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Anton V Proskurnikov
University of Groningen

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Anton V. Proskurnikov

ABSTRACT

Distributed algorithms for synchronization and consensus in multi-agent networks are considered. The agents are assumed to be linear of arbitrary order, the interaction topology may switch and the couplings are uncertain, assumed only to satisfy certain quadratic constraints. Using the Kalman-Yakubovich-Popov lemma and absolute stability theory techniques, consensus criteria for the networks of this type are obtained. Those criteria extend a number of known result for agents with special dynamics and are close in spirit to the celebrated circle criterion for the stability of Lurie systems.

Key Words: Networks, consensus, synchronization, absolute stability, KYP lemma.

I. Introduction.

The problem of multi-agent consensus or (called also agreement or controlled synchronization) has recently attracted the considerable attention of different research communities. The main subject of these researches is the synchronism achieved by means of local interactions between the agents (via communication or otherwise). Such a synchronism underlies numerous natural phenomena and is the cornerstone of many technical designs and engineering approaches. Examples, just to mention a few, include numerous forms of collective behavior in complex biological and man-made systems, such as flocking, swarming, rendezvouz etc. [1, 2, 3, 4, 5, 6, 7], synchronization in complex networks [8] and oscillator ensembles [9, 10, 11, 12]. The ideas of many consensus policies take their origin from distributed algorithms in computer sciences, averaging procedures in the theory of stochastic matrices, and agreement procedures in expert communities coming from applied statistics.

The most investigated consensus protocols are those with linear couplings (see [1, 2, 6, 13, 14, 15, 16, 17] among the others, as well as references therein) which are widespread and recognized to be simple and effective. Nevertheless, progressively growing number of control and physical applications require nonlinear consensus algorithms that can not be analyzed by standard linear tools (such as convergence criteria for infinite matrix products, Laplacian matrix decomposition, frequency-domain methods etc.) Such algorithms come from, for instance, oscillator networks with periodic couplings [9, 11], coordination with range-restricted sensoring [5, 7, 10], nonlinear agreement procedures [18]. In the real-world conditions linear protocols may be become nonlinear because of analog-digital conversions, quantization effects and nonlinear distortions in measurements. The mentioned challenges stimulated the rapid development of nonlinear consensus theory.

One of the first major breakthroughs in this area was concerned with the exploration of nonlinear averaging procedures for first-order agents [19, 7, 18, 20] etc. The common property of such algorithms is that the convex hull of the agents (considered as the points in a vector space) is provided to be nested due to a special choice of the velocities. Namely,

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The author is with St.Petersburg State University, Department of Mathematics and Mechanics and Institute for Problems of Mechanical Engineering RAS, Universitetsky pr. 28, St.Petersburg, 198504, RUSSIA, avp1982@gmail.com
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each agent moves into the relative interior of the convex hull of itself and the neighbors. The celebrated paper by Moreau [19] establishes the convergence of those contracting procedures in the discrete-time case by using the convex hull of the agents as a set-valued Lyapunov function (its diameter being common Lyapunov function). Analogous reasoning is applicable to rendezvouz algorithms on the plane [7] and continuous-time consensus protocols [18, 21]. Another class of results for nonlinear consensus protocols deals with passive agents [22, 23] and exploits Lyapunov functions constructed from individual storage functions of the agents. For non-passive nonlinearly coupled agents, e.g. double integrators, no common ways for establishing synchronization seem to be known but for special types of nonlinearities [2,10,24] (e.g. hyperbolic tangents) explored by ingenious Lyapunov functions.

While most of recent results on nonlinear consensus protocols concern agents with special dynamics (passive, multiple integrators etc.), in the present paper we consider networks of general MIMO agents with switching interaction topology. Unlike the mentioned works, the topology is neither assumed to have positive dwell-time, nor even be piecewise-continuous but supposed only to be measurable function of time. Two important kinds of consensus algorithms are considered: those of the first type require the interaction graph to be undirected and have nonlinear couplings satisfying symmetry conditions resembling the Newton's Third Law. Algorithms of the second type are linear and satisfy certain balance condition, they are applicable for directed topology. In both cases, the full knowledge of couplings may be unavailable, they are supposed only to satisfy some quadratic constraint [25]. Using the absolute stability approach, a condition for the robust consensus (in the mentioned class of uncertainties) is obtained, which is closely related to the celebrated circle criterion for stability of Lurie systems [25, 26]. The proposed criterion extends a number of known results on synchronization in nonlinearly coupled networks and is among the first addressing the issue of consensus robustness. It extends, for instance, the results from [22, 23] to the case of non-passive MIMO agents, allows to obtain synchronization criterion for harmonic oscillators from [12], new consensus results for double integrator agents and for strictly passifiable [27] agents.

The paper generalizes results of previous work [28] in several directions: on the one hand, we consider the case of MIMO agents and general quadratic constraints while the mentioned paper was devoted to the case of SISO agents and scalar couplings only; on the other hand, unlike [28] we consider not only undirected, but also directed interaction graphs. Also we consider new applications of the obtained criterion, concerning double integrator agents and strictly passifiable agents.

The paper is organized as follows. Section II introduces some basic concepts from the graph theory. Section III is devoted to the formulation of the problem, the main assumptions are given in Section IV. Section V presents the main result of the paper (the sufficient condition for consensus), and Section VI illustrates some applications to agents with special dynamics. Section VII offers the proof of the main result.

II. Preliminaries from the Graph Theory

A pair $G = (V, E)$ of two finite set V (the set of nodes) and $E \subset V \times V$ (the set of arcs) is called a *(directed) graph*. The node v is *connected to* the node w in G if $(v, w) \in E$. Any sequence of nodes $v_1, v_2, \ldots, v_k$ with $(v_i, v_{i+1}) \in E$ for $i = 1, 2, \ldots, k-1$ is called *a path* between v_1 and v_k. The graph is *strongly connected* if a path between any two different nodes exists. Given a graph $G = (V, E)$, the *mirror graph* $\hat{G} = (V, \hat{E})$ is obtained by inserting all inverted arcs, i.e. $\hat{E} = E \cup \{(w, v) : (v, w) \in E\}$. A graph coinciding with its mirror is said to be *undirected*. Throughout the paper $\mathbb{G}_N$ stands for the class of all graphs $G = (V_N, E)$ with the node set $V_N = \{1, 2, \ldots, N\}$ and set of arcs E containing no loops ($(v, v) \notin E$ for any $v \in V_N$). Define the *adjacency* matrix $(a_{jk}(G))$ of $G \in \mathbb{G}_N$ as follows: $a_{jk}(G)$ is 1 if $(k, j) \in E$ and 0 otherwise. The *Laplacian matrix* of G is given by

$$L(G) = \begin{bmatrix} \sum_{j=1}^N a_{1j} & -a_{12} & \ldots & -a_{1N} \\ -a_{21} & \sum_{j=1}^N a_{2j} & \ldots & -a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N1} & -a_{N2} & \ldots & \sum_{j=1}^N a_{Nj} \end{bmatrix}. \qquad (1)$$

If G is undirected then $L(G) = L(G)^T$ and $\lambda_1(G)$ (the least eigenvalue of $L(G)$) equals to 0, thus $L(G) \geq 0$ [1]. The second eigenvalue $\lambda_2(G)$ is called the *algebraic connectivity* of G and may be defined by [29]:

$$\lambda_2(G) = N \min_z \frac{\sum_{i,j=1}^N a_{ij}(G)(z_j - z_i)^2}{\sum_{i,j=1}^N (z_j - z_i)^2}. \qquad (2)$$

The minimum in (2) is over the set of all $z \in \mathbb{R}^N$ with $z_k \neq z_j$ for some j, k. One has $\lambda_2(G) > 0$ if and only if the undirected graph G is connected. Moreover, denoting by $e(G) \geq 0$ the minimal number of arcs one has to delete to break the graph connectivity, the inequality holds [29]

$$\lambda_2(G) \geq 2e(G)\left(1 - \cos\frac{\pi}{N}\right). \tag{3}$$

Following [1], we define the algebraic connectivity of a *directed* graph G by (2) and denote it as $\lambda_2(G)$ despite it is no longer eigenvalue of $L(G)$. Since $a_{jk}(G) + a_{kj}(G) \geq a_{jk}(\hat{G}) = a_{kj}(\hat{G})$ and thus $\lambda_2(G) \geq \frac{1}{2}\lambda_2(\hat{G})$, for a strongly connected graph $G \in \mathbb{G}_N$ one has

$$\lambda_2(G) \geq e(\hat{G})\left(1 - \cos\frac{\pi}{N}\right) \geq 1 - \cos\frac{\pi}{N}. \tag{4}$$

The estimates (3),(4) and many others [30] may be useful whenever precise computation of λ_2 is complicated (e.g. the graph is unknown or its size is too large for straightforward computation of the algebraic connectivity).

III. Problem formulation.

Throughout the paper we deal with a team of identical agents, indexed 1 through $N \geq 2$ and governed by a common MIMO state-space model

$$\dot{x}_j(t) = Ax_j(t) + Bu_j(t),\; y_j(t) = Cx_j(t). \tag{5}$$

Here $t \geq 0$, $x_j \in \mathbb{R}^d$, $u_j \in \mathbb{R}^m$, $y_j \in \mathbb{R}^k$ are the state, control, and output of the j-th agent, respectively. The model (5) is assumed to be controllable and observable. The control inputs are results of interactions (via communication, mechanical links or otherwise) between the agents. The interaction topology is time-varying and at time $t \geq 0$ is described by a graph $G(t) = \left[V_N, E(t)\right] \in \mathbb{G}_N$: the output $y_k(t)$ of k-th agent exerts influence on j-th one the if and only if $(k, j) \in E(t)$.

Specifically, we examine distributed control protocols of the following type:

$$u_j(t) = \sum_{k:(k,j)\in E(t)} \varphi_{jk}(t, y_k(t) - y_j(t)). \tag{6}$$

The mappings $\varphi_{jk} : [0; +\infty) \times \mathbb{R}^k \to \mathbb{R}^m$ are referred as *couplings* and determine interaction "strength" between the agents. The aim of the paper is to give conditions under which such a protocol establishes consensus among the agents in the following sense.

Definition 1 *The protocol (6)* provides the output consensus *if the following two claims hold for all initial states* $(x_j(0))$ *and some constant* $M > 0$*:*

$$\lim_{t\to+\infty} |y_j(t) - y_k(t)| = 0 \quad \forall k, j; \tag{7}$$

$$\mathcal{W}(t) := \sum_{j\neq k} |x_j(t) - x_k(t)|^2 \leq M\mathcal{W}(0)\; \forall t \geq 0. \tag{8}$$

If additionally $\mathcal{W}(t) \to 0$ *as* $t \to +\infty$*, we say that the protocol* (6) *provides* the state consensus. *We say that the* exponential state consensus *is established, if* $\mathcal{W}(t) \leq Me^{-\alpha t}\mathcal{W}(0)$ *for some constants* $M, \alpha > 0$.

The state consensus implies the output consensus. The converse is true under non-restrictive in practice condition of uniform continuity at zero point.

Remark 2 The output consensus implies the output consensus if $\lim_{y\to 0} \sup_{t\geq 0} |\varphi_{jk}(t, y)| = 0$ for any j, k.

Proof. Due to (6) and (7), the output consensus and the assumption $\lim_{y\to 0} \sup_{t\geq 0} |\varphi_{jk}(t, y)| = 0$ imply that $|u_j(t)| \to 0$ as $t \to +\infty$. Taking by definition $X_{jk} := x_k - x_j$, $U_{jk} := u_k - u_j$, and $Y_{jk} := y_k - y_j$, one obtains $\dot{X}_{jk} = AX_{jk} + BU_{jk}$ and $Y_{jk} = CX_{jk}$, where the pairs (A, B) and (A, C) are controllable and observable, respectively, $U_{jk}(t) \to 0$ and $Y_{jk}(t) \to 0$ as $t \to +\infty$. Thus $X_{jk}(t) \to 0$ for any k, j q.e.d. □

IV. Main Assumptions

The main assumptions basically come to the connectivity of the interaction topology (Assumption 3), the symmetry or balance condition on the couplings (Assumption 4) and the sector conditions for the couplings. We start with the assumption about the underlying graph.

Assumption 3 *The graph $G(t)$ is strongly connected for all $t \geq 0$. The function $G(\cdot)$ is Lebesgue measurable, i.e. $G^{-1}(\Gamma) \subset \mathbb{R}$ is a measurable set for any $\Gamma \in \mathbb{G}_N$.*

Maintaining connectivity (in some sense) is clearly necessary to prevent the agents from dissemination into separate clusters that do not interact and thus cannot be synchronized.

Our next assumptions concern the couplings φ_{jk}.

Assumption 4 *The closed-loop system (5), (6) has a solution defined for all $t \geq 0$ for any initial states $x_j(0)$, and at least one of the following two statements holds whenever $j \neq k$ and $t \geq 0$:*
a) The graph $G(t)$ is undirected and $\varphi_{jk}(t, y) = -\varphi_{kj}(t, -y)$;
b) The couplings are linear $\varphi_{jk}(t, y) = w_{jk}(t)y$ and the gains $w_{jk}(t) \in \mathbb{R}^{k\times m}$ satisfy the balance condition

$$\sum_{k:(k,j)\in E(t)} w_{jk}(t) = \sum_{k:(j,k)\in E(t)} w_{kj}(t)\; \forall j. \tag{9}$$

The symmetry condition from a) resembles the Newtons Third Law for couplings (since $\varphi_{jk}(t, y_k - y_j) = -\varphi_{kj}(t, y_j - y_k)$), whereas the balance condition is similar in spirit to the mass or energy preservation law (the summary inflow at the node equals the cumulative outflow). Moreover, in some applications, e.g. in oscillator networks [9, 11, 12] Assumption 4 holds due to exactly these laws.

Although in general the output consensus condition (7) says nothing about the asymptotics of individual outputs y_j, under Assumption 4 the condition (7) implies the *average output consensus* [1]: $y_j(t) - Ce^{tA}\tilde{x}_0 \to 0$ as $t \to \infty$ for all j, where $\tilde{x}_0 = \frac{1}{N}\sum_{j=1}^N x_j(0)$. To establish this, notice that $\sum_{j=1}^N u_j = 0$ due to Assumption 4. By summing up the equations from (5), we see that $\sum_{j=1}^N y_j(t) = Ce^{tA}\tilde{x}_0$, thus our claim is evident from (7).

In the present paper we focus on the case when the couplings may be unknown but satisfy some *quadratic constraint* [25]. In other words, $\varphi_{jk} \in \mathfrak{S}(\mathcal{F})$ for all j, k where $\mathcal{F} : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$ is a Hermitian form and $\mathfrak{S}(\mathcal{F})$ stands for the set of functions $\varphi : [0; +\infty) \times \mathbb{R}^k \to \mathbb{R}^m$ such that the claims hold:

i) $\varphi(t, 0) \equiv 0$ and $\varphi(t, y)$ is measurable in t for all y and continuous in y for almost all t;

ii) For any compact subset $K \subset \mathbb{R} \setminus \{0\}$,

$$\inf_{y\in K, t\geq 0} \mathcal{F}(\varphi(t, y), y) > 0. \tag{10}$$

In particular, the graph of $\varphi(t, \cdot)$ lies strictly in the cone $\mathcal{F}(\varphi, y) > 0$ except for the origin.

Thus the consensus criterion should be given in terms of $\mathcal{F}$ and the coefficients A, B, C, but not the couplings themselves. Such a criterion automatically ensures consensus for all couplings from the class $\mathfrak{S}(\mathcal{F})$ that satisfy Assumption 4. We note that one of the simplest and widespread class of quadratically constrained function is constituted by scalar-valued functions satisfying the conventional sector condition with given slopes [25, 26], see paragraph 5.2 for details.

V. Main results.

In the paragraph 5.1 below the main result of the paper is presented which establishes sufficient conditions for consensus in general MIMO case. Those conditions are especially transparent and convenient to use in the scalar case ($k = m = 1$) when the quadratic constraint represents the sector inequality. This particular case is subject of the paragraph 5.2

5.1. Consensus criterion for the MIMO agents case.

We start with introducing some auxiliary notations. Let $W_x(\lambda) = (\lambda I - A)^{-1}B$ and $W_y(\lambda) = CW_x(\lambda)$ be the transfer matrices of the plant (5) from u to y, x respectively and θ stand for the minimal algebraic connectivity of the interaction topology:

$$\theta = \min_{t\geq 0} \lambda_2\big(G(t)\big) \tag{11}$$

Given a Hermitian form $\mathcal{F}(u, y)$ as follows

$$\mathcal{F}(\varphi, y) = Re(\varphi^* q y) - y^* Q y - \varphi^* R \varphi, \tag{12}$$

(where $y \in \mathbb{C}^k$, $\varphi \in \mathbb{C}^m$ and $q \in \mathbb{R}^{m\times k}$, $Q = Q^* \in \mathbb{R}^{k\times k}$, $R = R^* \in \mathbb{R}^{m\times m}$) and a matrix $\Lambda \in \mathbb{C}^{k\times m}$, let

$$\Pi_{\mathcal{F}}(\Lambda) = q\Lambda + \Lambda^* q^* + \theta\Lambda^* Q\Lambda + \frac{1}{2(N-1)}R. \tag{13}$$

The following theorem is the main result of the paper.

Theorem 5 *Suppose that Assumptions 3 and 4 hold and $\varphi_{jk} \in \mathfrak{S}(\mathcal{F})\ \forall j, k$ where $\mathcal{F}$ is a quadratic form (12) with $Q \geq 0, \Gamma \geq 0$. Assume also that*

(i) *There exists matrix $K \in \mathbb{R}^{m\times k}$ such that $A - NBKC$ is Hurwitz and $\mathcal{F}(Ky, y) > 0\ \forall y \neq 0$;*

(ii) *for any frequency $\omega \in \mathbb{R}$ such that $\det(i\omega I - A) \neq 0$, the following inequality is true*

$$\Pi_{\mathcal{F}}(\Lambda) \geq 0 \text{ for } \Lambda := W_y(i\omega). \tag{14}$$

Then the protocol (6) provides the output consensus. Furthermore, if $\varepsilon > 0$ exists such that $\Pi(i\omega) \geq \varepsilon|W_x(i\omega)|^2$ for any ω, then the exponential state consensus is established.

The proof of this theorem will be given in Section VII.

By Theorem 9, conditions (a) and (b) in fact imply the robust consensus in the sense that (7) holds for all couplings $\varphi_{jk} \in \mathfrak{S}(\mathcal{F})$ and time-varying interaction graphs $G(t)$ satisfying Assumptions 3 and 4. In practice (a) is almost unavoidable for such a robust consensus. The existence of a matrix K such that the map $y \mapsto Ky$ belongs to $\mathfrak{S}(\mathcal{F})$ is typically a non-restrictive assumption which is fulfilled, for instance, for scalar case and sector inequalities (see paragraph 5.2). The following remark shows that all of those matrices, if exist, should satisfy (a).

Remark 6 Under the robust consensus, (a) holds with any $K \in \mathbb{R}^{m\times k}$ such that $\mathcal{F}(Ky, y) > 0\,\forall y \neq 0$.

Indeed, the output consensus for $G(t) \equiv G_0$ with complete graph G_0 and $\varphi_{jk}(t,y) = Ky$ implies the state consensus by Remark 2. So for a nonzero eigenvalue λ of the Laplacian $L(G_0)$, the matrix $A - \lambda BKC$ is Hurwitz [6]. It remains to note that $\lambda = N$ for the complete graph.

Remark 7 *In the conditions for consensus given by Theorem 5, the underlying topology of the network is concerned only by the multiplier θ in (13). Since by assumption $Q \geq 0$, formal replacement of θ by its lower estimate retains sufficiency for consensus. This observation is useful whenever the exact computation of θ is complicated. Then many available constructive estimates of the algebraic connectivity, like (3) or (4), may be used; we refer the reader to [30] for their survey. Moreover, the value of θ is unimportant if $Q = 0$.*

5.2. Consensus for scalar sectorial couplings.

In this paragraph we interpret the result of Theorem 5 for SISO agents and scalar couplings which satisfy *the sector inequalities* with known slopes [25, 26]. In other words, $k = m = 1$ and $\varphi_{jk} \in S[\alpha;\beta]$ where $S[\alpha;\beta]$ is a set of functions $\varphi : [0;+\infty) \times \mathbb{R} \to \mathbb{R}$ such that $\varphi(t,0) \equiv 0$, $\varphi(t,\sigma)$ is measurable in t for all σ, continuous in σ for almost all $t \geq 0$ and

$$\alpha < \inf_{\sigma \in K, t \geq 0} \frac{\varphi(t,\sigma)}{\sigma} \leq \sup_{\sigma \in K, t \geq 0} \frac{\varphi(t,\sigma)}{\sigma} < \beta, \qquad (15)$$

for any compact set $K \subset \mathbb{R} \setminus \{0\}$. It follows from (15) that the graph of $\varphi(t,\cdot)$ lies strictly between the lines $\xi = \alpha\sigma$ and $\xi = \beta\sigma$ everywhere except for the origin.

We note that if $\alpha < 0$ and $\beta > 0$, the protocol (6) obviously does not provide consensus unless A is a Hurwitz matrix, although all assumptions are satisfied. Since we are mostly interested in agents with unstable open-loop dynamics, we exclude the case $\alpha < 0, \beta > 0$ from consideration, thus focusing on the cases where either $0 \leq \alpha < \beta$ or $\alpha < \beta \leq 0$. Moreover, since the second of them is reduced to the first one by the substitution $(\alpha, \beta, B) \mapsto (-\beta, -\alpha, -B)$, we shall consider only the first case.

We first transform the inequalities (15) into quadratic constraint. We introduce the constants

$$\gamma = \frac{1}{\beta + \alpha} \geq 0, \quad \delta = \frac{\alpha}{1 + \alpha\beta^{-1}} \geq 0, \qquad (16)$$

and the quadratic forms as follows:

$$\mathcal{F}_{\alpha;\beta}(\varphi, y) := \varphi y - \delta y^2 - \gamma\varphi^2, \; \varphi, y \in \mathbb{R} \qquad (17)$$

$$\Pi_{\alpha;\beta}(\lambda) = Re\,\lambda + \theta\delta|\lambda|^2 + \frac{\gamma}{2(N-1)}, \; \lambda \in \mathbb{C}. \qquad (18)$$

Proposition 8 *$S[\alpha;\beta] = \mathfrak{S}(\mathcal{F}_{\alpha;\beta})$ and $\Pi_{\mathcal{F}_{\alpha;\beta}} \equiv \Pi_{\alpha;\beta}$.*

Indeed, the sector inequalities (15) may be rewritten as $(\varphi - \alpha y)(y - \beta^{-1}\varphi) = (1 + \alpha\beta^{-1})\mathcal{F}_{\alpha;\beta}(\varphi, y) > 0$, where $\varphi := \varphi(t,y)$ and the latter inequality is uniform in $t \geq 0$ and $y \in K$ (for any compact $K \subset \mathbb{R} \setminus \{0\}$). The second claim is easily seen from (13),(18). □

The proposition 8 allows one to obtain the following consensus criterion for SISO agents case.

Theorem 9 *Suppose that Assumptions 3 and 4 are satisfied, $\varphi_{jk} \in S[\alpha;\beta]\ \forall j,k$ where $0 \leq \alpha < \beta \leq \infty$, and the following two claims hold:*

(a) *There exists $\mu \in (\alpha;\beta)$ such that the matrix $A - \mu NBC$ is Hurwitz;*

(b) *for any frequency $\omega \in \mathbb{R}$ such that $\det(i\omega I - A) \neq 0$, the following inequality is true*

$$\Pi_{\alpha;\beta}(\lambda) \geq 0 \text{ for } \lambda := W_y(i\omega). \qquad (19)$$

Then the protocol (6) provides the output consensus. Moreover, if $\varepsilon > 0$ exists such that $\Pi_{\alpha;\beta}(W_y(i\omega)) \geq \varepsilon|W_x(i\omega)|^2$, the exponential state consensus is established.

This result immediately follows from Theorem 5 (applied for $\mathcal{F} := \mathcal{F}_0$, $R := \gamma$, $Q := \delta$) and Proposition 8. Indeed, (a) coincides with the assumption (i) from Theorem 5 and (b) is nothing more than (ii) from the same theorem.

Inequality (19) is non-trivial since the Hermitian form $\Pi_{\alpha;\beta}$ is not non-negative definite: its discriminant $\frac{1}{4}\left[\frac{2\delta\gamma\theta}{N-1} - 1\right] \leq 0$ since $\theta \leq N$ by (2) and $\delta\gamma \leq 1/4$.

Remark 10 Condition (b) means that the Nyquist curve $\{W_y(i\omega)\}$ lies outside the set $\mathcal{D}$ which is the open half-plane $\{z \in \mathbb{C} : Re\,z < -\frac{\gamma}{2(N-1)}\}$ for $\alpha = 0$ and the disk $\mathcal{D} = \{z \in \mathbb{C} : |z - z_0| < \rho_0\}$ for $\alpha > 0$,

$$z_0 = -\frac{1}{2\delta\theta}, \quad \rho_0 = \frac{1}{2\delta\theta}\sqrt{1 - \frac{2\theta\delta\gamma}{N-1}}.$$

Here the first claim is trivial since $\alpha = 0 \Rightarrow \delta = 0$ by (16). The second claim is valid since $Re\,z + \delta\theta|z|^2 + \frac{\gamma}{2(N-1)} \geq 0 \Leftrightarrow |z|^2 - 2z_0\,Re\,z + |z_0|^2 - \rho_0^2 \geq 0 \Leftrightarrow z \notin \mathcal{D}$.

The geometrical interpretation given by Remark 10 highlights the similarity between the conditions presented by Theorem 9 and the celebrated circle criterion for stability of Lurie systems [25, 26]. Theorem 9 can be viewed as extension of this criterion on consensus in multi-agent systems in the following sense. Consider two agents ($N = 2$) applying the protocol (6) with two-directional communication (E(t)={(1,2),(2,1)}) and the coupling functions $\varphi_{12} = -\varphi_{21} \in S[\alpha; \beta]$. Taking $X(t) = x_2(t) - x_1(t)$, $Y(t) = y_2(t) - y_1(t)$, $U(t) = u_2(t) - u_1(t)$ and $\Phi(t, Y) = -2\varphi_{12}(t, Y)$, we have $\Phi \in S[-2\beta; -2\alpha]$ and

$$\dot{X}(t) = AX(t) + BU(t), U(t) = \Phi(t, Y(t)). \quad (20)$$

By the circle criterion [25] the equilibrium $X = 0$ of the system (20) is exponentially stable if $A + kBC$ is Hurwitz for some $k \in [-2\beta; -2\alpha]$ (which follows from (a) since $\theta = 2$) and the Nyquist curve $\{W_y(i\omega) : \omega \in \mathbb{R}\} \subset \mathbb{C}$ lies strictly outside the disk based on the diameter $[(-2\alpha)^{-1} + 0i; (-2\beta)^{-1} + 0i]$ (or the half-plane $\{z : Re\, z < (-2\beta)^{-1}\}$ in the case of $\alpha = 0$). This disk (or half-plane) coincides with the set $\mathcal{D}$ introduced above.

VI. Illustrative examples

Now we illustrate that Theorems 5,9, applied to agents with special dynamics, provide improvements of recent results in the area, as well as some new results.

6.1. Consensus among passive agents

Let the transfer matrix W_y be square (thus $k = m$) and satisfy the positive realness condition:

$$W_y(i\omega) + W_y(i\omega)^* \geq 0 \qquad \forall \omega \in \mathbb{R}. \quad (21)$$

As is well known, this implies their *passivity* [22, 26] provided that (a) from Theorem 9 holds. Consensus and synchronization of passive systems have earned a considerable interest; see e.g. [22, 23] among the others. Most of the related results deal with nonlinear agents but the topology is either time-invariant or with positive dwell time, and the couplings are time-invariant. Though this paper is concerned with only linear agents, on the positive side, it considers the most general case of measurable time-varying interaction topology and admits non-stationary couplings.

Taking the quadratic form $\mathcal{F}(\varphi, y) := \varphi^T y$ and observing that (21) implies (14) for any θ and $\mathcal{F}(Ky, y) = \frac{1}{2} y^T (K + K^T) y/2 \quad \forall K \in \mathbb{R}^{m\times m}$, we obtain the following corollary of Theorem 5.

Theorem 11 *Let Assumptions 3 and 4 be valid, (21) hold, $\varphi_{jk} \in \mathfrak{S}(\mathcal{F})$ with $F(\varphi, y) = \varphi^T y$, and there exists matrix $K \in \mathbb{R}^{m\times m}$ such that $K + K^T > 0$ and $A - BKC$ is Hurwitz. Then the protocol (6) establishes the output consensus.*

In the scalar case the result of Theorem 11 becomes especially simple.

Theorem 12 *Let $k = m = 1$, Assumptions 3 and 4 be valid, (21) hold, $\varphi_{jk} \in S[0; +\infty]$, and $A - \mu BC$ be Hurwitz for some $\mu > 0$. Then the protocol (6) establishes the output consensus.*

Theorem 12 generalizes the results of [22, 18, 2], etc., which concern networks of the first-order integrators $\dot{y}_j = u_j \in \mathbb{R}$, as well as those of [11, 1], which deal with networks of identical Kuramoto oscillators with initial phases from $(-\pi/2; \pi/2)$. Another example is the result of [12]. It concerns a network of identical harmonic oscillators $\dot{q}_j = y_j \in \mathbb{R}, \dot{y}_j = -\omega_0^2 q_j + u_j$ and asserts that the y-output consensus is established by the linear balanced protocol (6) (Assumption 4b holds) with constant gains $w_{jk} > 0$. Theorem 12 generalizes this result on time-varying gains and nonlinear couplings. Indeed, $W_y(z) = z/(z^2 - \omega_0^2)$ and thus $W_y(i\omega) + W_y(i\omega)^* = 2\, Re\, W_y(i\omega) = 0$, and it is obvious that the feedback $u_j = -\mu y_j$ stabilizes the j-th agent for any $\mu > 0$.

6.2. Consensus among strictly passifiable agents

A natural generalization of passivity is *passifiability*: a linear agent is called *passifiable* if it may be made passive by an appropriate linear feedback. This is implied by the *strict passifiability* [27]. The problem of consensus among passifiable agents seems to have been unexplored up to now in the literature. To simplify matters, we bound ourselves with SISO agents only: $k = m = 1$. Under such an assumption, strict passifiability of the agent means that $CB > 0$ and the polynomial $\psi(\lambda) = \det(\lambda I - A) W_y(\lambda)$ is Hurwitz (i.e., the agent is minimum-phase).

The following theorem shows that such consensus holds whenever the couplings are strong enough.

Theorem 13 *Suppose that the agents are strictly passifiable, Assumptions 3 and 4 hold, and $\varphi_{jk} \in S[\alpha; +\infty]$, where for $\theta = \min_{t\geq 0} \lambda_2(G(t))$ one has*

$$\alpha \geq \alpha_* := \frac{1}{\theta} \sup_{\omega\in\mathbb{R}} f(\omega),\ f(\omega) := -\frac{Re\, W_y(i\omega)}{|W_y(i\omega)|^2}. \quad (22)$$

Then the output consensus is provided. Moreover, if $\alpha > \alpha_*$ and $\det(i\omega I - A) \neq 0$ for any $\omega \in \mathbb{R}$, then the exponential state consensus is provided.

The sup in (22) is finite since $f(\omega) = |\psi(i\omega)|^{-2}(\psi(i\omega)\chi(-i\omega) + \psi(-i\omega)\chi(i\omega))$ with

$\chi(z) := \det(zI - A)$ is well defined and continuous for all $\omega \in \mathbb{R}$, and $f(\omega) \to -CAB/(CB)^2$ as $\omega \to \pm\infty$. The quantity α_* is the coupling 'strength' threshold above which the consensus is necessarily achieved.

The first claim of Theorem 13 is immediate from Theorem 9 with $\beta = \infty$ (which yields that $\delta = \alpha, \gamma = 0$) since (a) of Theorem 9 is valid for sufficiently large μ thanks to the strict passifiability [27], and (19) holds due to (22). Moreover, $\Pi_{\alpha;\infty}(W_y(i\omega)) \geq (\alpha - \alpha_*)\theta|W_y(i\omega)|^2$. Since $CB > 0$ and $\det(i\omega I - A) \neq 0$, a constant $\xi > 0$ exists such that $|W_y(i\omega)| \geq \xi |W_x(i\omega)|$ for all $\omega \in \mathbb{R}$. This proves the second claim since $\Pi_{\alpha;\infty}(W_y(i\omega)) \geq \varepsilon |W_x(i\omega)|^2$ for $\varepsilon := \xi^2\theta(\alpha - \alpha_*)$.

6.3. Consensus among double-integrator agents.

Consensus problem for networks of double integrators have recently attracted considerable interest because of various applications to multi-vehicle formation control; see e.g., [2, 5, 24] among the others. In this case, the j-th agent is described by the following equations

$$\ddot{z}_j = u_j,\; y_j = q_0 z_j + q_1 \dot{z}_j, \tag{23}$$

where y_j is the output, and $q_0, q_1 \in \mathbb{R}$ are constants. So $W_y(i\omega) = q_0(i\omega)^{-2} + q_1(i\omega)^{-1}$.

It is easy to see that the agent is strictly passifiable if $q_0, q_1 > 0$ and that (19) shapes into $Kq_0^2\omega^{-4} + (Kq_1^2 - q_0)\omega^{-2} \geq 0$, where $K := \alpha\theta$. This permits us to apply Theorem 13, which gives rise to the following.

Corollary 14 *Suppose that the agents are described by equations (23) with $q_0, q_1 > 0$, Assumptions 3 and 4 hold, and $\varphi_{jk} \in S\left[\theta^{-1}q_1^{-2}q_0; +\infty\right]$, where $\theta := \inf_{t\geq 0} \lambda_2(G(t))$. Then the protocol (6) establishes the output consensus.*

It should be noticed that usually relative position and velocities $z_j - z_k$ and $\dot{z}_j - \dot{z}_k$ are measured rather than their linear combinations $y_j - y_k$. By choosing appropriate $q_0, q_1 > 0$, one may ensure establishing of the consensus for the coupling class $S[\alpha; +\infty]$ with arbitrary small $\alpha > 0$. However, to find q_0, q_1 an estimate of θ from below is required, moreover, Corollary 14 does not work for saturation-like bounded nonlinearities (that does not belong to $S[\alpha; \beta]$ unless $\alpha = 0$). These shortages may be overcome if absolute velocities $\dot{z}_j$ can be measured. In this case, the modified protocols may be applied [31]:

$$u_j = -p\dot{z}_j + \sum_{(k,j)\in E(t)} \varphi_{jk}(t, y_k(t) - y_j(t)). \tag{24}$$

The closed-loop system (23),(24) is the same as if the standard protocol (6) were applied to modified agents

$$\ddot{z}_j + p\dot{z}_j = u_j,\; y_j = q_0 z_j + q_1 \dot{z}_j. \tag{25}$$

Notice that if $p > 0$, $q_0 > 0, q_1 \geq 0$ then feedback $u_j = -\mu y_j$ stabilizes the plant (25) for any $\mu > 0$. Since $W_y(i\omega) = (q_0 + q_1(i\omega))[(i\omega)^2 + p(i\omega)]^{-1}$ and $Re\, W_y(i\omega) = (q_1 p - q_0)(p^2 + \omega^2)^{-2}$ for the agent (25), it is easily seen that (19) holds for $\alpha = \theta = 0$ if

$$\inf_{\omega\in\mathbb{R}} Re\, W_y(i\omega) = \min\left(\frac{q_1 p - q_0}{p^2}, 0\right) \geq -\frac{\beta^{-1}}{2(N-1)}.$$

Theorem 9 implies the following result:

Corollary 15 *Suppose that the agents are described by equations (23) with $q_1 \geq 0$, $q_0 > 0$, Assumptions 3 and 4 hold, $p > 0$ and $\varphi_{jk} \in S\,[0; \beta]$ with $\beta \leq +\infty$. Suppose also that $p^2\beta^{-1} + 2(N-1)(q_1 p - q_0) \geq 0$. Then the protocol (24) establishes the output consensus.*

Corollary 15, in particular, guarantees consensus for $q_1 p \geq q_0$ and $\beta = +\infty$ (which agrees with Theorem 12 as agents (25) are passive) and for $q_1 = 0$, $\beta \leq p^2/(2q_0(N-1))$ (the relative velocities are not used at all). Also it is easily noticed from (25) that in assumptions of Remark 2 the protocol (24) provides condition $\dot{z}_j(t) \to 0$ besides the state consensus.

VII. Proof of Theorem 5

Outline of the proof of Theorem 5 is as follows. Using individual quadratic constraints for the couplings (10), we derive the "global" quadratic constraint on the solutions y_j, u_j of the closed loop system (5),(6) (Lemma 16). This enable us to apply the Kalman-Yakubovich-Popov (KYP) lemma and establish existence of a quadratic Lyapunov function (Lemma 17). Theorem 5 is proved by means of this function.

Throughout the section, the assumptions of Theorem 5 are assumed to be valid. In particular, the Hermitian form $\mathcal{F}$ defined by (12) is given such that $\varphi_{jk} \in \mathfrak{S}(\mathcal{F})$. Let $\bar{\xi} := (\xi_1^T, \ldots, \xi_N^T)^T$ for any sequence of column vectors or scalars $\xi_1, \xi_2, \ldots, \xi_N$.

To start with, we introduce the Hermitian forms

$$P(y,u) = -Re(u^* q y) - \theta y^* Q y - \frac{1}{2(N-1)} u^* R y,$$

$$\mathfrak{P}(\bar{y}, \bar{u}) = \sum_{j,k=1}^{N} P(y_j - y_k, u_j - u_k).$$

Lemma 16 *For each solution of (5), (6) one has $\mathfrak{P}(\bar{y}(t),\bar{u}(t)) \geq 0\,\forall t \geq 0$. Moreover, for any $\nu > \mu > 0$, there exists $\rho > 0$ such that $\mathfrak{P}(\bar{y}(t),\bar{u}(t)) > \rho$ whenever $\nu > \max_{1\leq j,k\leq N} |y_k(t) - y_j(t)| > \mu$.*

Proof. We put $\xi_{jk}(t) := a_{jk}\big(G(t)\big)\varphi_{jk}\big(t, y_k(t) - y_j(t)\big)$ and $\eta_{jk}(t) := a_{jk}\big(G(t)\big)\big[y_k(t) - y_j(t)\big]$. By (6), $u_j(t) = \sum_{k=1}^N \xi_{jk}(t)$, where $\xi_{jj} := 0$. Therefore

$$|u_j(t)|^2 \leq (N-1)\sum_{k=1}^N |\xi_{jk}(t)|^2 \tag{26}$$

by the Cauchy-Schwartz inequality, and

$$\sum_{j,k=1}^N \xi_{jk}^T q y_j = \sum_{j=1}^N u_j^T q y_j. \tag{27}$$

Our next objective is to prove that

$$\sum_{j,k=1}^N \xi_{jk}^T q y_k = -\sum_{j=1}^N u_j^T q y_j. \tag{28}$$

Under (a) in Assumption 4, this is immediate from (27) and the equations $\xi_{jk} = -\xi_{kj}$. Under (b), one may put $w_{jk}(t) := 0$ if $a_{jk}(G(t)) = 0$ and notice that due to (9)

$$\sum_{j,k=1}^N y_k^T w_{jk}^T q y_k = \sum_{j,k=1}^N y_k^T \frac{w_{jk}^T q + q^T w_{jk}}{2} y_k =$$
$$= \frac{1}{2}\sum_{j,k=1}^N y_k^T w_{jk}^T q y_k + \frac{1}{2}\sum_{j,k=1}^N y_k^T q^T w_{kj} y_k \text{ and}$$
$$\sum_{j,k=1}^N y_j^T w_{jk}^T q y_k = \sum_{j,k=1}^N y_j^T \frac{w_{jk}^T q + q^T w_{jk}}{2} y_k =$$
$$= \frac{1}{2}\sum_{j,k=1}^N y_j^T w_{jk}^T q y_k + \frac{1}{2}\sum_{j,k=1}^N y_k^T q^T w_{kj} y_j.$$

Subtracting the above equalities, one obtains that

$$\sum_{j,k=1}^N \xi_{jk}^T q y_k = \sum_{j,k=1}^N (y_k - y_j)^T w_{jk}^T q y_k =$$
$$= \frac{1}{2}\sum_{j,k=1}^N \left[(y_k - y_j)^T w_{jk}^T q y_k + y_k^T q^T w_{kj}(y_k - y_j)\right] =$$
$$= \frac{1}{2}\sum_{j,k=1}^N \xi_{jk}^T q y_k - \frac{1}{2}\sum_{j,k=1}^N y_k^T q^T \xi_{kj},$$

which leads finally to the desired claim (28):

$$\sum_{j,k=1}^N \xi_{jk}^T q y_k = -\sum_{j,k=1}^N y_k^T q^T \xi_{kj} \overset{(27)}{=} -\sum_{k=1}^N u_k^T q y_k.$$

By definition of $\mathfrak{S}(\mathcal{F})$, $\mathcal{F}(\varphi_{jk}(t,y),y) \geq \varkappa(\mu,\nu) \geq 0$ whenever $0 \leq \mu \leq |\sigma| \leq \nu < \infty$, where $\varkappa(\mu,\nu) > 0$ if $\mu > 0$. By putting $y = y_k - y_j$, $a_{jk}(t) := a_{jk}(G(t))$ and elementary transformation, we se that

$$\overbrace{\xi_{jk}^T q(y_k - y_j)}^{a} - \overbrace{a_{jk}(t)(y_k - y_j)^T Q(y_k - y_j)}^{b} - \overbrace{\xi_{jk}^T R \xi_{jk}}^{c}$$
$$\geq a_{jk}(t)\varkappa(\mu_{kj},\nu_{kj})$$

whenever $\mu_{kj} \leq |y_k - y_j| \leq \nu_{kj}$. Here $a = -2\sum_{j=1}^N u_j^T q y_j$ by (27), (28). From (2) and $Q \geq 0$ it is easily found that $b = \sum_{j,k=1}^N a_{jk}(t)\left|Q^{1/2}(y_k - y_j)\right|^2 \geq N^{-1}\theta\sum_{j,k=1}^N (y_k - y_j)^T Q(y_k - y_j)$, and $c \geq (N-1)^{-1}\sum_{j=1}^N u_j^2$ by (26). Hence $N^{-1}\mathfrak{P}(\bar{y}(t),\bar{u}(t)) \geq \sum_{j,k} a_{jk}(t)\varkappa(\mu_{kj},\nu_{kj})$. By taking here $\mu_{jk} := 0$ and $\nu_{jk} := |y_j - y_k|$, we arrive at the first claim of the lemma. To prove the second one it suffices to note that whenever $\nu \geq \max_{j,k} |y_k - y_j| \geq \mu$, first, $|y_k - y_j| \leq \nu_{jk} := \nu\ \forall j,k$ and second, $|y_k - y_j| \geq \mu_{jk} := \mu/(2N)$ for at leat one arc (j,k), whereas we put $\mu_{jk} := 0$ for all other pairs (j,k). □

Lemma 17 *Suppose that $\Pi_{\mathcal{F}}(W_y(i\omega)) \geq \varepsilon|W_x(i\omega)|^2$ for some $\varepsilon \geq 0$ whenever $\omega \in \mathbb{R}$ and $\det(i\omega I - A) \neq 0$. Then there exists a symmetric matrix $H = H^T > 0$ such that $\dot{V}(\bar{x}(t)) + \mathfrak{P}(\bar{y}(t),\bar{u}(t)) \leq -\varepsilon\mathcal{W}(\bar{x}(t))$ for any solution of (5), where $V(\bar{x}) := \sum_{j,k=1}^N (x_k - x_j)^* H(x_k - x_j)$ and $\mathcal{W}(\bar{x})$ is from (8).*

Proof. It is easy to see by direct computation of $\dot{V}$ that $\dot{V}(\bar{x}) + \mathfrak{P}(\bar{y},\bar{u}) + \varepsilon\mathcal{W}(\bar{x}) \leq 0$ follows from the LMI $2x^*H(Ax + Bu) + P(Cx,u) + \varepsilon|x|^2 \leq 0\ \forall x,u$. By the KYP lemma (see [25, 32], the latter has a symmetric solution $H = H^T$ if and only if $P(C(i\omega I - A)^{-1}Bu, u) \overset{(13)}{=} -\Pi_{\mathcal{F}}(W_y(i\omega)) \leq -\varepsilon|W_x(i\omega)|^2$ for any $u \in \mathbb{C}^m$ and $\omega \in \mathbb{R}$ (such that $\det(i\omega I - A)^{-1} \neq 0$). Since the last requirement is nothing but (14), we see that the matrix H exists. We are going to show its positive definiteness.

Let (x_j^+, u_j^+, y_j^+) be the solution of (5), (6), where $G(t) \equiv G_0$, the graph G_0 is complete, $\varphi_{jk}(t,y) = Ky$, and K is taken from (i) of Theorem 5. By Lemma 16, $\mathfrak{P}(\bar{y}^+(t),\bar{u}^+(t)) \geq 0$, whereas $\lim_{t\to+\infty} |x_j^+(t) - x_k^+(t)| = 0$ due to (a). By integrating $\dot{V}(\bar{x}^+(t)) + \mathfrak{P}(\bar{y}^+(t),\bar{u}^+(t)) \leq 0$, we see that $V(\bar{x}^+(0)) \geq 0\, \forall \bar{x}^+(0) \Rightarrow H \geq 0$. Moreover, $V(\bar{x}^+(0)) = 0 \Rightarrow \mathfrak{P}(\bar{y}^+(\cdot), \bar{u}^+(\cdot)) \equiv 0 \Rightarrow y_1^+(\cdot) \equiv \ldots \equiv y_N^+(\cdot) \Rightarrow u_j(\cdot) \equiv 0$ and so $x_1^+(0) = \ldots = x_N^+(0)$ thanks to observability. Hence $H > 0$. □

Proof of Theorem 5. For the function $V(\bar{x}) = x^* H x$ from Lemma 17 (with $\varepsilon := 0$) and any solution of (5),(6),

$$0 \leq V(\bar{x}(t)) \leq V(\bar{x}(0)) - \int_0^t \mathfrak{P}(\bar{y}(\xi), \bar{u}(\xi)) d\xi, \quad (29)$$

where $\mathfrak{P}(\ldots) \geq 0$ by Lemma 16 and $H > 0$ by Lemma 17. It follows that $\mathcal{W}(\bar{x}(t)) = \sum_{j,k=1}^N |x_j(t) - x_k(t)|^2 \leq \frac{M(H)}{m(H)} \mathcal{W}(\bar{x}(0))$, where $M(H)$ and $m(H)$ are the maximal and minimal eigenvalues of H, respectively. Thus (8) does hold. To prove (7), suppose the contrary. Then there exist $\zeta > 0$ and a sequence $t_n \uparrow +\infty$ such that $\max_{j,k} |y_j(t_n) - y_k(t_n)| > 2\zeta$. Since $\sup_{t \geq 0} \|\dot{x}_j(t)\| < \infty$, a number $\Delta > 0$ exists such that $\max_{j,k} |y_j(t) - y_k(t)| > \zeta$ whenever $|t - t_n| \leq \Delta$. By Lemma 16, $\rho > 0$ exists such that $\mathfrak{P}(\bar{y}(t), \bar{u}(t)) > \rho$ if $|t - t_n| \leq \Delta$, which implies violation of (29). This contradiction proves (7). This proves the first claim of Theorem 5 which concerns the output consensus.

To prove the second claim, we again use Lemma 17 (with $\varepsilon > 0$ the same as in Theorem 5) and notice that besides (29) $\dot{V}(\bar{x}) + \mathfrak{P} \leq -\mu V$ for sufficiently small $\mu > 0$ thus $V(\bar{x}(t)) \leq M e^{-\mu t}$. Since $H > 0$, we conclude that the exponential state consensus is established.

VIII. Conclusion

Output consensus among identical high-order linear agents was examined in the case where the interaction topology is uncertain and switching but assumed to preserve its connectivity. The couplings among the agents are uncertain as well and satisfy the conventional quadratic constraint. A new criterion for robust output consensus is established which is to be extended on the leader-following formation control, reference-tracking consensus, and nonlinear agents of Lurie type during the ongoing research.

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