Behavior analysis of harmonically forced chain of pendulums

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***Abstract*— Behavior analysis of harmonically forced diffusively coupled chain of pendulums is given, results of simulation and experiments are presented.**

***Index Terms*— Complex systems; Mechanical systems; Cyberphysical systems**

I. INTRODUCTION

Nonlinear oscillations problem falls within the fields of nonlinear mechanics, nonlinear physics, as well as nonlinear control theory. During the latest years the subject of nonlinear oscillations control has attracted growing researchers' attention among the other fields of research where control theory methods were applied to physical phenomena exploration [1].

The resonant property of nonlinear systems was intensively studied in the areas of charged-particle acceleration physics [2], plasma physics [3], celestial mechanics [4], and automatic control. Various definitions of the resonance phenomenon have been introduced to describe such properties of the nonlinear oscillations as stochastic resonance [5], chaotic resonance [6], autoresonance [2], [7], and feedback resonance [8]–[11].

The resonance property is allied to the synchronization one. Synchronization is usually treated as corresponding in time behavior of two or more processes. General definitions of synchronization were proposed in [12], [13]. Starting with the work of Christian Huygens [14], the synchronization phenomena attracted attention of many researchers, see e.g. monographs [15], [16]. In his study, C. Huygens described synchronization of the pair of pendulum clocks weakly coupled one with another owing to the common basement. Huygens had found the pendulum clocks swung in exactly the same frequency and 180 degrees out of phase. After external disturbance was made, the antiphase state was restored within a half of an hour and remained indefinitely. Huygens' synchronization observations have served to inspire study of sympathetic rhythms of interacting nonlinear oscillators in many areas of science and technology. The onset of synchronization and the selection of particular phase relations have been studied in the numerous papers and monographs, see e.g. [5], [15], [17]–[20]. Exploration of the synchronization phenomena is usually focused on the study of the phase relations between motions of the coherent units. Different studies of the pair of coupled oscillators confirmed an observation that the *asynchronous* mode of motion is a predominant one. At the same time some experiments show that the *synchronous* mode can also be observed. In [21] the system of two coupled pendulums excited via relay differential feedback is studied. It is shown that there exists frequency synchronization in the steady-state mode. The pendulum deviation angles have the same oscillation frequency and some constant phase shift. In the case if only one pendulum is excited, for the small coupling gain and small angular deflections the phase shift is about a quarter-period. In the same case if the amplitudes are "large" the approximately inphase oscillations occur. The steady-state oscillations are antiphase if the coupling gain is not large enough and both the pendulums are excited, whereas for large coupling gain both inphase and antiphase motions can occur, depending on initial phase shift between pendulums. The problem of controlled excitation of oscillations in a chain of N coupled pendulums is considered in [22]. Based on speed-gradient method with the goal function including the energy term and the "synchronization" term the excitation control law is derived and its properties for "line" and "ring" chain topologies are studied.

In the present paper, phase relations between oscillators, diffusively connected in a chain, which is excited by means of the harmonic torque applied to its end, is studied.

The model of 1-DOF pendulums coupled by a torsion spring and controlled by a drive is presented in Section II. In Section III the phase relations between pendulums are studied analytically on the basis of the linear model of the chain dynamics. The computer simulation results are presented in Section IV. The design features of the laboratory set-up *Multipendulum Mechatronic Setup*, created in the IPME RAS, Saint Petersburg, [23] are briefly described, and some experimental results are presented in Section V. The summarized information is given in Conclusion.

II. MODEL OF THE CHAIN OF PENDULUMS

Despite the previous works [21], [22] in the present study we consider the case, when the control action, applied to a chain of pendulums, may be treated as a kind of a boundary condition, rather than a torque, applied to the first pendulum. Namely, let the control action be a *rotation angle* of the drive

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shaft, connected with the first pendulum via a torsion spring. Retroaction from the pendulum to drive is omitted. The last (Nth) pendulum in the chain is mechanically connected only with the previous one, no boundary conditions for Nth pendulum are specified. Note that the model of this kind corresponds to the mechanical multipendulum system, described in Section V-B. This leads to the following model of chain dynamics:

$$\begin{cases} \ddot\varphi_1 + \rho\dot\varphi_1 + \omega_0^2 \sin\varphi_1 - k(\varphi_2 - 2\varphi_1) = ku(t), \\ \ddot\varphi_2 + \rho\dot\varphi_2 + \omega_0^2 \sin\varphi_2 - k(\varphi_3 - 2\varphi_2 + \varphi_1) = 0, \\ \dots\dots\dots\dots\dots\dots \\ \ddot\varphi_i + \rho\dot\varphi_i + \omega_0^2 \sin\varphi_i - k(\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}) = 0, \\ \qquad (i = 3, 4, \dots, N-1), \\ \dots\dots\dots\dots\dots\dots \\ \ddot\varphi_N + \rho\dot\varphi_N + \omega_0^2 \sin\varphi_N - k(\varphi_N - \varphi_{N-1}) = 0, \end{cases} \tag{1}$$

where $\varphi_i = \varphi_i(t)$ ($i = 1, 2, \dots, N$) are the pendulum deflection angles; $u = u(t)$ is the controlling action: rotation angle of the drive shaft. The values ρ, ω_0, k are parameters of the system: ρ is the viscous friction parameter; ω_0 is the natural frequency of small oscillations of isolated pendulums; k is the parameter of coupling strength, for example, the stiffness of the spring connecting the pendulums. Schematic diagram of the chain is shown in Fig. 1.

In what follows it is assumed that the excitation action $u(t)$ is a harmonic waveform with a certain frequency ω and a magnitude u_0: $u(t) = u_0 \sin(\omega t)$.

In the absence of control and dissipation the model in question coincides with the classical Frenkel–Kontorova model, [5], [24], or the *sin-Gordon* equation in the continuum-limit approximation. However, it is distinct from the Frenkel–Kontorova model in the type of the control appearance: here control is localized and affects only one pendulum. In terms of control of distributed systems it corresponds to the boundary control.

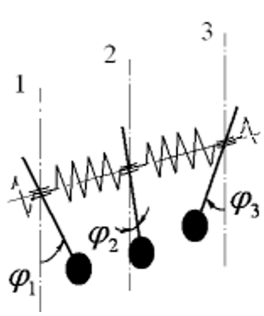

Fig. 1. Schematic diagram of the chain of pendulums

III. Examination of phase relations between pendulums via linearization method

A. Linearized model

Assuming the pendulum angles $\varphi_i(t)$ ($i = 1, \dots N$) be small, we obtain from (1) the following linearized model:

$$\begin{cases} \ddot\varphi_1 + \rho\dot\varphi_1 + \omega_0^2 \varphi_1 - k(\varphi_2 - 2\varphi_1) = ku(t), \\ \ddot\varphi_2 + \rho\dot\varphi_2 + \omega_0^2 \varphi_2 - k(\varphi_3 - 2\varphi_2 + \varphi_1) = 0, \\ \dots\dots\dots\dots\dots\dots \\ \ddot\varphi_i + \rho\dot\varphi_i + \omega_0^2 \varphi_i - k(\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}) = 0, \\ \qquad (i = 3, 4, \dots, N-1), \\ \dots\dots\dots\dots\dots\dots \\ \ddot\varphi_N + \rho\dot\varphi_N + \omega_0^2 \varphi_N - k(\varphi_N - \varphi_{N-1}) = 0. \end{cases} \tag{2}$$

Let us study phase relations in the steady-state mode for the system (2), forced by the harmonic input signal $u(t) = u_0 \sin(\omega t)$. To simplify exposition, consider, for example, the case of $N = 5$ pendulums. We obtain that the transfer functions $W_i(s) = \frac{B_i(s)}{A(s)}$ of the system (2) from input u to outputs φ_i, $i = 1, \dots, 5$ have a polynomial $A(s)$ of degree ten as the denominator and the following numerator $B_i(s)$:

$$\begin{aligned} B_1(s) &= k\big(s^8 + 4s^7\rho + (7k + 6\rho^2 + 4{\omega_0}^2)s^6 \\ &+ (21\rho k + 12\rho{\omega_0}^2 + 4\rho^3)s^5 + (15k^2 + 21k{\omega_0}^2 \\ &+ 21k\rho^2 + 6{\omega_0}^4 + 12\rho^2{\omega_0}^2 + \rho^4)s^4 + (4\rho^3{\omega_0}^2 + 12{\omega_0}^4\rho \\ &+ 30k^2\rho + 42{\omega_0}^2\rho k + 7k\rho^3)s^3 + (30k^2{\omega_0}^2 + 10k^3 \\ &+ 6\rho^2{\omega_0}^4 + 21{\omega_0}^4 k + 21k\rho^2{\omega_0}^2 + 15k^2\rho^2 + 4{\omega_0}^6)s^2 \\ &+ (30k^2\rho{\omega_0}^2 + 21{\omega_0}^4\rho k + 10k^3\rho + 4{\omega_0}^6\rho)s \\ &+ {\omega_0}^8 + 10k^3{\omega_0}^2 + 15k^2{\omega_0}^4 + 7{\omega_0}^6 k + k^4\big), \\ B_2(s) &= k^2\big(s^6 + 3s^5\rho + (3{\omega_0}^2 + 3\rho^2 + 5k)s^4 \\ &+ (\rho^3 + 10\rho k + 6\rho{\omega_0}^2)s^3 + (3{\omega_0}^4 + 6k^2 + 5k\rho^2 + 10k{\omega_0}^2 \\ &+ 3\rho^2{\omega_0}^2)s^2 + (6k^2\rho + 10{\omega_0}^2\rho k + 3{\omega_0}^4\rho)s + 6k^2{\omega_0}^2 \\ &+ 5{\omega_0}^4 k + {\omega_0}^6 + k^3\big), \\ B_3(s) &= k^3\big(s^4 + 2s^3\rho + (2{\omega_0}^2 + 3k + \rho^2)s^2 \\ &+ (2\rho{\omega_0}^2 + 3\rho k)s + 3k{\omega_0}^2 + {\omega_0}^4 + k^2\big), \\ B_4(s) &= k^4\big(s^2 + s\rho + k + {\omega_0}^2\big), \\ B_5(s) &= k^5. \end{aligned}$$

Since all the transfer functions $W_i(s)$, $i = 1, \dots, 5$ have a common denominator $A(s)$, phase relations between outputs $\varphi_i(t)$ in the steady-state mode are defined by fractions $w_{i,l}(s) = \frac{B_l(s)}{B_i(s)}$. Functions $\psi_{i,l}(\omega) = \arg w_{i,l}(j\omega)$ (where $j^2 = -1$ is an imaginary unit) for $l = i+1$, $i = 1, \dots, 4$ give the required phase shifts between system outputs as functions of the excitation frequency ω. The relative degree of each fraction $w_{i,i+1}(s)$ is two. Since the polynomials $B_i(s)$ are Hurwitz, the following relations are valid: $\lim\limits_{\omega\to 0} \psi_{i,i+1}(\omega) = 0$ and $\lim\limits_{\omega\to\infty} \psi_{i,i+1}(\omega) = -\pi$. The first case (the case of small frequency of the excitation signal) corresponds *inphase* motion of the pendulums, while the second one (when the excitation frequency is large), corresponds *antiphase* motion of neighboring pendulums. This general observation is illustrated numerically in Section IV and experimentally confirmed in Section V.

IV. Numerical analyses of the phase relations

For getting more detailed information about system behavior let us consider the following numerical example. Assume that the system (1) has the following parameters: $k = 7.22\ \mathrm{s}^{-2}$, $\rho = 0.11\ \mathrm{s}^{-1}$, $\omega_0 = 4.3\ \mathrm{s}^{-1}$. Then then the fractions $w_{i,i+1}$ have following zero-pole representation with 3-digits accuracy:

$$w_{1,2}(s) = \frac{7.22(s^2+0.11s+19.9)(s^2+0.11s+30)}{(s^2+0.11s+19.3)(s^2+0.11s+25)} \times \frac{(s^2+0.11s+41.9)}{(s^2+0.11s+36)(s^2+0.11s+44)},$$

$$w_{2,3}(s) = \frac{7.22(s^2+0.11s+21.2)}{(s^2+0.11s+19.9)(s^2+0.11s+30)} \times \frac{s^2+0.11s+37.3}{s^2+0.11s+41.9},$$

$$w_{3,4}(s) = \frac{7.2254(s^2+0.1084s+25.65)}{(s^2+0.1084s+21.19)(s^2+0.1084s+37.35)},$$

$$w_{4,5}(s) = \frac{7.2254}{s^2+0.1084s+25.65}.$$

This result shows that there exist certain frequencies ω_1 and ω_2 such that motion of the pendulums close to inphase if $\omega < \omega_1$ and is approximately antiphase if $\omega > \omega_2$. The frequency response plots (functions $\psi_{i,l}(\omega)$) are depicted in Fig. 2.

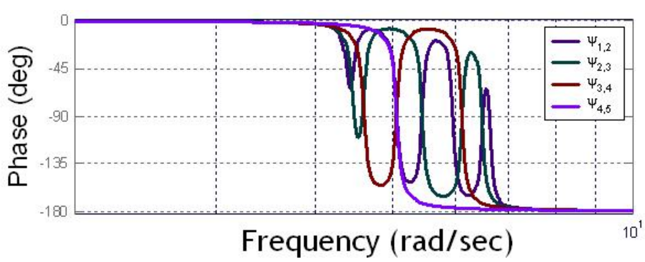

Fig. 2. Frequency response plots $\psi_{i,i+1}(\omega)$ v.s. ω, $i = 1, \ldots, 4$.

Remark. It is interesting to derive an analytical estimation of the frequencies ω_1 and ω_2 from the analytical expressions of the phase relations between the neighboring oscillators. This problem looks rather complex since it is related to the zero-poles estimation problem for a high order dynamical system. The similar problem is study of the system parameters influence on both position and size of the "uncertain region" $[\omega_1, \omega_2]$. These problems are intended for future researches.

The logarithmically scaled magnitude responses $H_i(\omega) = \big|W_i(j\omega)\big|$, $i = 1, 2, \ldots, 5$, are depicted in Fig. 3, demonstrating the resonant property of the system. It is seen that in the intermediate frequency range $[\omega_1, \omega_2]$, the magnitude responses $H_i(\omega)$ are close to one, and that they become lower as the excitation frequency ω goes apart this interval. Therefore, the pendulum chain may play a role of a mechanical band pass filter.

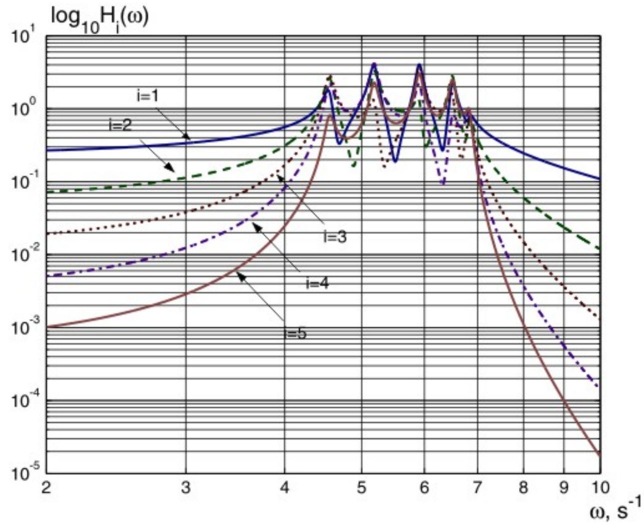

Fig. 3. Magnitude response plots $H_i(\omega)$ v.s. ω, $i = 1, 2, \ldots, 5$.

V. Experimental study

Below the multipendulum mechatronic setup, designed in the Institute for Problems of Mechanical Engineering of RAS (Saint Petersburg, Russia) and some experiments with harmonically forced motion for this setup are presented.

A. Multipendulum Mechatronic Setup of IPME RAS

The Multipendulum Mechatronic Setup of IPME RAS includes [23]:

– a modular multi-section mechanical oscillating system;
– an electrical equipment (with computer interface facilities);
– the personal computer for experimental data processing, representation of the results the real-time control.

For data exchange via standard In-Out ports of the computer, the special exchange routine is written. The devices are connected by means of the elastic link. In Sec. V-B a brief description of the construction is presented. For making laboratory experiments and on-line control, electrical design, data exchange interface and software tools were created. Their description is given in Sec. V-C.

B. Design of the mechanical part

The setup consists of a number of identical pendulum sections connected with springs. The schematics of a pendulum section is presented in Fig. 4, while a photo of five sections and an electric motor is given in Fig. 5. The foundation of the section is a hollow rectangular body. Inside the body an electrical magnet and electronic controller board are mounted. On the foundation the figure support containing the platform for placing the sensors in its middle part is mounted. The pendulum itself possesses a permanent magnet tip in the bottom part. The working ends of the permanent magnet and the electrical magnet are posed exactly opposite each other and separated with a non-magnetic plate in a window of the body. The idea behind control of the pendulum is changing the poles of the electrical magnet by means of switching the direction of the current in the windings of the electrical magnet. In order to allow changing the eigenfrequency of the pendulum oscillations the pendulum is endowed with additional plummets and counterparts changing its effective length (the distance between the suspension point and the center of mass. On the rotation axis of the pendulum the optical encoder disk for measuring the angle (phase) of the pendulum is mounted. It has 90 slits. The peripheral part of the disk is posed into the slit of the sensor support. The sensor consists of a radiator (emitting diode) and a receiver (photodiode). The obtained sequences of signals allow to measure angle (phase) and angular velocity of the pendulum, evaluate amplitude and crossing times and other variables related to the pendulum dynamics.

Axes of the neighbor sections are connected with the torsion springs, arranging force interaction and allowing exchanging energy between neighbor sections. The set of interconnected pendulum sections represents a complex oscillatory dynamical system, characterized by nonlinearity and high number of degrees of freedom. Such a mechanical system can serve as a basis for numerous educational and research experiments related to dynamics, control and synchronization in the networks of multidimensional nonlinear dynamical systems. In principle, any number of sections can be connected. At the moment mechanical parts of 50 sections are manufactured.

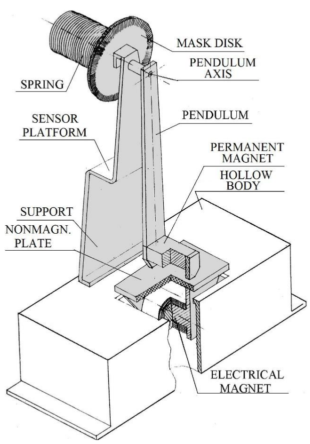

Fig. 4. Schematics of the pendulum section.

Fig. 5. Photo of the chain consisting of five pendulum sections.

Fig. 6. Inphase ($\omega = 6$ s^{-1}) and antiphase ($\omega = 9$ s^{-1}) synchronization of pendulums.

C. Electronics of the multipendulum setup

Oscillation control is provided on the basis of combined hardware/software method [23]. The energy for excitation is transmitted by the pulse-width modulated (PWM) signal with the constant level and variable duty cycle. From the programming point of view, hardware is represented by the write-only registers (WO) for putting in the prescribed duty cycle of control signal from the computer, and the read-only registers (RO) for transfer oscillation half period duration values to the computer. The PWM based method provides more precise control than the number-pulse one, because of averaging the high frequency pulses by the mechanical subsystem. The *control unit* generates the exciting action applied to the pendulums via the opposite magnetic fields. It includes bi-channel *asynchronous pulse-width modulator* (APWM), the Data Exchange System and the power amplifiers to drive the electromagnets.

The Data Exchange System of the setup is intended to transfer data and control commands from the Control Computer to the interface board of the pendulum sections. Each interface board is an intelligent measuring/controlling electronic devise, assigned for unloading processor of the Control Computer from chores of forming the control signal and preventing the Control Computer from a wasteful wait state of the sensor replies.

D. Results of experiments

A number of the experiments have been fulfilled with the multipendulum setup to verify results of theoretical analysis, given in Section III. In our experiments the chain of $N = 5$ pendulums (see Fig. 5) has been taken. The sinusoidal signal $u(t) = u_0 \sin(\omega t)$ with the magnitude u_0 and the frequencies $\omega \in [2, 10]$ s^{-1} has been applied to the control circuit of the electric motor. The experiments confirm theoretical results. It was obtained by the experiments, that for the considered setup, the boundary frequencies are $\omega_1 \approx 6.5$ s^{-1}, $\omega_2 \approx 8$ s^{-1}. The results are illustrated by Fig. 6, where the chain photos for the cases of inphase ($\omega = 6$ s^{-1}) and antiphase ($\omega = 9$ s^{-1}) modes of motion are presented.

VI. Conclusions

In the paper behavior analysis of harmonically forced diffusively coupled chain of pendulums is given. Based on the linear model is obtained that there exist certain frequencies ω_1 and ω_2 such that motion of the pendulums close to inphase if the excitation frequency ω less than ω_1 and is approximately antiphase if $\omega > \omega_2$. It is also obtained that the pendulum chain may play a role of a mechanical band pass filter. This result is confirmed by computer simulation and experiments on the *Multipendulum mechartonic setup* of the IPME RAS.

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