SYNCHRONIZATION ANALYSIS OF NONLINEAR OSCILLATORS *

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ABSTRACT
The possibilities of studying nonlinear physical systems by small feedback action are discussed. Analytical bounds of possible system energy change by feedback are established. The feedback resonance and synchronization phenomena are studied for 2-DOF system consisting of two coupled pendulums. Computer simulation results are presented.

KEY WORDS
Nonlinear Control, Oscillations, Synchronization

1 Introduction: physics and control

Physics and mechanics provided generations of mathematicians with both problems to solve and inspiration for solution. Same is true with respect to the control theory which essential part is actually a branch of mathematics. However the reverse influence was not noticeable until recently.

The situation changed dramatically in 90s after it was discovered that even small feedback introduced into chaotic system can change its behavior significantly, e.g. turn the chaotic motion into the periodic one [1]. The seminal paper [1] gave rise to an avalanche of publications demonstrating metamorphoses of numerous systems – both simple and complicated – under action of feedback. However the potential of modern nonlinear control theory (e.g. [2, 3, 4]) still was not seriously demanded although the key role of the system nonlinearity was definitely appreciated. On the other hand the new problems have some specific features for control theorists: the desired point or the trajectory of the system is not prescribed whilst the "small feedback" requirement is imposed instead. It took some time to realize that the problems of such kind are typical for control of more general oscillatory behavior and to work out the unified view of nonlinear control of oscillations and chaos [5].

It needs only one more effort to make the next step and to start systematic studying the properties of physical (as well as chemical, biological, etc.) systems by means of small feedback actions.

The first consequences of such an approach for physics and mechanics were demonstrated in [6], where the mechanism of creating resonant behavior in oscillatory system by feedback was examined for 1-DOF (one-degree-of-freedom) nonlinear oscillator. This phenomenon was called a *feedback resonance*. Feedback resonance phenomenon has been investigated in the paper [7], where some results concerning feedback resonance in 1-DOF oscillators and its application to studying escape from a potential well have been refined. Extension to the n-DOF systems can be found in [8]. In the papers [6]–[9] the concept of feedback resonance is used for introducing an *excitability index*, that characterizes excitability properties of the nonlinear system. Some properties of the excitability index have been found on the base of computer simulations and real-world experiments in [10, 11].

In this paper an investigation of feedback resonance phenomenon is continued. In Sec. 3 the speed-gradient method, useful for feedback design in oscillatory systems is outlined. In Sections 2, 4 some results concerning feedback resonance in 1-DOF oscillators and its application to studying escape from a potential well are refined. In Sec. 5 the feedback resonance is demonstrated for 2-DOF system consisting of two coupled pendulums. The results are illustrated by computer simulations.

2 Excitation of nonlinear oscillator

Consider the controlled 1-DOF oscillator modeled after appropriate rescaling by the differential equation

$$\ddot{\varphi} + \frac{\partial \Pi(\varphi)}{\partial \varphi} = u, \tag{1}$$

where φ is the phase coordinate, $\Pi(\varphi)$ is potential energy function, u is controlling variable. The state vector of the system (1) is $x = (\varphi, \dot{\varphi})$ and its important characteristics is the total energy $H(\varphi, \dot{\varphi}) = 0.5\dot{\varphi}^2 + \Pi(\varphi)$. The state vector of the uncontrolled (free) system moves along the energy surface (curve) $H(\varphi, \dot{\varphi}) = H_0$. The behavior of the free system depends on the shape of $\Pi(\varphi)$ and the value of H_0. E.g. for simple pendulum one has $\Pi(\varphi) = \omega_0^2(1 - \cos\varphi) \geq 0$. Obviously, choosing $H_0 :\ 0 < H_0 < 2\omega_0^2$ one obtains oscillatory motion with amplitude $\varphi_0 = \arccos(1 - H_0/\omega_0^2)$. For $H_0 = 2\omega_0^2$ the motion along the separatrix including upper equilibrium is observed, while for $H_0 > 2\omega_0^2$ the energy curves get infinite and the system exhibits

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the permanent rotation with the average angular velocity $\bar{\dot{\varphi}} \approx \sqrt{2H_0}$.

Let us put the question: is it possible to significantly change the energy (i.e. behavior) of the system by means of arbitrarily small controlling action?

The answer is well known when the potential is quadratic, $\Pi(\varphi) = 0.5\omega_0^2\varphi^2$, i.e. the systems dynamics are linear: $\ddot{\varphi}(t) + \omega_0^2\varphi(t) = u(t)$. In this case we may use the harmonic external action $u(t) = u_m \sin\omega t$ and for $\omega = \omega_0$ watch the resonance unbounded solution $\varphi(t) = -\dfrac{u_m t}{2\omega_0}\cos\omega_0 t$.

However for nonlinear oscillators the resonant motions are more complicated with interchange of energy absorption and emission. It is well known that even for simple pendulum the harmonic excitation can even give birth to chaotic motions. The reason is, roughly speaking, in that the natural frequency of a nonlinear system depends on the amplitude of oscillations.

Therefore the idea comes: to create resonance in a nonlinear oscillator by changing the frequency of external action as a function of amplitude of oscillations. To implement this idea we need to make the value $u(t)$ depending on the current measurements $\varphi(t)$, $\dot{\varphi}(t)$ which exactly means introducing the feedback

$$u(t) = U\left(\varphi(t), \dot{\varphi}(t)\right). \tag{2}$$

Now the problem is: how to find the feedback law (2) in order to achieve the energy surface $H(\varphi, \dot{\varphi}) = H_*$. This problem falls into the field of control theory. To solve it we suggest to use the so called Speed-Gradient (SG) method [5, 12, 13, 14] which is outlined below.

3 Speed-gradient algorithms and energy control

Various algorithms for control of nonlinear systems were proposed in the literature, see e.g. [2, 3, 4, 20]. For purposes of "small control" design the following "speed-gradient" procedure is convenient [5, 12, 13, 14].

Let the controlled system be modeled as

$$\dot{x} = F(x, u), \tag{3}$$

where $x \in \Re^n$ is the state and $u \in \Re^m$ is input (controlling signal). Let the goal of control be expressed as the limit relation

$$Q(x(t)) \to 0 \quad \text{when} \quad t \to \infty. \tag{4}$$

In order to achieve the goal (4) one may apply the SG-algorithm in the finite form

$$u = -\Psi(\nabla_u \dot{Q}(x, u)), \tag{5}$$

where $\dot{Q} = (\partial Q/\partial x)F(x, u)$ is the speed of changing $Q(x(t))$ along the trajectories of (3), vector $\Psi(z)$ forms a sharp angle with the vector z, i.e. $\Psi(z)^{\mathrm{T}} z > 0$ when $z \neq 0$ (superscript "T" stands for transpose). The first step of the speed-gradient procedure is to calculate the speed $\dot{Q}$. The second step is to evaluate the gradient $\nabla_u \dot{Q}(x, u)$ with respect to controlling input u. Then the vector-function $\Psi(z)$ should be chosen to meet sharp angle condition. E.g. the choice $\Psi(z) = \gamma z, \gamma > 0$ yields the standard *proportional* (P) feedback

$$u = -\gamma \nabla_u \dot{Q}(x, u), \tag{6}$$

while the choice $\Psi(z) = \gamma \mathrm{sign} z$, where the signum function $sign$ is understood componentwise, yields the *relay algorithm*

$$u = -\gamma \mathrm{sign}(\nabla_u \dot{Q}(x, u)). \tag{7}$$

The integral (I) form of SG-algorithm

$$\frac{du}{dt} = -\gamma \nabla_u \dot{Q}(x, u), \tag{8}$$

also can be used as well as combined, e.g. proportional-integral (PI) forms.

The underlying idea of the choice (6) is that moving along the antigradient of the speed $\dot{Q}$ provides decrease of $\dot{Q}$. It may eventually lead to negativity of $\dot{Q}$ which, in turn, yields decrease of Q and, eventually, achievement of the primary goal (4). However, to prove (4) some additional assumptions are needed, see [5, 12, 13, 14].

Let us illustrate derivation of SG-algorithms for the Hamiltonian controlled system of the form

$$\begin{aligned} \dot{q} &= \nabla_p H(q, p) + \nabla_p H_1(q, p) u, \\ \dot{p} &= -\nabla_q H(q, p) - \nabla_q H_1(q, p) u, \end{aligned} \tag{9}$$

where H is Hamiltonian of the free system, H_1 is interaction Hamiltonian. In order to control the system to the desired energy level H_*, the energy related goal function $Q(q, p) = (H(q, p) - H_*)^2$ is worth to choose. First step of speed-gradient design yields

$$\begin{aligned} \dot{Q} = 2(H - H_*)\dot{H} = 2(H - H_*)[(\nabla_q H)^{\mathrm{T}} \nabla_p H_1 \\ - (\nabla_p H)^{\mathrm{T}} \nabla_q H_1] u = 2(H - H_*)\{H, H_1\} u, \end{aligned}$$

where $\{H, H_1\}$ is *Poisson bracket*. Since $\dot{Q}$ is linear in u, the second step yields $\nabla_u \dot{Q} = 2(H - H_*)\{H, H_1\}$. Now different forms of SG-algorithms can be produced. For example proportional (P) form (6) looks as follows

$$u = -\gamma (H - H_*)\{H, H_1\}, \tag{10}$$

where $\gamma > 0$ is gain parameter. For the special case $H_1(q, p) = q$, $q = \varphi$ it turns into the algorithm

$$u = -\gamma\left(H - H_*\right)\dot{\varphi}. \tag{11}$$

Similarly, the relay form (7) gives the algorithm

$$u = -\gamma\, \mathrm{sign}\, \left(H - H_*\right) \cdot \mathrm{sign}\, \dot{\varphi}. \tag{12}$$

Analysis of the behavior of the system containing the feedback is based on the following theorem.

Theorem [5]. Let functions H, H_1 and their partial derivatives be smooth and bounded in the region $\Omega_0 =$

$\{(q,p) : |H(q,p) - H_*| \le \Delta\}$. Let the unforced system (for $u = 0$) have only isolated equilibria in Ω_0.

Then any trajectory of the system with feedback either achieves the goal or tends to some equilibrium. If, additionally, Ω_0 does not contain stable equilibria then the goal will be achieved for almost all initial conditions from Ω_0. ∎

Similar results are also valid for the goals expressed in terms of several integrals of motions and for the general nonlinear systems with SG-algorithms (see [14]).

4 Escape from a potential well

The feedback resonance phenomenon is related to escape from the potential wells which is important in many fields of physics and mechanics [16, 17]. Sometimes escape is an undesirable event and it is important to find conditions preventing it (e.g. buckling of the shells, capsize of the ships, etc.). In other cases escape is useful and the conditions guaranteeing it are needed. In all cases the conditions of achieving the escape by means of as small external force as possible are of interest.

For the system (1) the SG-method with the choice of the goal function $Q(x) = \big(H(x) - H_*\big)^2$ produces simple feedback laws: (11), (12).

It follows from the Theorem of Sec. 3 that the goal $H(x(t)) \to H_*$ in the system (1), (11) (or (1), (12)) will be achieved from almost all initial conditions provided that the potential $\Pi(\varphi)$ is smooth, its stationary points are isolated and there is no stable equilibria of the unforced system within initial energy layer $\{(\varphi,\dot\varphi) : H_0 \le H(\varphi,\dot\varphi) \le H_*\}$, where $H_0 = H(\varphi(0),\dot\varphi(0))$ is initial energy level (we assume $H_0 \le H_*$).

In [17] such a possibility (optimal escape) has been studied for typical nonlinear oscillators

$$\ddot\varphi + \rho\dot\varphi + \frac{\partial \Pi}{\partial \varphi} = u. \tag{13}$$

with a single-well potential $\Pi_e(\varphi) = \varphi^2/2 - \varphi^3/3$ (so called "escape equation") and a twin-well potential $\Pi_d(\varphi) = -\varphi^2/2 + \varphi^4/4$ (Duffing oscillator). The least amplitude of a harmonic external forcing $u(t) = u_m \sin\omega t$ for which no stable steady state motion exists within the well was determined by intensive computer simulations. For example, for escape equation with $\rho = 0.1$ the optimal amplitude was evaluated as $u_m \approx 0.09$, while for Duffing twin-well equation with $\rho = 0.25$ the value of amplitude was about $u_m \approx 0.212$. Our simulation results agree with [17]. The typical time histories of input and output for $u_m = 0.209$ are shown in Fig. 1. It is seen that escape does not occur.

Using feedback forcing one may expect reducing the escape amplitude. In fact using the results of Sec. 2, the amplitude of feedback (11), (12) leading to escape can be easily calculated, just substituting the height of the potential barrier $\max_\Omega \Pi(\varphi) - \min_\Omega \Pi(\varphi)$ for H_m into equation $H_m = \frac{1}{2}\left(\frac{u_m}{\rho}\right)^2$, where Ω is the well corresponding to the initial state. For example taking $H = 1/6, \rho = 0.1$ for $\Pi_e(\varphi)$ gives $u_m = 0.0577$, while $H_m = 1/4, \rho = 0.25$ gives $u_m = 0.1767$, the values which are substantially smaller than those evaluated in [17]. The less is the damping, the bigger is the difference between the amplitudes of feedback and nonfeedback signals leading to escape. Simulation exhibits still stronger difference: escape for Duffing oscillator occurs even for $u_m = \gamma = 0.122$ if the law (12) is applied, see Fig. 2. Note that the oscillations in the feedback systems have both variable frequency and variable shape.

5 Excitation of two coupled pendulums

Consider the special case of the diffusively coupled oscillator model (used for modeling various physical and mechanical systems, see [18]): the two pendulums model. For the case of torsion spring it has the form

$$\begin{cases} \ddot\varphi_1 + \rho\dot\varphi_1 + \omega^2\sin\varphi_1 + k\big(\varphi_1 - \varphi_2\big) = u(t), \\ \ddot\varphi_2 + \rho\dot\varphi_2 + \omega^2\sin\varphi_2 + k\big(\varphi_2 - \varphi_1\big) = 0, \end{cases} \tag{14}$$

where $\varphi_i(t)$ are the rotation angles of pendulums ($i = 1, 2$); $u(t)$ is the external torque (control action), applied to the first pendulum; ω, k, ρ are the system parameters: ω is the natural frequency of small oscillations, k is the coupling parameter (e.g. stiffness of the string), ρ is the viscous friction gain.

The total energy of the system (14) $H(x)$ can be written as follows

$$\begin{aligned} H(x) = \frac{1}{2}\dot\varphi_1 + \omega^2(1-\cos\varphi_1) + \frac{1}{2}\dot\varphi_2^2 \\ + \omega^2(1-\cos\varphi_2) + \frac{k}{2}\big(\varphi_1 - \varphi_2\big)^2 \end{aligned} \tag{15}$$

where $x(t) \in \mathcal{R}^4$ stands for the state vector $x(t) = [\varphi_1, \varphi_2, \dot\varphi_1, \dot\varphi_2]^{\mathrm{T}}$.

Consider the problem of excitation a "wave" with the desired amplitude by means of small feedback. The problem can be understood as achieving the given energy level of the system with additional requirement that pendulums should have either coincident or opposite phases of oscillation.

In order to apply the Speed-gradient procedure of Sec. 3, let us introduce objective functions as

$$\begin{aligned} Q_\varphi(\dot\varphi_1, \dot\varphi_2) &\triangleq \frac{1}{2}\big(\delta_\varphi\big)^2 \\ Q_H(x) &\triangleq \frac{1}{2}(H(x) - H_*)^2. \end{aligned} \tag{16}$$

where $\delta_\varphi = \dot\varphi_1 + \sigma\dot\varphi_2$, $\sigma \in \{-1, 1\}$; H_* is the prescribed value of the total energy. The minimum value of the function Q_φ meets the "coincident/opposite phases" requirement (at least for small initial phases $\varphi_1(0), \varphi_2(0)$) : $Q_\varphi(\dot\varphi_1, \dot\varphi_2) \equiv 0$ iff $\dot\varphi_1 \equiv -\sigma\dot\varphi_2$. Hence option $\sigma = 1$ sets

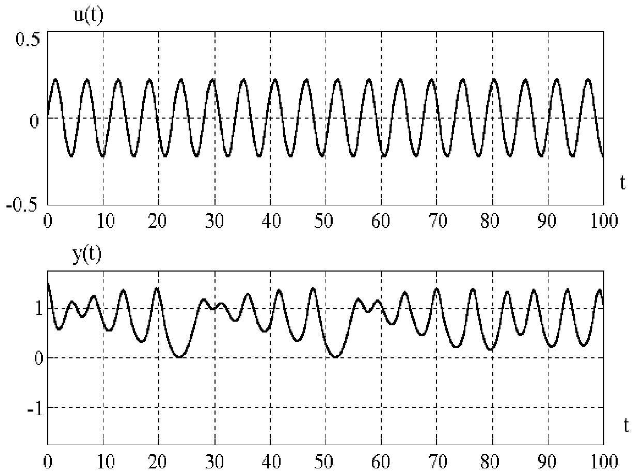

Figure 1. Time histories of input and output of Duffing oscillator for $u(t) = u_m \sin \omega t$; $u_m = 0.209$. Escape does not occur.

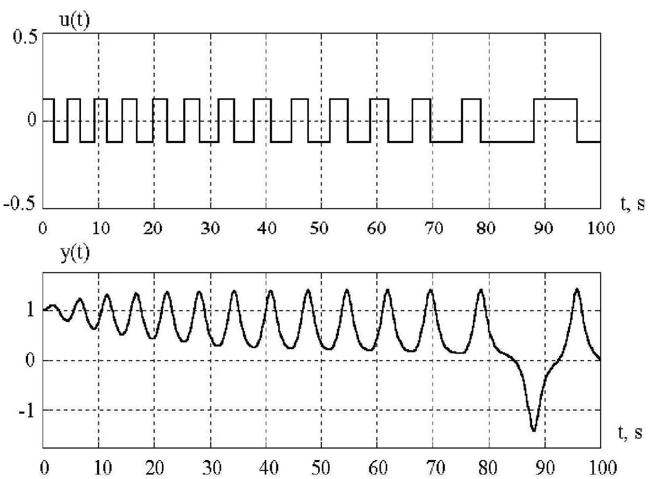

Figure 2. Escape of Duffing oscillator by means of the feedback forcing (12); $u_m = 0.122$.

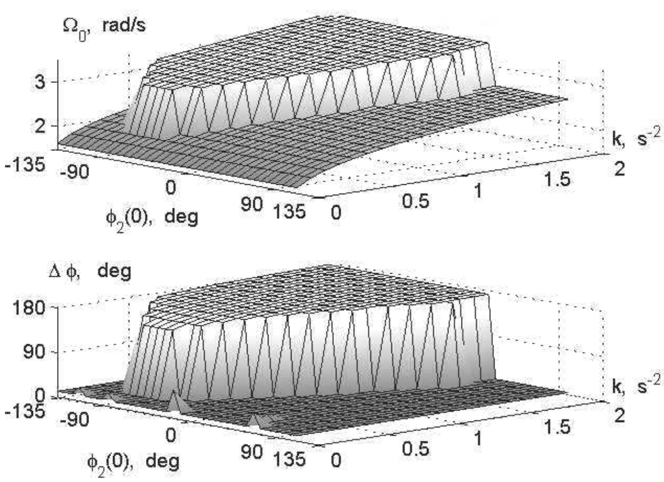

Figure 3. Oscillations frequency Ω_0 and the phase shift $\Delta\psi$ versus k, $\varphi_2(0)$. The control law (20), $\gamma = 1$ s.

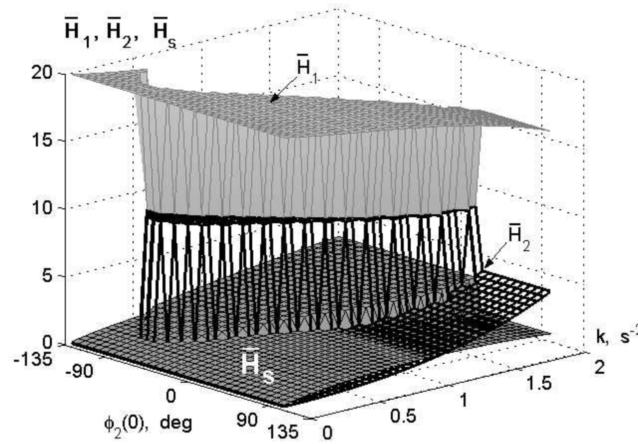

Figure 4. The averaged energies $\bar{H}_1$, $\bar{H}_2$, $\bar{H}_s$ versus k, $\varphi_2(0)$. The control law (20), $\gamma = 1$s.

antiphase desired pendulums oscillations, while $\sigma = -1$ sets *inphase* oscillations. The minimization of Q_H means achievement of the desired amplitude of the oscillations.

In order to design the control algorithm, the weighted objective function $Q(x)$ is introduced as the weighted sum of Q_φ and Q_H, namely

$$Q(x) \stackrel{\Delta}{=} \alpha Q_\varphi(\dot{\varphi}_1, \dot{\varphi}_2) + (1-\alpha) Q_H(x), \qquad (17)$$

where α, $0 \le \alpha \le 1$ is a given weighting coefficient.

Accordingly to the Speed-gradient procedure of Sec. 3, the following control law can be obtained:

$$\begin{aligned} u(t) &= -\gamma \left(\alpha \delta_\varphi(t) + \big(1-\alpha\big)\delta_H(t)\dot{\varphi}_1(t)\right), \\ \delta_\varphi(t) &= \dot{\varphi}_1(t) + \sigma \dot{\varphi}_2(t), \\ \delta_H(t) &= H\big(x(t)\big) - H_*, \end{aligned} \qquad (18)$$

where $\sigma \in \{-1, 1\}$ is a phase shift parameter; α is a weighting coefficient; $\gamma > 0$ is a gain coefficient. Applying the results of [5, 14] yields that for $\sigma = 1$ sufficient conditions for the achievement of the control goal $Q(x(t)) \to 0$ are valid if the desired level of energy does not exceed the value $H_* = 2\omega^2$, corresponding to the upper equilibrium of one pendulum and lower equilibrium of another one.

By analogy with (12), the following relay SG-control law can be written:

$$u(t) = -\gamma \operatorname{sign} \Big(\alpha \delta_\varphi(t) + \big(1-\alpha\big)\delta_H(t)\dot{\varphi}_1(t)\Big). \qquad (19)$$

The control law (18) with $\sigma = 1$ has been proposed and numerically examined in [7]. In the paper [15] the case of $\sigma = -1$ and nonlinear coupling function in (14) is considered and the results of analytical, numerical and experimental study of the closed-loop system are presented.

In this paper let us focus our attention to properties of the system with the algorithm (19). At first let us assume $\alpha = 0$, i.e. consider the closed-loop system with a pure *energy-control* algorithm

$$u = -\gamma \operatorname{sign}(H - H_*) \operatorname{sign} \dot{\varphi}_1(t), \qquad (20)$$

where the total energy H is given by (15), H_* is a desired value of H. Note that the control law (20) has the same form that the law (12).

The question under consideration is: if no requirement on the phase shift is given (i.e. it is taken $\alpha = 0$ in (19)), what will be the phase shift in the steady-state oscillation mode? To answer that question the system (14), (20) has been numerically studied by means of computer simulations. Following parameter values and the initial conditions were taken: $\omega^2 = 10$ s^{-2}, $\rho = 0.1$ s^{-1}, $\gamma = 1$ s, $H_* = 20$ s^{-2}, $\varphi_1(0) = \pi/2$, $\dot{\varphi}_1(0) = \dot{\varphi}_2(0) = 0$. Initial value $\varphi_2(0)$ varied in the segment $[-3/4\pi, 3/4\pi]$, the coupling coefficient k changed from 0.1 to 2 s^{-2}. The simulation time was equal to 450 s; the fixed-step Dormand-Prince method with a step 0.025 s was used. Some results are shown in Figs. 3, 4. In Fig. 3 the plots of the *frequency of oscillation* Ω and the *phase shift* $\Delta\psi$ in the steady-state mode versus the coupling parameter k and the initial condition $\varphi_2(0)$ are pictured. It is shown that in the some domain of the plane $\big(k, \varphi_2(0)\big)$ the steady phase shift is about zero, i.e. the pendulums fall into inphase synchronous oscillations, while in the complement domain the phase shift is about π – the motions are antiphase. Note, that the oscillation frequencies for antiphase motion exceed those ones for the inphase motion. This effect has a lucid physical explanation: in the antiphase mode the spring torque is added to the torque of a gravitational force. Fig. 4 shows how the averaged energies $\bar{H}_i$ ($i = 1, 2, s$) distribute between the different units; $i = 1, 2$ relates to the pendulums, $i = s$ relates to the spring. One can see that in the antiphase mode the amplitudes of φ_1 and φ_2 are almost equal with each other, while in the inphase mode the amplitude of φ_1 is noticeably greater that one of φ_2.

Let us consider now the general form (19). The simulation results for $\alpha = 0.7$ are shown in Figs. 5–8. Two cases of the damping parameter ρ were considered: $\rho = 0$ (the lossless case), and $\rho = 0.1$ s^{-1}. It was taken $\varphi_1(0) = \varphi_2(0) = 0$, $\dot{\varphi}_1(0) = \dot{\varphi}_2(0) = 10^{-5}$ s^{-1}. It is seen that both pendulums fall in antiphase oscillations if $\sigma = 1$ and inphase oscillations if $\sigma = 1$. The relation between transient times for H and for Q_φ can be changed by means of changing the weight coefficient α. In the lossless case the control amplitude can be arbitrarily decreased by means of decreasing the gain γ.

Conclusions

The fundamental question of physics, mechanics and other natural sciences is: what is possible and why? In this paper we attempted to investigate what is possible to do with a physical system by feedback. It was shown that if system is close to conservative, its energy can be changed in a broad range by small feedback. Moreover, for multidimensional (e.g. 2-DOF) systems some additional symmetry (synchronization) properties can be preserved by feedback.

The nature of such an efficiency of feedback is very similar to the case of control of chaos [1, 19]. The method

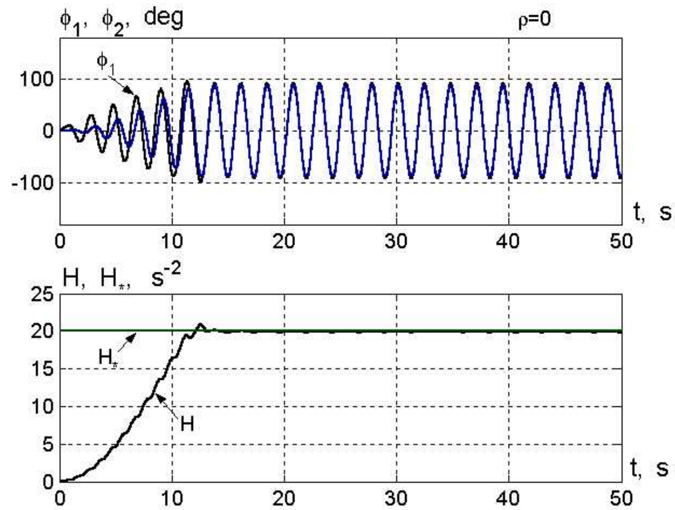

Figure 5. Excitation of inphase oscillations; the lossless case ($\rho = 0$). The control law (19), $\gamma = 1$ s, $\alpha = 0.7$, $\sigma = -1$.

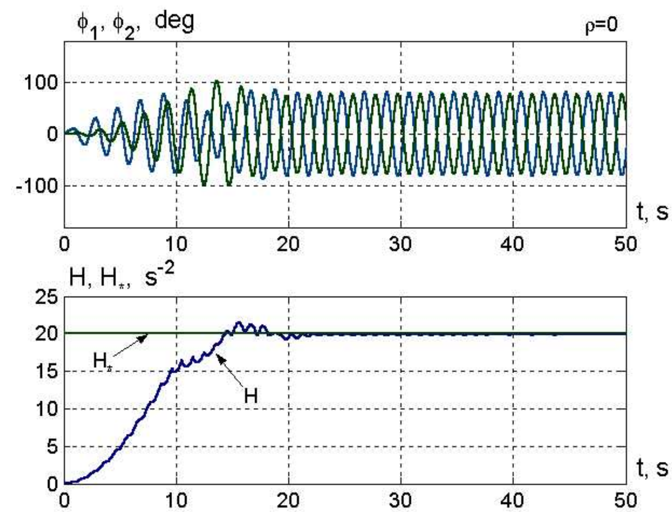

Figure 6. Excitation of antiphase oscillations; the lossless case ($\rho = 0$). The control law (19), $\gamma = 1$ s, $\alpha = 0.7$, $\sigma = -1$.

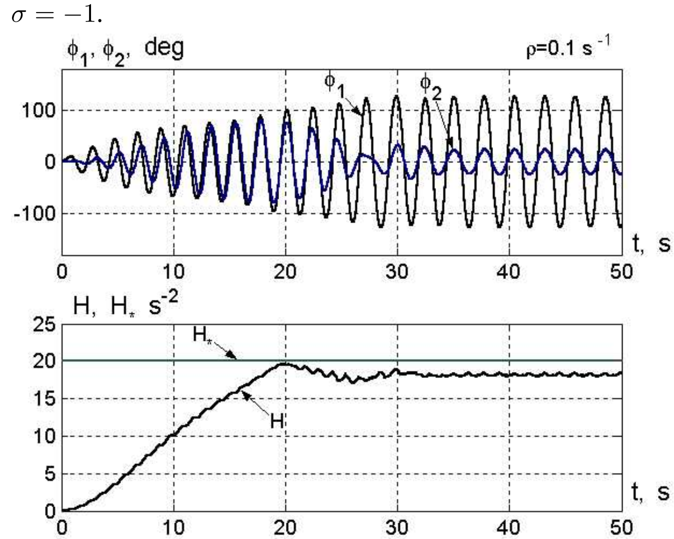

Figure 7. Excitation of inphase oscillations; $\rho = 0.1$ s^{-1}.

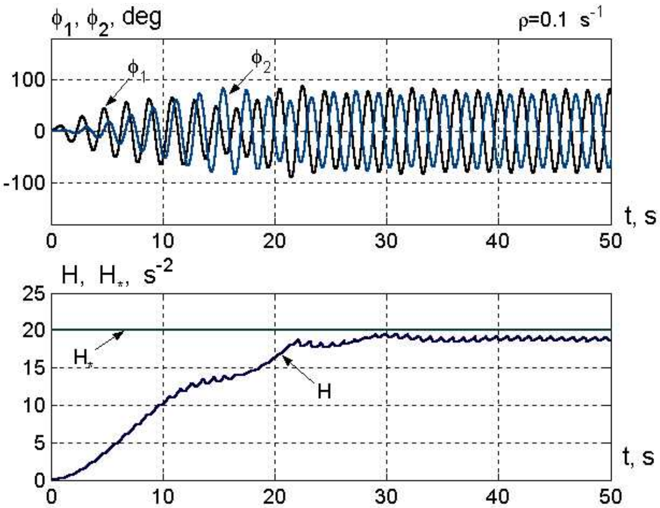

Figure 8. Excitation of antiphase oscillations; $\rho = 0.1\ \mathrm{s}^{-1}$. The control law (19), $\gamma = 1$ s, $\alpha = 0.7$, $\sigma = 1$.

of [1] and other related methods apply small control on the cross sections when the trajectory passes near an (unstable) periodic orbit which trace on the section is just the fixed point. By virtue of recurrence property of chaotic motion the trajectory will return into the vicinity of the fixed point. It can be interpreted as a kind of approximate conservativity for the discretized system considered at the sample instances. Therefore in this case it is also possible to achieve large changes in system behavior by means of small control.

Thus the mechanism, possibilities and limitations of feedback are understood for the two broad classes of processes — conservative and chaotic oscillations — which are of importance in physics. It motivates further study of this phenomenon which belongs to the boundary area of physics and control science (in a broader sense — cybernetics) and In fact one may constitute this field as the new field of physics: *cybernetical physics*. Its subject is investigation of the features of the natural system by admitting (weak) feedback interactions with the environment. Its methodology heavily relies on the design methods developed in cybernetics. However the approach of cybernetical physics differs from the conventional usage of feedback in control applications (e.g. robotics, mechatronics, see [21]) aimed at driving the system to the prespecified position or the given trajectory.

Other related phenomena which are already under investigation are: controlled synchronization, excitation of waves in nonlinear media, controlling energy exchange of subsystems, etc. We believe that the cybernetical methodology will also gain new insights in chemistry, biology and environmental studies.

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