Measurements, *IEEE Trans. Automat. Control*, 2007, vol. 52, no. 2, pp. 767–781.

129. Verriest, E. and Egerstedt, M., Control with Delayed and Limited Information: A First Look, in: *Proc. 41st IEEE Conf. Decision Control*, Las Vegas, N.V., 2002, pp. 1231–1236.

130. Petersen, I.R. and Savkin, A.V., Multi-rate Stabilization of Multivariable Discrete-time Linear Systems via a Limited Capacity Communication Channel, in: *Proc. 40th IEEE Conf. Decision Control*, vol. TuA11, Orlando, Florida USA, IEEE, 2001, pp. 304–309.

131. Sontag, E.D. and Wang, Y., New Chracterizations of Input-to-State Stability, *IEEE Trans. Automat. Control*, 1996, vol. 41, no. 9, pp. 1283–1294.

132. Matveev, A.S. and Savkin, A.V., Stabilization of Stochastic Linear Plants via Limited Capacity Stochastic Communication Channels, in: *Proc. 45th IEEE Conf. Decision Control*, San Diego, C.A., 2006, pp. 484–489.

133. Körner, J. and Orlitsky, A., Zero-Error Information Theory, *IEEE Trans. Inform. Theory*, 1998, vol. 44, no. 6, pp. 2207–2229.

134. Shannon, C.E., The Zero Error Capacity of a Noisy Channel, *IRE Trans. Inform. Theory*, 1956, vol. IT-2, pp. 8–19.

135. Ahlswede, R., Cai, N., and Zhangm Z., Erasure, List, and Detection Zero-error Capacities for Low Noise and a Relation to Identification, *IEEE Trans. Inform. Theory*, 1996, vol. 42, no. 1, pp. 55–62.

136. Csiszar, I. and Narayan, P., Secrecy Capacities for Multiple Terminals, *IEEE Trans. Inform. Theory*, 2004, vol. 50, no. 2, pp. 3047–3061.

137. Ovseevich, I.A., Capacity of Random Feedbacked Channel and Coordination of Sources with such Channels, *Probl. Peredachi Inf.*, 1968, vol. 4, pp. 52–59.

138. Verdú, S., Fifty Years of Shannon Theory, *IEEE Trans. Inform. Theory*, 1998, vol. 44, no. 6, pp. 2057–2078.

139. Dobrushin, R.L., Information Transmission through Feedbacked Channel, *Teor. Veroyatn. Primen.*, 1958, vols. 3, 4, pp. 395–412.

140. Ebert, P.M., The Capacity of the Gaussian Channel with Feedback, *Bell Syst. Tech. J.*, 1970, vol. 49, pp. 1705–1712.

141. Pinsker, M.S. and Dobrushin, R.L., Memory Increases Capacity, *Probl. Peredachi Inf.*, 1969, vol. 5, pp. 94–95.

142. Zigangirov, K.Sh.,Upper Estimates of the Error Probability for the Feedbacked Channels, *Probl. Peredachi Inf.*, 1970, vol. 6, no. 2, pp. 87–92.

143. Tatikonda, S. and Mitter, S., Control over Noisy Channels, *IEEE Trans. Automat. Control*, 2004, vol. 49, no. 7, pp. 1196–1201.

144. Witsenhausen, H., Separation of Estimation and Control for Discrete Time Systems, *Proc. IEEE*, 1971, vol. 59, no. 11, pp. 1557–1566.

145. Kobayashi, H., Hanafusa, H., and Yoshikawa, T., Controllability under Decentralized Information Structure, *IEEE Trans. Automat. Control*, 1978, vol. 23, pp. 182–188.

146. Matveev, A.S. and Savkin, A.V., On a Problem Related to Application of Digital Networked Communication Technology to Stabilization of Noisy Plants over Noisy Channels, in: *Proc. IEEE Conf. Control Appl.*, Munich, Germany, 2006, pp. 2072–2077.

147. Matveev, A.S. and Savkin, A.V., An Analogue of Shannon Information Theory for Networked Control Systems. Stabilization via a Noisy Discrete Channel, in: *Proc. 43rd IEEE Conf. Decision Control*, vol. FrA11, Atlantis, Paradise Island Bahams, IEEE, 2004, pp. 4491–4496.

148. Matveev, A.S. and Savkin, A.V., An Analogue of Shannon Information Theory for Networked Control Systems: State Estimation via a Noisy Discrete Channel, in: *Proc. 43rd IEEE Conf. Decision Control*, vol. FrA11, Atlantis, Paradise Island Bahams, IEEE, 2004, pp. 4485–4490.

149. Matveev, A.S., State Estimation via Limited Capacity Noisy Communication Channels, *Math. Control, Signal., Syst.*, 2008, vol. 20, no. 1, pp. 1–35.

150. Matveev, A.S. and Savkin, A.V., Zero Error Capacity as the Border of the Domain of Almost Sure Observability over Noisy Channels, in: *Proc. 44th IEEE Conf. Decision Control Eur. Control Conf. CDC-ECC'05*, Seville, Spain, 2005, pp. 3219–3224.

151. Matveev, A.S. and Savkin, A.V., Shannon Zero Error Capacity in the Problems of State Estimation and Stabilization via Noisy Communication Channels, *Int. J. Control*, 2007, vol. 80, no. 2, pp. 241–255.

152. Matveev, A.S. and Savkin, A.V., Almost Sure Nonobservability and Nonstabilizability of Unstable Noisy Plants via Communication Channels with Packet Losses, in: *Proc. 44th IEEE Conf. Decision Control Eur. Control Conf. CDC-ECC'05*, Seville, Spain, 2005, pp. 7338–7341.

153. Matveev, A.S. and Savkin, A.V., Comments on “Control over Noisy Channels” and Relevant Negative Results, *IEEE Trans. Automat. Control*, 2005, vol. 50, no. 12, pp. 2105–2110.

154. Ling, Q. and Lemmon, M.D., Stability of Quantized Control Systems under Dynamic Bit Assignment, *IEEE Trans. Automat. Control*, 2005, vol. 50, no. 5, pp. 734–740.

155. Şimşek, T. and Varaiya, P., Noisy Data-Rate Limited Estimation: Renewal Codes, in: *Proc. 42nd IEEE Conf. Decision Control*, vol. ThA03, Maui, Hawaii USA, IEEE, 2003, pp. 3148–3154.

156. Jain, R., Şimşek, T., and Varaiya, P., Control under Communication Constraints, in: *Proc. 41st IEEE Conf. Decision Control*, vol. ThP02, Las Vegas, Nevada, USA, IEEE, 2002, pp. 3209–3216.

157. Martins, N.C., Dahleh, M.A., and Elia, N., Feedback Stabilization of Uncertain Systems Using a Stochastic Digitat Link, in: *Proc. 43rd IEEE Conf. Declsion Control*, vol. WeB02, Atlantis, Paradise Island, Bahamas, IEEE, 2004, pp. 1889–1895.

158. Martins, N.C., Dahleh, M.A., and Elia, N., Feedback Stabilization of Uncertain Systems in the Presence of a Direct Link, *IEEE Trans. Automat. Control*, 2006, vol. 51, no. 3, pp. 438–447.

159. Martins, N.C., Information Theoretic Aspects of the Control and Mode Estimation of Stochastic Systems, *PhD Thesis*, MA, USA: Massachusetts Institute of Technology, 2004.

160. Minero, P., Franceschetti, M., Dey, S., and Nair, G., Data Rate Theorem for Stabilization over Fading Channels, in: *Proc. 45th Ann. Allerton Conf. on Communic., Control and Comput*, Urbana-Champaign, USA: Univ. Illinois, 2007.

161. Hamming, R.W., *Coding and Information Theory*, Prentice Hall. Translated under the title *Teoriya kodirovaniya i teoriya informatsii*, Moscow: Radio i svyaz’, 1983.

162. Sidel’nikov, V.M., *Teoriya kodirovaniya* (Coding Theory), Moscow: Fizmatlit, 2008.

163. Gong, Zh. and Aldeen, M., Stabilization of Decentralized Control Systems, *J. Math. Syst., Estimat., Contr.*, 1997, vol. 7, no. 1, pp. 1–16.

164. Anderson, B.D.O. and Moore, J.B., Time-Varying Feedback Laws for Decentralized Control, *IEEE Trans. Automat. Control*, 1981, vol. 26, no. 5, pp. 1133–1138.

165. Lee, S., Meerkov, M., and Runolfsson, T., Vibrational Feedback Control: Zero Placement Capabilities, *IEEE Trans. Automat. Control*, 1987, vol. 52, no. 5, pp. 604–611.

166. Matveev, A.S. and Savkin, A.V., *State Estimation and Control over Communication Networks*, Boston: Birkhäuser, 2009.

167. Nahi, N.E., Optimal Recursive Estimation with Uncertain Observations, *IEEE Trans. Inform. Theory*, 1969, vol. 15, pp. 457–462.

168. Yu, M., Wang, L., Chu, T., and Xie, G., Stabilization of Networked Control Systems with Data Packet Dropout and Network Delays via Switching System Approach, in: *Proc. IEEE Conf. Decision Control*, Atlantis, Paradise Island, Bahamas, 2004, pp. 3539–3544.

169. Xiong, J. and Lam, J., Stabilization of Linear Systems over Networks with Bounded Packet Loss, *Automatica*, 2007, vol. 43, vol. 43. pp. 80–87.

170. Sahai, A. and Mitter, S., Source Coding and Channel Requirements for Unstable Processes, 2007. Manuscript at `http//:www.eecs.berkeley.edu/ sahai/Papers/anytime.pdf`

171. Feinstein, A., A New Basic Theorem of Information Theory, *IEEE Trans. Inform. Theory*, 1954, vol. 4, no. 4, pp. 2–22.

172. Feinstein, A., *Foundations of Information Theory*, McGraw-Hill. Translated under the title *Osnovy teorii informatsii*, Moscow: IL, 1960.

173. Sahai, A. and Xu, Q., The Anytime Reliability of Constrained Packet Erasure Channels with Feedback, in: *Proc. 42nd Allerton Conf. on Communicat., Control, Comput.*, Monticello, IL: 2004, pp. 200–209.

174. Sahai, A. and Xu, Q., The Anytime Reliability of the AWGN+erasure Channel with Feedback, in: *Proc. 42nd Allerton Conf. on Communicat., Control, Comput.*, Monticello, IL: 2004, pp. 300–309.

175. Sahai, A., Avestimehr, S., and Minero, P., Anytime Communication over the Gilbert–Elliott Channel with Noiseless Feedback, in: *Proc. IEEE Int. Simp. Inform. Theory*, Adelaide, Australia, 2005, pp. 1783–1787.

176. Savkin, A.V., Detectability and Output Feedback Stabilizability of Nonlinear Networked Control Systems, in: *Proc. 44th IEEE Conf. Decision Control Eur. Control Conf. CDC-ECC'05*, Seville, Spain, 2005, pp. 8174–8178.

177. Yakubovich, V.A., *Metody issledovaniya nelineinykh sistem upravleniya* (Methods to Study Nonlinear Control Systems), Chs. 2,3, Moscow: Nauka, 1975.

178. Cheng, T.M. and Savkin, A.V., Output Feedback Stabilization of Nonlinear Networked Control Systems with Non-Decreasing Nonlinearities: A Matrix Inequalities Approach, *Int. J. Robust Nonlinear Control*, 2007, vol. 17, pp. 387–404.

179. Malyavej, V. and Savkin, A.V., The Problem of Optimal Robust Kalman State Estimation via Limited Capacity Digital Communication Channels Malyavej, V. and Savkin, A.V., *Syst. & Control Lett.*, 2005, vol. 54, pp. 283–292.

180. Slepian, D. and Wolf, J.K., Noiseless Coding for Correlated Information Sources, *IEEE Trans. Inform. Theory*, 1973, vol. 19, no. 4, pp. 471–480.

181. Tatikonda, S.C., Some Scaling Properties of Large Distributed Control Systems, in: *Proc. 42nd IEEE Conf. Decision Control*, Maui, Hawaii, USA, 2003, pp. 3142–3147.

182. Nair, G.N. and Evans, R.J., Stabilizing Decentralized Linear Systems under Data Rate Constraints, in: *Proc. 43rd IEEE Conf. Decision Control*, Atlantis, Bahamas, 2004, pp. 3992–3997.

183. Liu, J. and Elia, N., Quantized Control with Applications to Mobile Vehicles, in: *Proc. 41st IEEE Conf. Decision Control*, vol. ThA01, Las Vegas, Nevada USA, IEEE, 2002, pp. 2391–2396.

184. Arcak, M. and Kokotovic, P.V., Nonlinear Observers: A Circle Criterion Design and Robustness Analysis, *Automatica*, 2001, vol. 37, no. 12, pp. 1923–1930.

185. Liberzon, D. and Hespanha, J.P., Stabilization of Nonlinear Systems with Limited Information Feedback, *IEEE Trans. Automat. Control*, 2003, vol. 50, no. 6, pp. 910–915.

186. De Persis, C., n-Bit Stabilization of n-Dimensional Nonlinear Systems in Feedforward Form, *IEEE Trans. Automat. Control*, 2005, vol. 50, no. 3, pp. 299–311.

187. De Persis, C., On Stabilization of Nonlinear Systems under Data Rate Constraints Using Output Measurements, *Int. J. Robust Nonlinear Control*, 2006, vol. 16, pp. 315–332.

188. De Persis, C. and Nešić, D., Practical Encoders for Controlling Nonlinear Systems under Communication Constraints, in: *Proc. 44th IEEE Conf. Decision & Control and Eur. Control Conf. CDC-ECC'05*, vol. MoA13.3, Seville, Spain, Dec. 2005, pp. 434–439.

189. Savkin, A.V. and Cheng, T.M., Detectability and Output Feedback Stabilizability of Nonlinear Networked Control Systems, *IEEE Trans. Automat. Control*, 2007, vol. 52, no. 4, pp. 730–735.

190. Fradkov, A.L., Andrievsky, B., Evans, R.J., Chaotic Observer-Based Synchronization under Information Constraints, *Physic. Rev. E.*, 2006, vol. 73, pp. 066209.

191. Fradkov, A.L., Andrievsky, B., and Evans, R.J., Adaptive Observer-Based Synchronization of Chaotic Systems with First-Order Coder in Presence of Information Constraints, *IEEE Trans. Circuits Syst. I*, 2008, vol. 55, no. 6, pp. 1685–1694.

192. Pecora, L.M. and Carroll, T.L., Synchronization in Chaotic Systems, *Phys. Rev. Lett.*, 1990, vol. 64, p. 821.

193. Fradkov, A.L., Nijmeijer, H., and Markov, A., Adaptive Observer-Based Synchronization for Communications, *Int. J. Bifurcat. Chaos.*, 2000, vol. 10, no. 12, pp. 2807–2814.

194. Miroshnik, I.V., Nikiforov, V.O., and Fradkov, A.L., *Nelineinoe i adaptivnoe upravlenie slozhnymi dynamicheskimi sistemami* (Nonlinear and Adaptive Control of Complex Dynamic Systems), St. Petersburg: Nauka, 2000.

195. Andrievskii, B.R., Blekhman, I.I., Bortsov, Yu.A., *et al.*, Blekhman, I.I. and Fradkov, A.L., Eds., *Upravlenie mekhatronnymi vibratsionnymi ustanovkami* (Control of Mechanotronic Vibrational Stands), St. Petersburg: Nauka, 2001.

196. Fradkov, A.L. and Andrievskii, B.R., Method of Passifications in the Problems of Adaptive Control, Observation, and Synchronization, in: Gelig, A.Kh., Leonov, G.A., and Fradkov, A.L., Eds., *Nelineinye sistemy. Chastotnye i matrichnye neravenstva* (Nonlinear Systems. Frequency and Matrix Inequalities), Moscow: Fizmatlit, 2008, pp. 452–499.

197. Andrievskii, B.R. and Fradkov, A.L., Method of Passification in Adaptive Control, Estimation, and Synchronization, *Automation and Remote Control*, 2006, vol. 67, no. 11, pp. 1699–1731.

198. Andrievskii, B.R. and Fradkov, A.L., *Izbrannye glavy teorii avtomaticheskogo upravleniya s primerami na yazyke MATLAB* (Selected Chapters of the Automatic Control Theory with examples in MATLAB), St. Petersburg: Nauka, 1999.

199. Andrievskii, B.R. and Fradkov, A.L., *Elementy matematicheskogo modelirovaniya v programmnykh sredakh MATLAB 5 and Scilab. Uch. pos.* (Textbook of the Elements of Mathematical Modeling in the Software Environments MATLAB 5 and Scilab), St. Petersburg: Nauka, 2001.

200. Fradkov, A.L., Andrievsky, B., and Evans, R.J., Synchronization of Passifiable Lurie Systems via Limited-Capacity Communication Channel, *IEEE Trans. Circuits Syst. I*, 2009, vol. 56, no. 2, pp. 430–439.

201. Fradkov, A.L., Synthesis of Adaptive System of Stabilization of Linear Dynamic Plants, *Automation and Remote Control*, 1974, vol. 35, no. 12, pp. 96–103.

202. Fomin, V.N., Fradkov, A.L., and Yakubovich, V.A., *Adaptivnoe upravlenie dinamicheskimi ob"ektami*, (Adaptive Control of Dynamic Plants), Moscow: Nauka, 1981.

203. Derevitskii, D.P. and Fradkov, A.L., Studying the Discrete Adaptive Control Systems of Continuous Plant Using Continuous Models, *Izv. Akad. Nauk SSSR, Tekh. Kibern.*, 1975, vol. 5, pp. 93–99.

204. Derevitskii, D.P. and Fradkov, A.L., *Prikladnaya teoriya diskretnykh adaptivnykh sistem upravleniya* (Applied Theory of Discrete Adaptive Control Systems), Moscow: Nauka, 1981.

205. Fradkov, A.L. and Andrievskii, B.R., Adaptive Synchronization of Nonlinear Systems, *Izv. Vuzov, Prib.*, 2007, vol. 50, no. 10, pp. 17–23.

206. Fradkov, A.L., Andrievsky, B., and Evans, R.J., Controlled Synchronization of One Class of Nonlinear Systems under Information Constraints, Dec. 2007. Manuscript at `http://arxiv.org/abs/0712.0636v1`

207. Fradkov, A.L., Andrievsky, B., Andrievsky, A., Observer-Based Synchronization of Discrete-Time Chaotic Systems under Communication Constraints, in: *Proc. 17th IFAC World Congr.*, Seoul, Korea, IFAC, 2008. Manuscript at `http://www.ifac-papersonline.net/`

208. Andrievskii, B.R. and Fradkov, A.L., Adaptive Synchronization of a Class of Nonlinear Systems under Constrained Capacity of the Communication Channel, in: *Upravlenie bol'shimi sistemami* (Control of Large Systems), vol. 25, Moscow: IPU RAN, 2009, pp. 48–83.

209. Fradkov, A.L., Andrievsky, B., and Evans, R.J., Chaotic Observer-Based Synchronization under Information Constraints, *Phys. Rev. E.*, 2008, vol. 78, pp. 036210 (1–6).

210. Fradkov, A.L., Andrievsky, B., and Evans, R.J., Hybrid Quantized Observer for Multi-Input-Multi-Output Nonlinear Systems, in: *Proc. 2008 IEEE Multi-conf. Syst. Control*, USA, 2008, FrB03.3.

211. Fradkov, A.L., Andrievsky, B., and Evans, R.J., Synchronization of Nonlinear Systems under Information Constraints, *Chaos*, 2008, vol. 18, no. 3, pp. 037109.

212. Fradkov, A.L. and Andrievsky, B., Application of Passification Method to Controlled Synchronization of Tree Networks under Information Constraints, in: *Proc. 3rd IEEE Multi-conf. Syst. Control (MSC 2009)*, St. Petersburg, Russia, 2009, pp. 513–518.

213. Hou, L., Michel, A.N., and Ye, H., Some Qualitative Properties of Sampled-data Control Systems, *IEEE Trans. Automat. Control*, 1997, vol. 42, no. 12, pp. 1721–1725.

214. Brockett, R.W., Stabilization of Motor Networks, in: *Proc. 34th IEEE Conf. Decision Control*, Dec. 1995, pp. 1484–1488.

215. Lee, K.-H. and Petersen, D.P., Optimal Linear Coding for Vector Channels, *IEEE Trans. Commun.*, 1976, vol. COM-24, no. 12, pp. 1283–1290.

216. Braslavsky, J.H., Middleton, R.H., and Freudenberg, J.S., Feedback Stabilization over Signal-to-Noise Ratio Constrained Channels, in: *Proc. 2004 Amer. Control Conf.*, Boston, MA, 2004, vol. 6, pp. 4903–4908.

217. Braslavsky, J.H. and Middleton, R.H., and Freudenberg, J.S., Feedback Stabilization over Signal-to-Noise Ratio Constrained Channels, *IEEE Trans. Automat. Control*, 2007, vol. 52, no. 8, pp. 1391–1403.

218. Freudenberg, J.S., Braslavsky, J.H., and Middleton, R.H., Control over Signal-to-Noise Ratio Constrained Channels: Stabilization and Performance, in: *Proc. 44th IEEE Conf. Decision Control Eur. Control Conf. CDC-ECC'05*, Seville, Spain, 2005, pp. 191–196.

219. Zaborsky, J., An Information Theory Viewpoint for the General Identification Problem, *IEEE Trans. Automat. Control*, 1966, vol. 11, no. 1, pp. 130–131.

220. Saridis, G.N., Entropy Formulation of Optimal and Adaptive Control, *IEEE Trans. Automat. Control*, 1988, vol. 33, no. 8, pp. 713–721.

221. Tsai, Y.A., Casiello, F.A., and Loparo, K.A., Discrete-Time Entropy Formulation of Optimal and Adaptive Control Problems, *IEEE Trans. Automat. Control*, 1992, vol. 37, pp. 1083–1088.

222. Saridis, G.N., *Entropy in Control Engineering*, Singapore: World Scientific, 2001.

223. Vladimirov, I.G., Kurdyukov, A.P., and Semenov, A.V., Stochastic Problem of H_∞-Optmimization, *Dokl. Rossisk. Akad. Nauk*, 1995, vol. 343, no. 5, pp. 607–609.

224. Kurdyukov, A.P. and Maksimov, E.A., Solution of the Stochastic H_∞-optimization Problem for Discrete Time Linear Systems under Parametric Uncertainty, *Automation and Remote Control*, 2006, vol. 67, no. 8, pp. 1283–1310.

225. Petersen, I.R., James, M.R., and Dupuis, P., Minimax Optimal Control of Stochastic Uncertain Systems with Relative Entropy Constraints, *IEEE Trans. Automat. Control*, 2000, vol. 45, pp. 398–412.

226. Fradkov, A.L., Toward a Unique Theory of Control, Computation, and Communication. Paper at the Session of the Scientific Council for the Theory of Controllable Processes and Automation, Russian Academy of Sciences. IPU RAN, April 3, 2008. (in collaboration with Andrievskii, B.R. and Matveev, A.S.). Manuscript at `www.ipme.ru/ipme/labs/ccs/alf/f_apr08.pdf`

227. Hristu, D. and Morgansen, K., Limited Communication Control, *Syst. & Control Lett.*, 1999, vol. 37, no. 4, pp. 193–205.

228. Evans, R., Krishnamurthy, V., Nair, G., and Sciacca, L., Networked Sensor Management and Data Rate Control for Tracking Maneuvering Targets, *IEEE Trans. Signal Proc.*, 2005, vol. 53, no. 6, pp. 1979–1991.

229. La Scala, B.F. and Evans, R.J., Minimum Necessary Data Rates for Accurate Track Fusion, in: *Proc. 44th IEEE Conf. Decision Control Eur. Control Conf. CDC-ECC'05*, 2005, vol. ThIA20, Seville, Spain, IEEE, 2005, pp. 6966–6971.

230. Malyavej, V., Manchester, I.R., and Savkin, A.V., Precision Missile Guidance Using Radar/Multiple-Video Sensor Fusion via Communication Channels with Bit-Rate Constraints, *Automatica*, 2006, vol. 42, pp. 763–769.

231. Ashby, W.R., *An Introduction to Cybernetics*, London: Chapman & Hall, 1956. Translated under the title *Vvedenie v kibernetiku*, Moscow: IL, 1959.

232. Wiener, N., *Cybernetics or Control and Communication in the Animals and the Machines*, New York: MIT Press, 1961. Translated under the title *Kibernetika ili upravlenie i svyaz' in zhivotnom i mashine*, Moscow: Sovetskoe Radio, 1958, 1968, Nauka, 1983.

233. Stephenson, A., On a New Type of Dynamical Stability, *Memoirs & Proc. Manchester Literary Philosophical Soc.*, 1908, vol. 52, no. 8, pp. 1–10.

234. Stephenson, A., On Induced Stability, *Philos. Mag.*, 1909, vol. 17, pp. 765–766.

235. Kapitsa, P.L., Dynamic Stability of Pendulum at Oscillating Suspension Point, *Zh. Eksp. Teor. Fiz.*, 1951, vol. 21, no. 5, pp. 588–597.

236. Bogolyubov, N.N., Theory of Perturbations in Nonlinear Mechanics, in: *Sb. tr. In-ta stroit. mekh. AN USSR* (Collected Papers. Inst. Constr. Mechanics), 1950 (14), pp. 9–34.

237. Blekhman, I.I., *Vibratsionnaya mekhanika* (Vibrational Mechanics), Moscow: Nauka, 1994.

238. Meerkov, S.M., Vibrational Control, *Automation and Remote Control*, 1973, vol. 34, no. 2, Part 1, pp. 201–209.

239. Leonov, G.A. and Shumafov, M.M., *Metody stabilizatsi lineinykh upravlyaemykh sistem* (Methods of Stabilization of Linear Controllable Systems), St. Petersburg: Izd-vo Spbgu, 2005.

240. Lovász, L., On the Shannon Capacity of a Graph, *IEEE Trans. Inform. Theory*, 1979, vol. IT-25, pp. 1–7.

This paper was recommended for publication by A.P. Kurdyukov, a member of the Editorial Board