ISSN 0012-2661, Differential Equations, 2012, Vol. 48, No. 13, pp. 1–21. © Pleiades Publishing, Ltd., 2012.
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CONTROL THEORY

Aircraft Control with Anti-Windup Compensation

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Abstract—We consider an anti-windup compensation method ensuring the convergence of the closed-loop system for a class of master controls. An application of the method to an aircraft flight control problem is shown.

DOI: 10.1134/S001226611213

INTRODUCTION

Under external disturbances, the aircraft control system may, in addition to the desired stable solution corresponding to the desired flight, produce other stable and unstable solutions corresponding to undesired dangerous behavior of the aircraft. In addition, the desired solution may lose its stability for large amplitudes of the input signal, which may lead to catastrophic consequences [1].

There were a number of aircraft crashes caused by incorrect control algorithm synthesis, including the crash of the American advanced fighter *YF*-22 Raptor (manufactured by Lockheed Martin/Boeing), which crashed when landing at the Edwards air base in April 1992 [2], and the Swedish fighter *JAS*39 *Gripen* (manufactured by SAAB) [3]. These catastrophes were caused by incorrect control algorithm synthesis, which was carried out without concerning saturation type nonlinearities, whose influence can result in so-called *pilot induced oscillations* deranging the pilotage [4]. In these crashes, the *pitch flatter* effect was observed during landing (i.e., there occurred pitching oscillations with increasing amplitude).

Note that so-called *hidden oscillations* (whose attraction domain does not contain any neighborhood of equilibria) [5–9], which arise in such systems, dramatically complicate numerical analysis and can lead to erroneous conclusions: "*Since stability in simulations does not imply stability of the physical control system* (*an example is the crash of the YF22*) *stronger theoretical understanding is required*" [10].

There are also well-known cases in which spacecraft entered uncontrolled rotation [11, 12]. The study of transient modes for such damping necessitates developing a mathematical theory of global analysis of attitude systems. The demand for such a theory was indicated by Academician B.V. Raushenbakh, who noted that it is difficult to control a spacecraft during rapid turns.

Undesired situations can be depicted most visually for the case in which a system describes a plant in the form of a "pure" integrator with input saturation closed with a PID negative feedback. Here the control error is integrated by the controller, but for large mismatches it cannot be counteracted because of saturation. This leads to onset of oscillation processes in the system, which correspond to maximum possible input amplitudes for the plant (integrator). In the literature, this phenomenon has been dubbed *windup*. Accordingly, measures for counteracting this phenomenon by introducing additional feedbacks and/or compensators are known as *anti-windup*.

Recall that Keldysh [13] developed mathematical methods for the analysis of various flatter damping systems used in aircraft in the 1940. In the present paper, we develop and modify Keldysh's methods for anti-windup compensation problems.

 (Reg. No. 13-le, 26.1.2012)

The present paper is organized as follows. A brief survey of publications on anti-windup compensation methods is given in Section 2. In Section 3, we present general information on convergent systems and describe the possibility of application of the harmonic balance method for their investigation. An anti-windup compensation method based on the convergence property for neutrally stable plants is described in Section 3. Section 5 deals with its application to the control of the aircraft yaw angle. Concluding remarks are given in Section 6.

2. ANTI-WINDUP COMPENSATION METHODS

Earlier methods for preventing integrator excitation (anti-windup compensation) were mainly heuristic and lacked mathematical rigor. Surveys of these methods can be found in [14, 15], and a list of literature is given in [16]. The modern state of anti-windup compensation methods is reflected in the monographs [17, 18] and the survey [19].

It was noted in [20] that controller synthesis under control constraints can be carried out on the basis of optimal (for example, time- or energy-optimal) control methods [21, 22]. However, even if such solutions, which lead to an on-off (discontinuous) control, can sometimes be specified in a feedback form in principle, the implementation of these methods encounters substantial computational difficulties even for the simplest systems. Therefore, optimal solutions of this kind are not used in most applications; instead, a "nominal" linear controller is synthesized under the no-saturation assumption, and some compensation signal is introduced in it as the saturation becomes "active."

One of the first attempts at the theoretical justification of existing methods for damping the integrator excitation in the controller in the case of control saturation was suggested in [23], where the abbreviation ARW (antireset-windup) was used, because the integral component in PI- and PID-controllers is sometimes referred to as the "reset component" in engineering publications. It was noted in that paper that, by then, the main idea of how to suppress the integrator excitation was to constrain the output signal of the linear controller by using additional feedbacks so as to ensure that the constrained variable (for example, the actuator rod displacement signal) stays within a given range. Nonlinear adders, multipliers, or selectors of minimum and maximum values were used for this purpose in early papers. Despite the broad and fruitful use of such structures in practice, the analysis of properties of the closed-loop system was clearly insufficient from the theoretical viewpoint, leaving space for intuition, designers' experience, modeling, and tweaking adjustment methods. The following problem was aimed at in [23] : use methods of the theory of nonlinear systems to prove the robustness of the system with antireset-windup suggested in [24] in comparison with a nominal linear system in the case of a scalar control.

Anti-windup compensation was studied in [25] for systems with a cascade (two-loop) discrete controller in which constraints are imposed on the magnitude of the control signal produced by the external loop PI-controller (for example, a constraint on the current in the winding of a motor in a velocity control system). It was noted that, as a result of discretization, large values of the compensating feedback factor leads to stability loss. The stability analysis carried out in [25] is based on the Popov criterion for discrete systems [26–28]. A numerical example of a control system with a first-order inertial plant was considered there.

It was suggested in [29] to use an antireset-windup method for systems with several actuators that have saturation, and with a vector control formed by a control law with an integral component. The nonlinear blocks in actuators are assumed to be described by static dependences with upper and lower boundary values that define the saturation levels. Within this range, the outputs of the nonlinear blocks coincide with the input signals. In this antireset-windup scheme, the integrators in the controller are embraced by simple nonlinear insensitivity feedbacks: if the output signal of some integrator lies in the admissible range, then the compensating feedback signal is zero; if the output signal is outside this range, then there appears a negative feedback signal proportional to the output of the integrator. Therefore, for each component of the control, the system acquires the Lur'e form with two nonlinear blocks that have a common input. (The output of the linear part of the system is the output of the integrator unit of the controller.) The linear part of the system has two inputs; one is the control itself (the output of the unit with saturation), and the other is the output of the nonlinear compensating feedback unit. In addition, the system is subjected to a master control formed outside the feedback.

A systematic technique for synthesizing cross-coupled (multi-inputs–multi-output) systems for asymptotically stable plants (if the integral occurs in the control law, then the asymptotic stability means that the linear part of the system is neutrally stable) was presented in [20] in the case of several saturation units. It was noted in this paper that, for the case of a vector control, saturation may lead not only to integrator windup but also to a change in the direction of the vector control signal, which also results in a system operation failure. The method suggested in [20] is based on the introduction of a supervising feedback such that if the master control and the disturbance are sufficiently small, then the control system operates as a "nominal" linear system (synthesized without regard of saturation). In the case of relatively large inputs leading to saturation, the control law is modified so as to ensure stability and, if possible, preserve the characteristics of the nominal linear system. To solve the posed problem, it was suggested in [20] to augment the error signal loop with an "Error Governor" ensuring that, as far as possible, the control signal does not reach saturation for any master controls and disturbances. One example considered in [20] is the control of longitudinal motion of the *F8* aircraft.

Apparently, the paper [30] was the first to study the combined influence of control signal saturation in the level and rate of change. Just as in [20], a systematic technique for synthesizing controllers ensuring the stability and admissible behavior of the closed-loop system was suggested in [30] for cross-coupled linear systems neutrally stable as open-loop systems.

Anti-windup compensation methods with *anti-windup bumpless transfer* were suggested in [31, 32]; in these methods, the nonlinearities in the input signal are taken into account by implementing the following two-step synthesis procedure: first, one designs a linear controller without concerning the nonlinearities at the input of the plant, and then one introduces an anti-windup bumpless transfer correction so as to minimize the harmful influence of the input nonlinearities to the behavior of the closed-loop system. Thus, standard controllers developed for constraint-free systems are adjusted to the constraints. Controller synthesis is based on the*passivity* conception [33] and the *multiplier theory* [34, 35]. For an appropriate choice of multipliers, sufficient stability conditions are reduced to equivalent linear matrix inequalities (LMI). Sector nonlinearities, especially saturation type nonlinearities, are considered in the paper. It is noted that if there is a known bound for the signal fed to the input of the nonlinearity, then one can obtain less conservative conditions by shrinking the sector considered.

The windup problem, which arises for manual piloting of a statically unstable aircraft if there are constraints on the control surface deflection and slew rate, was considered in [36], where one solution was suggested and compared with the optimal solution. A short-periodic longitudinal motion of a tailless aircraft was considered in [37] for arbitrary balancing flight modes. It was shown that the suggested anti-windup compensator admits more vigorous maneuvering than the one provided by the standard *command limiting*. The suggested compensation mechanism guarantees the stability of the piloted aircraft for arbitrary control commands of the pilot and ensures the desired flight performance characteristics to the extent to which they are possible under the given control constraints.

In [38], the authors considered an anti-windup compensation mechanism for linear stationary systems in the case of nonlinear constraints on the control surface deflection and slew rate. A procedure for synthesizing a convex anti-windup control was developed on the basis of an extended version of the circle criterion for linear fractional transformation (LFT) systems, that is, the class of linear time-dependent systems with *fractional* dependence of the parameters. The compensator equations were obtained in closed form, which simplified the implementation. The efficiency of the suggested control method was demonstrated for the linearized model of the aircraft *AF*-8 from [20]. An anti-windup compensation method was suggested in [39] for the case of actuator rate limitation, and a compensator adjustment algorithm was developed to achieve a trade-off between the control performance and the size of the estimated attraction domain of the stable mode. The application of the method was demonstrated for a realistic example of a flight control system for a nonlinear model of a longitudinal and transverse motion of the experimental aircraft *ATTAS* (Advanced Technology Testing Aircraft) used by the German Aerospace Center (Deutsche Zentrum für Luft und Raumfahrt, DLR). The possibility of the use of anti-windup compensation to diminish the sensitivity of aircraft to pilot-induced oscillations (PIO) was illustrated. The results of the research were later justified in a number of test flights. The synthesis and analysis were performed in [40], and the results of test flights for the experimental aircraft *ATTAS* obtained by the German Aerospace Center were represented. Further results can be found in [31], where a comparative analysis of dynamic anti-windup compensators of diminished order was performed to estimate the value of various design parameters. The static anti-windup compensation problem linear unstable aircraft in the case of saturation in the control channel was considered in [42]. Standard approaches based on Lyapunov functions, the S-procedure, and nonlinearities with sector constraints were used. The suggested approach was analyzed from the viewpoint of enlarging the domain of safe initial conditions for which the stability of the closed-loop system can be guaranteed.

A robust anti-windup compensation scheme was suggested in [43] to improve the aircraft lateral control performance, and its efficiency was demonstrated. The problem of the synthesis of an anti-windup controller that takes into account the trade-off between the control performance in the presence and absence of an the actuator saturation nonlinearity was considered in [44]. The results were used for the bank control for the $F8$ aircraft. The case of large parametric uncertainty in an aircraft model with actuator saturation was analyzed in [45, 46]. The authors suggested a robust adaptive linear-quadratic synthesis of a control law with adaptive anti-windup compensation for counteracting the changes in the aircraft parameters in the course of time. It was shown that the airframe follows the trajectory formed by the navigation algorithm despite the presence of large parametric indeterminacies. The anti-windup compensation problem was considered in [47] in discrete time. These results were used for a model control problem for a promising fighter in [48]. A procedure for synthesizing anti-windup compensation for linear systems described by regular transfer functions was suggested in [49] for magnitude and rate constraints in the actuators. Using generalized sector conditions and *linear matrix inequalities* (LMI), the authors suggested a procedure for finding the anti-windup compensator gain ensuring stability for given initial conditions. This approach was illustrated by an example of control of the longitudinal motion of the $F8$ fighter in the longitudinal channel. An anti-windup compensation procedure was suggested in [50] on the basis of the response of a nonlinear system with saturation type nonlinearity to a step input. This procedure was used in the control problem for the longitudinal motion of the M-2000 aircraft. The approach was further developed in [51, 52] to ensure the fastest offset-free tracking of the desired angle of attack with high control performance. For dead-band nonlinearities, the papers [53, 54] suggest a solution of the anti-windup compensation problem with the use of a modified sector condition on the basis of LMI for constructing dynamic compensators of full and reduced order.

In numerous publications, the anti-windup compensation problem is stated as the problem of providing the global asymptotic stability of the equilibrium of a system in the absence of external disturbances. This approach is fundamentally wrong and dangerous in applications. The paper [55] provides an example of a second-order system with a saturation type nonlinearity that satisfies the critical case of the Popov criterion and hence is globally asymptotically stable in the absence of external disturbances. The same system subjected to an external (periodic) disturbance may have a multitude of periodic modes, while the desired low-amplitude mode may lose stability. Similar results for an aircraft course control system can be found in [56]. This observation shows that one needs a sound and rigorous mathematical definition of the anti-windup compensation synthesis problem.

3. CONVERGENT SYSTEMS AND THE HARMONIC BALANCE METHOD

3.1. Statement of the Problem of Anti-Windup Compensation

As is seen from the survey part of the present paper, nowadays there is no unified generally accepted approach to the description of the problem; numerous authors suggest their own definitions and the corresponding solution methods. Let us use a simplified example to outline an alternative approach to the description of the problem.

Let us consider a control system in which the loop error signal is formed as the difference between the master control and the plant (integrator) output. The loop error signal is fed to the input of a linear PI (proportionally integrating) controller and then to the plant input via a saturation type nonlinearity. By using the Popov criterion, one can readily show that, for zero master control, the closed-loop system is globally asymptotically stable if the coefficients of the PI-controller are positive. At the same time, for a harmonic master control, one can use the harmonic linearization method (which is not rigorous in general) to predict the existence of multiple periodic solutions in the closed-loop system.

To prove the existence of multiple periodic solutions in such a system, consider the system of differential equations

$$\dot{x}_1 = -K_i x_2 + K_i w(t),$$
$$\dot{x}_2 = \operatorname{sat}(x_1 - K_p x_2 + K_p w(t))$$

for the parameter values $K_p = 10$ and $K_i = 20$. Here $\operatorname{sat}(\cdot) := \operatorname{sgn}(\cdot)\min\{|\cdot|, 1\}$ is the saturation function. These parameter values ensure the asymptotic stability of the zero equilibrium for the zero input $[w(t) = 0]$. Moreover, an application of the Popov criterion permits one to prove the global stability of this equilibrium. However, a simulation of the system for the harmonic input $w(t) = \sin t$ shows that in the system there exist at least two periodic stable solutions corresponding to the same input. This situation can be illustrated by another classical example, the Duffing oscillator with friction. It is well known that the Duffing oscillator is globally asymptotically stable if the input is zero. At the same time, one can observe multiple periodic and even chaotic solutions under a harmonic input.

These examples show that the approach based on ensuring global asymptotic stability for zero input may prove inadequate for practical use. In this connection, we deem the following statement of the anti-windup compensation problem reasonable: by using additional feedbacks, ensure the Demidovich convergence property [57] in the system. This approach is used in the present paper. In addition, to analyze the control performance in the closed-loop system for a nonzero master control, it is of interest to find the scope of the harmonic linearization method and estimate the error of this method.

3.2. Harmonic Balance Method for Nonautonomous Convergent Systems

In this section, we present some definitions and properties of *uniformly convergent systems* [55, 58–60] to be used below.

Consider systems of the form

$$\dot{x}(t) = f(x, w(t)) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector and $w(t) \in \mathbb{R}^m$ is an input belonging to a subclass $\mathcal{W}$ of the class $\bar{\mathbb{P}C}_m$ of bounded piecewise continuous functions, $w(t) \in \mathcal{W} \subset \bar{\mathbb{P}C}_m$. In addition, we assume that the vector function $f(x, w)$ satisfies some regularity conditions ensuring the existence of local solutions $x(t, t_0, x_0)$ of system (1) for any input $w \in \mathcal{W}$.

Definition 1. System (1) is said to be *uniformly convergent* for the class $\mathcal{W}$ of inputs if for each input $w(t) \in \mathcal{W}$ there exists a solution $\bar{x}(t) = x(t, t_0, \bar{x}_0)$ satisfying the following conditions.

1. $\bar{x}(t)$ is defined and bounded for all $t \in (-\infty, +\infty)$.
2. $\bar{x}(t)$ is globally uniformly asymptotically stable.

Note that the *uniformity* in this definition is understood as the time uniformity; i.e., if a system is uniformly convergent for a class $\mathcal{W} \subset \bar{\mathbb{P}C}_m$ of inputs, then for an arbitrary input $w(t) \in \mathcal{W}$, there exists a unique solution $\bar{x}(t)$ globally asymptotically uniform in time.

The solution $\bar{x}(t)$ is called the *limit solution*. As follows from the definition, every solution of a uniformly convergent system "forgets" its initial conditions and converges to the limit solution. The following assertions characterize some properties of the limit solution.

Property 1. [61]. Consider system (1) for a given input $w(t)$ defined for all $t \in \mathbb{R}$. Let $\mathcal{D} \subset \mathbb{R}^n$ be a compact set positively invariant with respect to the dynamics of system (1). Then there exists at least one solution $\bar{x}(t)$ such that $\bar{x}(t) \in \mathcal{D}$ for all $t \in (-\infty, +\infty)$.

Property 2 [62]. The limit solution $\bar{x}(t)$ of a uniformly convergent system is unique; i.e., it is the unique solution bounded for all $t \in (-\infty, +\infty)$.

Property 3 [63]. If system (1) is uniformly convergent, then, for a constant input signal $w(t) \equiv w$, the corresponding limit solution $\bar{x}(t)$ is constant as well, $\bar{x}(t) \equiv \bar{x}$. If the input $w(t)$ is a T-periodic function, then the corresponding limit solution $\bar{x}(t)$ is a T-periodic function as well.

Unlike nonlinear systems of general form, convergent nonlinear systems have the important property that they can be studied almost in the same way as linear systems. Indeed, while the study of general nonlinear systems is complicated by the possible occurrence of various steady-state processes depending on the initial state of the system, convergent systems have a unique limit solution for a given input, and hence their limit behavior can be uniquely determined. In particular, this permits one to analyze such systems in the frequency domain on the basis of the behavior of the limit solution under a harmonic input [64, 65]. In addition, since the limit solution of a convergent system depends only on the input and is independent of the initial conditions, we find that it can be numerically found by the *modeling the system* for fixed initial conditions, while for general nonlinear systems it is necessary to analyze the behavior for infinitely many initial states. These properties of convergent systems are used below to obtain their sensitivity functions and when analyzing systems with *saturation*.

The analysis of properties of linear control systems is substantially based on the derivation of frequency characteristics like the *sensitivity function* $\mathcal{S}(i\omega)$ and the *additional sensitivity function* $\mathcal{T}(i\omega)$ [66, 67]. Generalized modifications of these functions were introduced in [68] for nonlinear systems in Lur'e form with the convergence property. Let us present the main assertions.

Consider a nonlinear system of the form

$$\dot{x}(t) = f(x, r), \qquad y = h(x, r), \qquad e(t) = r(t) - y(t), \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, the scalar functions $y(t)$ are the output of the system, $r(t)$ is the master control, $e(t)$ is the shadowing error, $f(x, r) \in \mathbb{R}^{(n+1)\times n}$ is the system vector function, and $h(x, r) \in \mathbb{R}^{(n+1)\times 1}$ is the output function. Consider harmonic master controls $r(t) = a\sin(\omega t)$ with amplitude a and frequency ω. Suppose that system (2) is *uniformly convergent* for inputs of this class. Then for each a and ω, there exists a unique steady-state T-periodic solution $\bar{x}(t)$ of system (2) with period $T = 2\pi/\omega$ and the corresponding output process $\bar{y}(t)$ and error $\bar{e}(t) = r(t) - \bar{y}(t)$ [55, 69].

Definition 2 [68]. The functions

$$\mathcal{S}(a, \omega) = \|\bar{e}\|_2 / \|r\|_2, \qquad \mathcal{T}(a, \omega) = \|\bar{y}\|_2 / \|r\|_2,$$

where $\|z\|_2 = \left((\omega/2\pi)\int_0^{2\pi/\omega} z(\tau)^2\, d\tau\right)^{1/2}$, are called the *generalized sensitivity function* and *generalized additional sensitivity function*, respectively, of the convergent system (2).

For linear systems, the functions $\mathcal{S}(a, \omega)$ and $\mathcal{T}(a, \omega)$ coincide with the ordinary amplitude-frequency characteristics $|S(i\omega)|$ and $|T(i\omega)|$, respectively. For nonlinear systems, the functions $\mathcal{S}(a, \omega)$ and $\mathcal{T}(a, \omega)$ depend on the frequency ω of the input as well as its amplitude a. The generalized sensitivity functions can be obtained numerically by integrating system (2) $N_a \cdot N_\omega$ times for the harmonic input $r(t)$ with various amplitudes a_i, $i = 1, 2, \ldots, N_a$ and frequencies ω_j, $j = 1, 2, \ldots, N_\omega$. The necessary amount of computations can be reduced dramatically with the use of the *harmonic balance method* for nonautonomous convergent systems that have the Lur'e form [65] (a linear dynamical subsystem closed by a static nonlinearity; see[66, 70]). Let us present the main assertions of the method.

Suppose that system (2) is represented in the Lur'e form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\psi(y) + Fw(t), \\ y(t) &= Cx(t) + Dw(t), \qquad z(t) = Hx(t), \end{aligned} \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t)$ is the scalar input, $y(t)$ is the scalar output of the system, $\psi(y)$ is a scalar continuous function, and A, B, C, D, F, and H are matrices of appropriate sizes.

In addition, it is assumed that the function $\psi(\cdot)$ satisfies the following *sectorial growth condition* for some finite $\mu > 0$:

$$0 \le \frac{\psi(y_1) - \psi(y_2)}{y_1 - y_2} \le \mu \quad \text{for all} \quad y_1, y_2, \quad y_1 \ne y_2. \tag{4}$$

Let us study the behavior of system (4) for the harmonic input $w(t) = b\sin(\omega t)$ with amplitude b and frequency ω. By [68, 71], in the steady-state (after the termination of transient processes), the output $y(t)$ can be approximately represented by a harmonic process with frequency ω and amplitude a satisfying the *harmonic balance equation*

$$|1 - K(a)G(i\omega)|^2 a^2 = |H(i\omega)|^2 b^2, \tag{5}$$

where $G(i\omega) = C(i\omega I_n - A)^{-1}B$, $H(i\omega) = C(i\omega I_n - A)^{-1}F + D$, $K(a)$ is the *harmonic linearization coefficient* (or the *describing function*) of the nonlinearity $\psi(\cdot)$ and I_n is the $n \times n$ identity matrix.

For an odd nonlinearity $\psi(x)$, the harmonic linearization coefficient $K(a)$ is given by the expression (see also [66])

$$K(a) = \frac{2}{\pi a} \int_0^\pi \psi(a \sin\vartheta) \sin\vartheta \, d\vartheta. \tag{6}$$

3.3. Accuracy of the Method

An estimate for the accuracy of the analysis by the generalized harmonic linearization method was obtained in [68], where bounds were found for the $\mathcal{L}_2$-norm of the difference between the outputs of a nonlinear system and a model obtained by the harmonic linearization of the system under a harmonic external input. The main result in [68] is stated in the following theorem.

Theorem 1. *Consider system* (3) *with periodic input* $w(t) = b\sin(\omega t)$*. Assume that the following conditions are satisfied.*

1. *The pair* (A, B) *is completely controllable*, *and the pair* (A, C) *is completely observable.*
2. *The harmonic balance equation* (5) *has a unique positive real solution* $a(b, \omega)$*.*
3. *The frequency condition* $\operatorname{Re} G(ik\omega) < 1$ *holds for all* $k = \pm 1, \pm 3, \pm 5, \ldots$
4. $\psi(\cdot)$ *is an odd function.*

Then system (3) *has a unique half-period-symmetric* $2\pi/\omega$*-periodic solution*, *and the* $\mathcal{L}_2$*-norm of the difference between the outputs of the nonlinear system and its model linearized by the harmonic balance method is bounded by* $\gamma\nu(a(b, \omega))$, *where*

$$\begin{aligned} \gamma &= 2\varrho_2/(2 - \mu\varrho_1), \\ \varrho_1 &= \sup_{k=\pm1,\pm3,\pm5,\ldots} \left| C \left(ik\omega I - A - \frac{\mu}{2} BC \right)^{-1} B \right| \\ \varrho_2 &= \sup_{k=\pm1,\pm3,\pm5,\ldots} \left| H \left(ik\omega I - A - \frac{\mu}{2} BC \right)^{-1} B \right|, \\ \nu(a) &= \sqrt{\frac{1}{2\pi} \int_0^{2\pi} (aK(a) \cdot \sin\vartheta - \psi(a \sin\vartheta))^2 \, d\vartheta}. \end{aligned} \tag{7}$$

Here $\psi(\cdot)$ *stands for the nonlinearity in the feedback of the system in its Lur'e form*, *and* $K(a)$ *is the harmonic linearization coefficient of the function* $\psi(\cdot)$.

3.4. Harmonic Linearization for Systems with Saturation

For the symmetric saturation type nonlinearity $\psi(y) = \mathrm{sat}_m(y)$, where

$$\mathrm{sat}_m(z) = \begin{cases} b & \text{for} \quad z \geq m \\ a & \text{for} \quad z \leq -m \\ z & \text{otherwise} \end{cases} \qquad (m > 0), \tag{8}$$

the expression (6) acquires the form

$$K(a) = \begin{cases} 1 & \text{for} \quad a \leq m \\ \dfrac{2}{\pi}\left(\arcsin\left(\dfrac{m}{a}\right) + \dfrac{m}{a}\sqrt{1 - \dfrac{m^2}{a^2}}\right) & \text{for} \quad a > m. \end{cases} \tag{9}$$

Let us estimate the accuracy of the method by evaluating the function $\nu(a)$ in closed form.

If $a \leq m$, then $K(a) = 1$, and obviously $\nu(a) = 0$. Now consider the integrand in the expression (7),

$$\chi(a, \vartheta) = (aK(a) \cdot \sin\vartheta - \psi(a \sin\vartheta))^2.$$

One can readily show that if $a > m$, then

$$\int_0^{2\pi} \chi(a, \vartheta)\, d\vartheta = 4\left(\int_0^{\alpha} \chi(a, \vartheta)\, d\vartheta + \int_{\alpha}^{\pi/2} \chi(a, \vartheta)\, d\vartheta\right), \tag{10}$$

where $\alpha = \arcsin\dfrac{m}{a}$. By integrating by parts in (10), we obtain

$$\int_0^{2\pi} \chi(a, \vartheta)\, d\vartheta = 2\alpha a^2(1 - 2K(a)) - 2m(1 + 2K(a))\sqrt{a^2 - m^2} + K(a)^2 a^2 \pi + 2m^2(\pi - 2\alpha), \tag{11}$$

where the harmonic linearization coefficient $K(a)$ is given by (9). Therefore, the function $\nu(a)$ can be evaluated by the substitution of (11) into (7).

Note that for the *on-off* nonlinearity $\psi(x) = m\,\mathrm{sgn}(x)$, the value of ν is independent of a, $\nu(a) \equiv \nu^*$, because the corresponding harmonic linearization coefficient has the form $K(a) = \dfrac{4m}{\pi a}$.

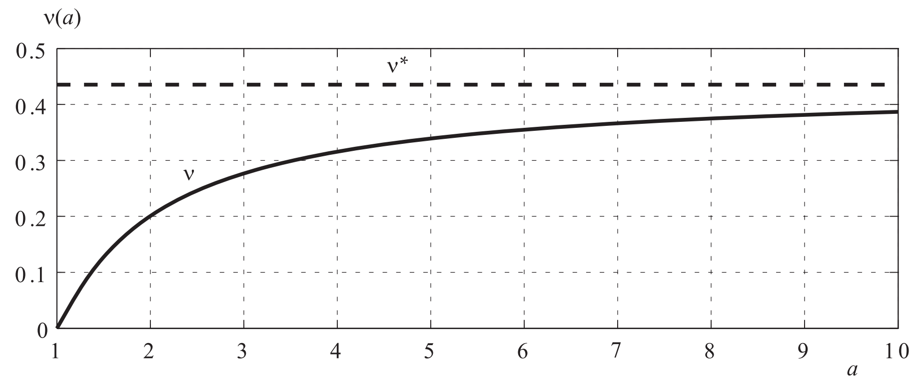

Fig. 1. The graphs of $\nu(a)$ and ν^* for the nonlinearity $y = \mathrm{sat}_m(a)$ with $m = 1$.

The substitution gives

$$\nu^* = m\frac{\sqrt{\pi^2 - 8}}{\pi} \approx 0.435\, m.$$

In addition, for the *saturation* type nonlinearity (7) and (11), the function $\nu(a)$ tends asymptotically to ν^* as $a \to \infty$, because the relative value of the difference between the values of the on-off function and the saturation function decreases with increasing a.

The graphs of $\nu(a)$ and ν^* for $m = 1$ and $a \in [1, 10]$ are shown in Fig. 1.

4. ANTI-WINDUP COMPENSATION ON THE BASIS OF THE CONVERGENCE PROPERTY

4.1. Uniform Convergence of Neutrally Stable Lur'e Systems with Saturation

Consider the following systems in Lur'e form with a saturation type nonlinearity:

$$\dot{x} = Ax + B\,\mathrm{sat}(u) + Fw, \qquad u = Cx + Dw, \qquad y = Hx, \tag{12}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}$ is the control input, $w \in \mathbb{R}^m$ is the external input (for example, a reference signal or a disturbance), and $y \in \mathbb{R}^p$ is the output of the system. The *saturation function* $\mathrm{sat}(\cdot)$ is defined as $\mathrm{sat}(u) = \mathrm{sgn}(u)\min(1, |u|)$. We assume that A is a *neutrally stable* matrix; i.e., there exists a symmetric positive definite matrix $P = P^{\mathrm{T}} > 0$ satisfying the Lyapunov matrix inequality $PA + A^{\mathrm{T}}P \le 0$, and moreover, the matrix A has at least one zero eigenvalue.

Without loss of generality, we assume that if the state vector x is represented in the form $x = [x_1, x_2]^{\mathrm{T}}$, where $x_1 \in \mathbb{R}^1$ and $x_2 \in \mathbb{R}^{n-1}$, then the matrices A, B, and F in (12) have the block structure

$$A = \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \tag{13}$$

where, obviously, B_1 is a scalar. If the original system (12) does not satisfy condition (13), then, by virtue of the above-mentioned properties of the matrix A, there always exists a similarity transformation reducing (12) to the form (13). Note that the convergence property is invariant under the choice of a basis and hence is preserved under similarity transformations.

In what follows, we assume that the input $w(t)$ belongs to the class $\mathcal{W}$ defined as follows.

Definition 3. We say that a continuous function $t \mapsto w(t)$, $w(t) \in \mathbb{R}^m$, *belongs to the class* $\mathcal{W}$ if it is bounded and satisfies the following conditions.

1. The function $Dw(t)$ is uniformly continuous for all $t \in \mathbb{R}$.
2. The inequality $|F_1 w(t)| \le \alpha_1 |B_1|$ holds for all $t \in \mathbb{R}$ with some constant $\alpha_1 < 1$.

The following assertion is true.

Theorem 2. *If there exists a symmetric positive definite Lyapunov matrix* $P = P^{\mathrm{T}} > 0$ *such that*

$$PA + A^{\mathrm{T}}P \le 0, \tag{14}$$

$$P(A + BC) + (A + BC)^{\mathrm{T}}P < 0, \tag{15}$$

then system (12) *is uniformly convergent for any function* $w \in \mathcal{W}$.

If there exists a symmetric positive definite Lyapunov matrix $P = P^{\mathrm{T}} > 0$ *such that the strict inequality* $PA + A^{\mathrm{T}}P < 0$ *holds* [*instead of* (14)] *and, in addition,* $P(A+BC) + (A+BC)^{\mathrm{T}}P < 0$, *then one can show that the corresponding system is quadratically convergent. However, the systems considered below are neutrally stable; therefore, the inequality* $PA + A^{\mathrm{T}}P < 0$ *cannot be achieved.*

Remark 1. As follows from the assumptions of the theorem, the derivative of the quadratic function $V(e) = e^{\mathrm{T}}Pe$, where $e \in \mathbb{R}^n$ is the difference between two arbitrary solutions of the system, is nonpositive. Moreover, the derivative is zero if two solutions coincide and/or lie simultaneously in the saturation zone. In the case of a periodic input, the proof of the theorem can be carried out directly by generalizing the LaSalle principle. Although this is not a goal of the present paper, one can formally show that the assumption imposed on the input influence in the periodicity condition can be weakened.

Remark 2. As follows from the Yakubovich–Kalman lemma [61, 72, 73], if the *frequency inequality*

$$\operatorname{Re} W(i\omega) < 1 \tag{16}$$

holds for all $\omega \neq 0$, where $W(s) = C(sI - A)^{-1}B$, then there exists a positive definite matrix P satisfying conditions (14) and (15).

4.2. Uniform Convergence of Systems with Anti-Windup Compensation for Neutrally Stable Plants

Consider the following model of dynamics of the plant:

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p\left(\operatorname{sat}(u) + w_1\right), \\ y_p &= C_p x_p, \end{aligned} \tag{17}$$

where the matrix A_p is neutrally stable. (Some of its eigenvalues are simple and have zero real part, while the remaining eigenvalues have negative real part.) Let the controller dynamics be given by the equations

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c(w_2 - y_p) + k_A\left(\operatorname{sat}(u) - u\right), \\ u &= C_c x_c + D_c(w_2 - y_p), \end{aligned} \tag{18}$$

where k_A is the anti-windup compensation coefficient.

The equations of the closed-loop system (17), (18) can be reduced to the Lur'e form (12), where $x = [\ x_p,\ x_c\ ]^{\mathrm{T}} \in \mathbb{R}^n$, $w = [\ w_1,\ w_2\ ]^{\mathrm{T}} \in \mathbb{R}^m$, and

$$A = \begin{bmatrix} A_p & 0 \\ k_A D_c C_p - B_c C_p & A_c - k_A C_c \end{bmatrix},$$

$$B = \begin{bmatrix} B_p \\ k_A \end{bmatrix}, \quad F = \begin{bmatrix} B_p & 0 \\ 0 & B_c - k_A D_c \end{bmatrix},$$

$$C = \begin{bmatrix} -D_c C_p & C_c \end{bmatrix}, \quad D = \begin{bmatrix} 0 & D_c \end{bmatrix}, \quad H = \begin{bmatrix} C_p & 0 \end{bmatrix}.$$

To ensure the uniform convergence of this system, we use Theorem 2; this is done in Section 5 for the problem of yaw angle control of an aircraft. In particular, the procedure of choosing the coefficient k_A providing the convergence of the system is presented there.

5. APPLICATION TO THE PROBLEM OF YAW ANGLE CONTROL OF AN AIRCRAFT

Let us apply the method described in Section 4 and based on the convergence method to the problem of yaw angle control of an asircraft.

5.1. Control of the Aircraft Yaw Angle with a Static Anti-Windup Compensation

Consider the problem in which the aircraft yaw angle $\psi(t)$ should shadow the master control $\psi^*(t)$ and the goal is to ensure that the system is offset-free in the first-order, i.e., that it has no offset with respect to the component of $\psi^*(t)$ varying linearly in time. To ensure this, we use the PID-controller

$$u_r(t) = k_D\omega_y(t) - k_P\Delta\psi(t) - k_I\int\limits_0^t \Delta\psi(t)\,dt, \tag{19}$$

where $u(t)$ is the control fed to the rudder driver, $\Delta\psi = \psi^*(t) - \psi(t)$ is the shadowing error, and $\omega_y(t)$ and $\psi(t)$ are the angular velocity and the yaw angle, respectively. The parameters k_D, k_P, and k_I are the *differential*, *proportional*, and *integral* transfer ratios of the autopilot, which are chosen when synthesizing the control law. The control signal $u_r(t)$ fed to the rudder servo is limited owing to the constraints for the slideslip angle, the lateral acceleration, and mechanical characteristics of the rudder by some number $\bar{u}_r$; i.e., the following condition should be satisfied:

$$-\bar{u}_r \le u_r(t) \le \bar{u}_r. \tag{20}$$

To satisfy condition (20), we introduce a saturation type nonlinearity in the command signal σ_r of the rudder servo δ_r as

$$\sigma_r = \mathrm{sat}_{\bar{u}}(u), \tag{21}$$

where $\mathrm{sat}_{\bar{u}}(u) = \{\bar{u}\,\mathrm{sgn}(u) \text{ for } |u| > \bar{u} \mid u \text{ for } |u| \le \bar{u}\}$.[1]

Equations (19) and (21) describe a PID-controller with rudder servo input saturation. (To simplify the exposition, we assume that the static transfer ratio of the rudder servo is equal to unity.)

5.2. Model of the Dynamics of Aircraft Course Motion

By way of example, consider a linearized model of lateral angular motion of a hypothetical aircraft [74, 75]. Following [56], we assume that the bank angle γ is stabilized by a rapidly acting channel of the autopilot (i.e., course turn is performed with a bank-free sliding). Therefore, we use the model of isolated course motion, which is obtained from the model described in [75] if we take $\gamma(t) \equiv 0$. By omitting external disturbances, we obtain the following equations of the aircraft dynamics:

$$\begin{cases} \dot{\beta}(t) = a_z^{\beta}\beta(t) + \cos\alpha^* \cdot \omega_y(t) + a_z^{\delta_r}\delta_r(t), \\ \dot{\omega}_y(t) = a_{m_y}^{\beta}\beta(t) + a_{m_y}^{\omega_y}\omega_y(t) + a_{m_y}^{\delta_r}\delta_r(t), \\ \dot{\psi}(t) = \omega_y(t), \end{cases} \tag{22}$$

where $\beta(t)$ is the aircraft yaw angle; $\psi(t)$ and $\omega_y(t)$ is the angle and angular velocity of hunting; α^* is the reference value of the angle of attack corresponding to the considered flight mode; a_z^{β}, $a_z^{\delta_r}$, $a_{m_y}^{\beta}$, $a_{m_y}^{\omega_y}$, and $a_{m_y}^{\delta_r}$ are the coefficients of the linearized model of the aircraft. Next we use the following numerical values of the coefficients (in SI units) [75]: $a_z^{\beta} = -0.152$, $a_z^{\delta_r} = -0.032$, $a_{m_y}^{\beta} = -1.757$, $a_{m_y}^{\omega_y} = -0.136$, $a_{m_y}^{\delta_r} = -1.46$, $\alpha = 0.4363$ rad (25°), whence we obtain $\cos\alpha^* = 0.906$.

The rudder servo can be modelled with sufficient accuracy by the second-order system

$$\begin{cases} \dot{\delta}_r(t) = \omega_r(t), \\ \dot{\omega}_r(t) = k_\sigma(\sigma_r(t) - \delta_r) - k_2\omega_r(t), \end{cases} \tag{23}$$

where $\delta_r(t)$ and $\omega_r(t)$ are the rudder deflection angle and rate, respectively,, and $k_{\sigma 1}$ and k_ω are the transfer ratios chosen in the servo synthesis. Next, we take $k_\sigma = 67.2\ \mathrm{s}^{-2}$ and $k_2 = 11.5\ \mathrm{s}^{-1}$, which corresponds to the driver transfer function

$$W_{\delta_r}^{\sigma_r}(s) = \frac{1}{T_r^2 s^2 + 2\xi_r T_r s + 1},$$

where $T_r = k_\sigma^{-\frac{1}{2}} = 0.122$ s is a time constant, $\xi_r = 0.5k_\sigma^{-\frac{1}{2}}k_\omega = 0.7$ is a damping constant, and $s \in \mathbf{C}$ is the argument of the Laplace transform.

[1] Rigorously speaking, the validity of (21) does not provide the validity of (20) at all times for an arbitrary input signal owing to the possible overshoot in the rudder drive. However, the overshoot is usually small in servo drives and can be neglected in the context of the considered problem. One can also take a liberal value $\bar{u}_r$. In addition, control surface actuators have constructive restrictions on the shaft displacement, for example, in the form of a screw-nut pair with stops or terminal switches, which cat the energy supply in the extreme positions.

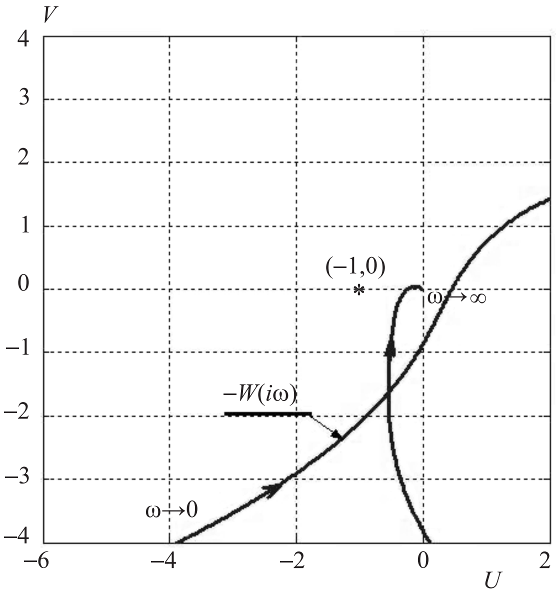

Fig. 2. The Nyquist curve of the open-loop control system of an aircraft. The control law.

Therefore, the transfer function from the control signal σ_r to the hunting angle ψ for the "extended" plant (22), (23) acquires the form

$$W_\psi^{\sigma_r}(s) = \frac{\psi(s)}{\sigma_r(s)} = \frac{-98.2(s+0.113)}{s(s^2+0.288s+1.61)(s^2+11.5s+67.2)}. \tag{24}$$

5.3. Analysis of the Aircraft Course Control System

5.3.1. Synthesis of the nominal controller. Consider the "nominal" (saturation-free) mode for which $\sigma_r^*(t) \equiv u(t)$, and the closed-loop aircraft control system is described by the linear equations (19), (22). [Note that, in such a mode, the anti-windup compensation component of the control signal $u_r(t) - \sigma_r(t)$ in (26) is zero; therefore, by (19) and (26), the output signals coincide.] By using the frequency stability criterion for the linear system (19), (22) and the Nelder–Nead search optimization procedure in the Matlab software package [76], we obtain the following parameters of the PID-controller (19): $k_I = 0.46\ \mathrm{s}^{-1}$, $k_P = 0.37$, and $k_D = 1.8$ s. For such coefficients, the transfer function of the open system from the servo input signal $\sigma_r(t)$ to the output $u(t)$ of the PID-controller (19) has the form

$$W_u^{\sigma_r}(s) = \frac{-173(s+0.113)(s^2+0.208s+0.261)}{s^2(s^2+0.288s+1.61)(s^2+11.5s+67.2)}, \tag{25}$$

which corresponds to the gain stability margin $G_m = 2.8$ dB, the phase stability margin $\phi_m = 60°$, and the index of oscillation (the value of the H_∞-norm) $M = 1.26$ in the closed-loop system. The Nyquist curve $W_u^{\sigma_r}(i\omega)$ and the transfer function of the closed-loop system are shown in Figs. 2 and 3.[2]

[2] The frequency characteristic $W_u^{\sigma_r}(i\omega)$ is shown in Fig. 2 with the opposite sign, because the gain factor of the transfer function (25) for an aircraft of standard aerodynamic scheme is negative for high frequencies.

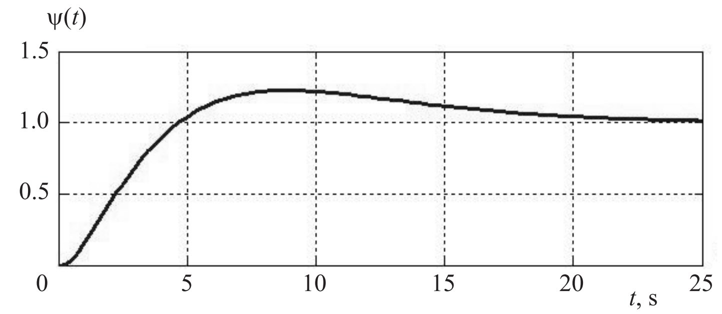

Fig. 3. The transfer function of the closed-loop "nominal" control system of an aircraft. The control law (19), $\sigma_r(t) \equiv u_r(t)$.

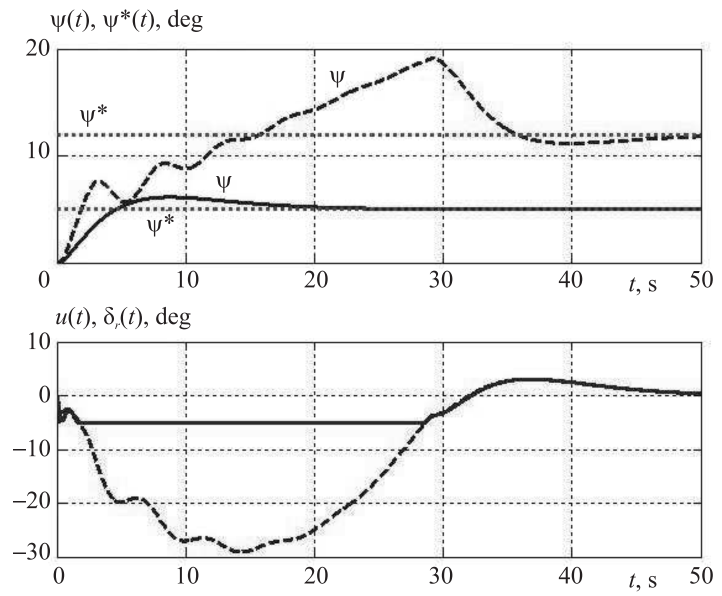

Fig. 4. Transient processes in the aircraft control system for a step master control $\psi^* = 5°$ (solid line) and $\psi^* = 12°$ (dashed line) and the graphs of the output signal of the PID-controller $u_r(t)$ (solid line) and rudder deflections$\delta_r(t)$ (dashed line) for $\psi^* = 12°$.

5.3.2. Influence of saturation. Let us now take into account the saturation of the input signal in the feedback circuit. The plant is neutrally stable; therefore, the open linear system including the PID-controller is unstable owing to the presence of zero poles of multiplicity 2 in (25). Consequently, the frequency inequality (16) cannot hold, and the convergence of system (19), (21), (22) cannot be justified on the basis of Theorem 2. The failure of inequality (16) for the considered numerical example can be seen in the Nyquist curve for $-W(i\omega)$ shown in Fig. 2. For these parameters of the aircraft, inequality (16) fails for all $\omega \le 0.19\ \mathrm{s}^{-1}$.

As was mentioned above, the presence of saturation in a control contour with integral component can result in a fall of the contour performance until stability loss; moreover, this effect is not necessarily observed in the analysis of the motion of the isolated system (under the action of initial conditions alone) and appears only for specific forms of the input and even for specific combinations of inputs and initial conditions. Let us analyze this property in more detail for the considered aircraft control system.

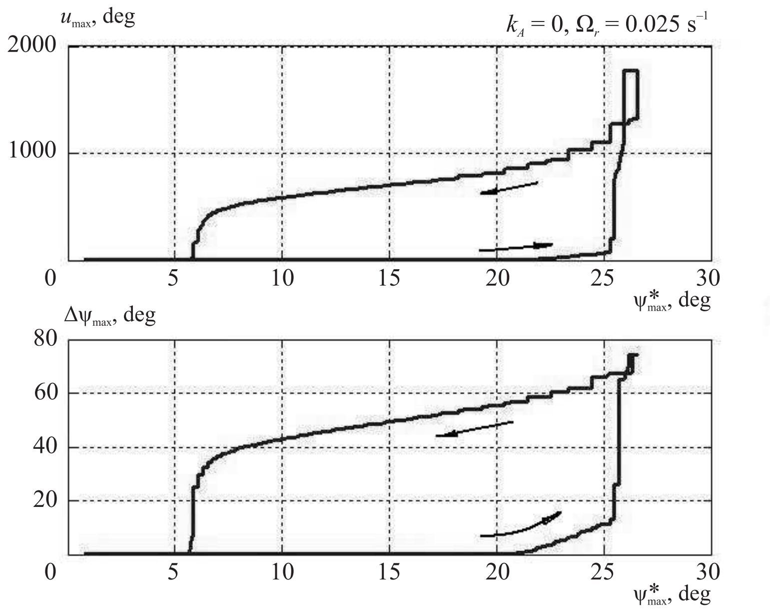

Fig. 5. Dependences of the amplitudes of the control signal u_r and the shadowing error $\Delta\psi(t)$ on the master control amplitude in the system for the control law (19), (21).

Figure 4 presents the transfer processes in the closed-loop aircraft control system for a step master control ψ^* of distinct levels, 5° and 12°. [Note that, owing to the neutral stability of the aircraft, these processes are equivalent to the motions of the isolated system for the corresponding $\psi(0)$.] As one can see in the figures, there is no saturation for $\psi^* = 5°$, and the process has the form of the transfer function of the nominal system shown in Fig. 3. For $\psi^* = 12°$, the process performance is substantially deteriorated as a result of saturation. Although the equilibrium remains asymptotically stable, the overshoot achieves the level of 60%, and the transient process duration is doubled. In this figure, one can also see that the amplitude of the control signal $u_r(t)$ at the output of the PID-controller (19) is several times larger than the servo saturation level $\bar{u}$.

Now consider the response of the system to a harmonic input of the form $\psi^*(t) = \psi^*_{\max}\sin(\Omega t)$. Let us vary the value $\psi^*_{\max} = 0 \div 30°$ in the growth and decay direction in the range $\psi^*_{\max} = 0 \div 26°$, we take $\Omega = 0.025$ s^{-1}. The results of computations are shown in Fig. 5 in the form of the dependences of the control signal amplitude u_r and the shadowing error $\Delta\psi(t)$ on the master control amplitude.[3] In these graphs, one can see that as $\psi^*_{\max}$ grows, the amplitudes $u_{\max}$ and $\Delta\psi_{\max}$ gradually increase until some jump appears. Then, as $\psi^*_{\max}$ decreases, the amplitudes $u_{\max}$ and $\Delta\psi_{\max}$ monotonically decrease until a second jump. A pronounced hysteresis shape of the dependences $u_{\max}$ and $\Delta\psi_{\max}$ is clear in Fig. 5. A similar result was obtained in [77] for the control of a brushless direct current motor. Here one can see that the windup can destruct the stability of systems under the action of an input signal: the steady-state processes with the same input can be different. This effect is illustrated in Fig. 6, which represents the graphs of $\psi(t)$ and $u(t)$ obtained for $\psi^*(t) = \psi^*_0 + \psi^*_{\max}\sin(\Omega t)$, $\psi^*_0 = 11°$, $\psi^*_{\max} = 25°$, $\Omega = 0.01$ s^{-1}, and the initial conditions $\psi(0) \in [-40, 40]°$. (The remaining initial conditions were zero.) The appearance of distinct steady-state processes in the system for the same master control is shown in the graphs. In addition, the main part of processes converges to a steady-state motion for which the saturation is not achieved, and the amplitude of the shadowing error is small (fractions of a degree of angle), but there are individual implementations with the "failure" of the shadowing in which the amplitude error achieves a level of 100 degrees. It is difficult to reveal the possibility of such processes even by intensive computer simulation; however, they can arise in practice.

[3] In the computations, we take a modeling range of $10^4\pi$ s. On the first half of this interval, the value $\psi^*_{\max}$ is defined

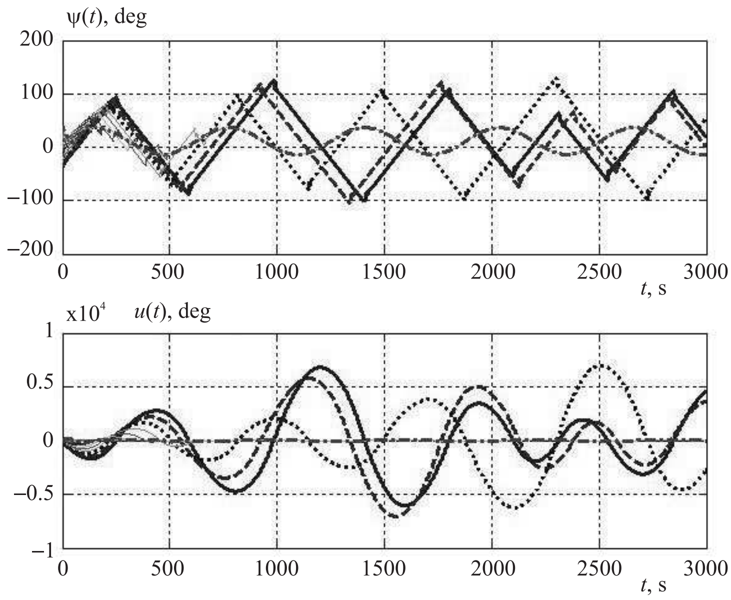

Fig. 6. The set of processes $\psi(t)$ and $u_r(t)$ for the control law (19), (21) for $\psi^*(t) = \psi_0^* + \psi_{\max}^* \sin(\Omega t)$ and for various $\psi(0)$.

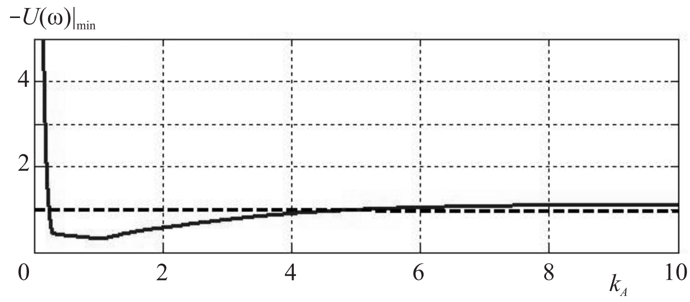

Fig. 7. The dependence $\min(-U(\omega))$ for the linear part of system (21)–(23), (26) on the coefficient k_A.

5.3.3. Aircraft control systems with anti-windup compensation. Now consider the following control law with anti-windup compensation:

$$u_r(t) = k_D\omega_y(t) - k_P\Delta\psi(t) - k_I \int\limits_0^t (\Delta\psi(\tau) + k_A(u(\tau) - \sigma_r(\tau)))\, d\tau, \qquad (26)$$

where k_A is the anti-windup compensation coefficient chosen in the synthesis and the remaining parameters are the same as in (19) and (21).

Now, instead of the control law (19), we use the control law (21), (26) with anti-windup compensation. In Fig. 7, one can see that inequality (16) holds for $0.24 \le k_A \le 5.0$. Consequently, by Theorem 2, system (26), (21), (22) is convergent for $k_A \in [0.24, 5.0]$. The transfer function of the

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to be linearly increasing from 0 to 26°, and on the second half it is defined to be decreasing.

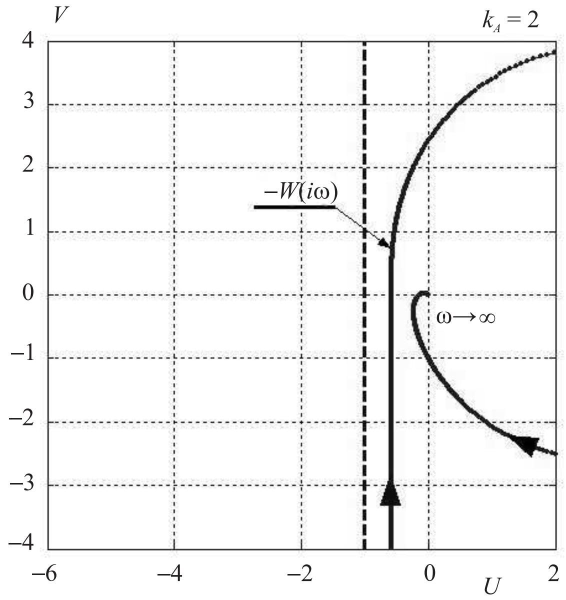

Fig. 8. The Nyquist curve for the aircraft control system with the anti-windup compensation for the control law (21), (26) and for $k_A = 2$.

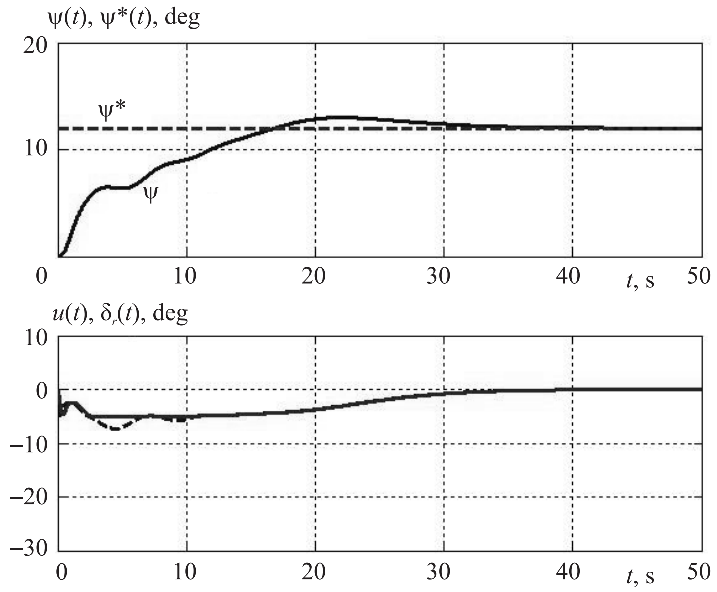

Fig. 9. Transient processes in aircraft control systems for a step master control $\psi^* = 12°$ and the graphs of the output signal of the PID-controller $u_r(t)$ (solid line) and the rudder deflection $\delta_r(t)$ (dashed line) for the control law (21), (26) and for $k_A = 2$.

linear part of the system with regard of anti-windup compensation for $k_A = 2$ has the form

$$W_u^{\sigma_r}(s) = \frac{-111(s+0.90)(s-0.44)(s-0.11)}{s(s+0.92)(s^2+0.29s+1.6)(s^2+11.5s+67.2)}.$$

The corresponding Nyquist curve is shown in Fig. 8.

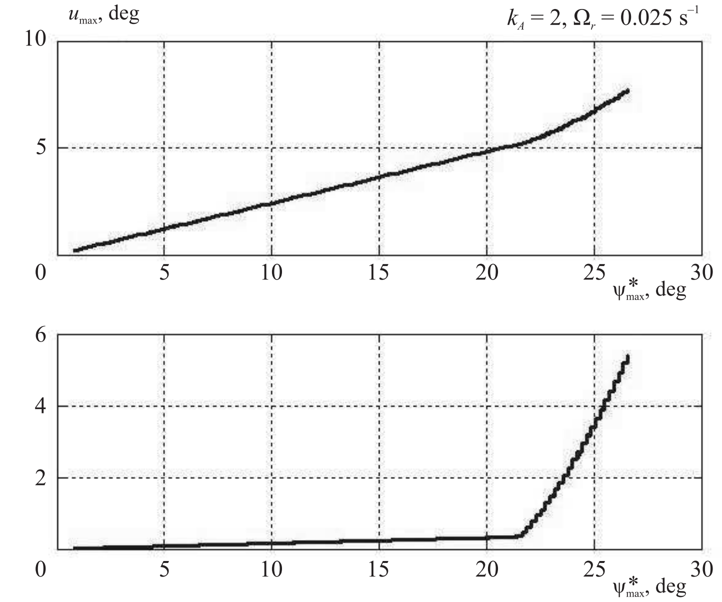

Fig. 10 Dependences of the amplitudes of the control signal u_r and the shadowing error $\Delta\psi(t)$ on the master control amplitude in the system for the control law (21), (26) and for $k_A = 2$.

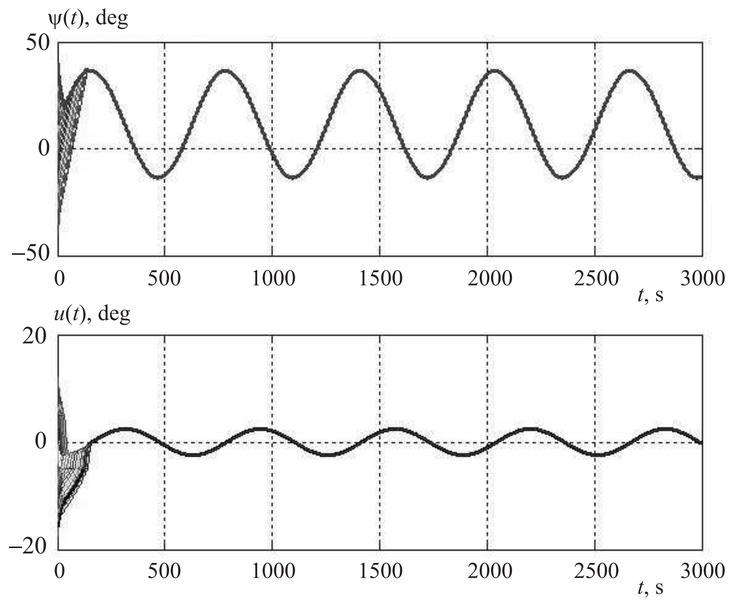

Fig. 11. The processes $\psi(t)$ and $u_r(t)$. The control law (21), (26) ($k_A = 2$) for $\psi^*(t) = \psi^*_0 + \psi^*_{\max}\sin(\Omega t)$ and for various $\psi(0)$.

We simulate the aircraft control system with anti-windup compensation for $k_A = 2$ for the same inputs and initial conditions that have been used above for the control system without anti-windup compensation. The results are presented in Figs. 9–11.

The transient processes in the aircraft control system for the constant master control $\psi^* = 12°$ and the graphs of the output signal of the PID-controller $u_r(t)$ and the rudder deflection $\delta_r(t)$ are shown in Fig. 9. By comparing the obtained graphs with those shown in Fig. 4 for $\psi^* = 12°$, we observe an essential improvement of the process performance. Needless to say, the saturation of the control signal leads to some deceleration of control processes compared with the "nominal"

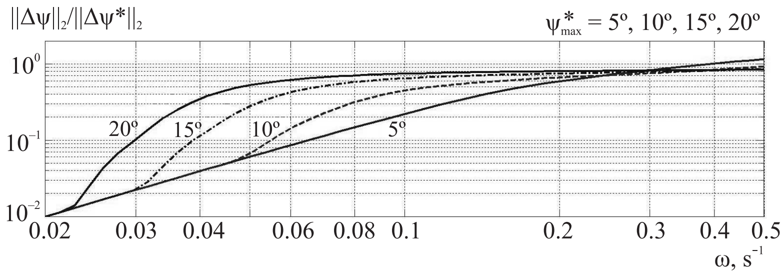

Fig. 12. The sensitivity function $\mathcal{S}(\psi^*_{\max}, \omega) = \|\Delta\psi\|_2/\|\psi^*\|_2$ for the control law (21), (26) with $k_A = 2$ and with $\psi^*_{\max} = 5° \div 25°$.

aircraft control system" (cf. Fig. 3); however, this deterioration does not have a fatal level like in control systems without anti-windup compensation.

The response of the aircraft control system to the harmonic input $\psi^*(t) = \psi^*_{\max}\sin(\Omega t)$ with amplitude $\psi^*_{\max}$ varying in the range $0 \div 30°$ for $\Omega = 0.025$ s^{-1} is shown in Fig. 10 in the form of the dependences of the amplitudes of the control law u_r and the shadowing error $\Delta\psi(t)$ on the amplitude of the master control. Unlike the aircraft control system without anti-windup compensation (see Fig. 5), there is no hysteresis, and, in addition, the amplitudes of the control signals and the shadowing error in hunting lie in the range of practical requirements.

The graphs of the functions $\psi(t)$ and $u(t)$ obtained for $\psi^*(t) = \psi^*_0 + \psi^*_{\max}\sin(\Omega t)$, $\psi^*_0 = 11°$, $\psi^*_{\max} = 25°$, and $\Omega = 0.01$ s^{-1} and for the initial values $\psi(0) \in [-40, 40]°$ are shown in Fig. 11. The figures demonstrate the coincidence of the steady-state processes in the aircraft control system for a given command input.

Since the aircraft control system with the anti-windup compensation (21), (26) is convergent for $k_A = 2$, it follows from Section 3.2 that one can compute the sensitivity function $\mathcal{S}(\psi^*_{\max}, \omega)$, which specifies the accuracy of shadowing of the harmonic input depending on its frequency and amplitude. The graph of $\mathcal{S}(\psi^*_{\max}, \omega)$ for the considered system is shown in Fig. 12.

6. CONCLUSION

The present paper dealt with control problems in the case of a nonlinear effect of saturation of the control signal, which leads to the appearance of undesired oscillatory modes in systems with isodromic governor. We have reviewed publications on anti-windup compensation methods. We have suggested an anti-windup compensation method ensuring the convergence of the closed-loop system for master controls of a specific class in the case of neutrally stable plants.

We illustrate the application of the method to the control problem for the flight of an aircraft. The results of modeling of aircraft control systems have been represented for various forms of controllers, and the accuracy of the suggested algorithms has been analyzed on the basis of the harmonic balance method for conservative systems. We have demonstrated the application of the developed analytic-numerical methods to the analysis of undesired oscillatory modes in control systems with saturation. We have modelled aircraft control systems for large amplitudes of the master control and the effect of saturation of nonlinear control elements.

ACKNOWLEDGMENTS

The research was supported by the Ministry of Education and Science of the Russian Federation, St. Petersburg State University, Academy of Finland, and the Russian Foundation for Basic Research (project no. 12-08-01183-a).

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