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Frequency-Domain Criteria for Consensus in Multiagent Systems with Nonlinear Sector-Shaped Couplings

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Abstract—Consideration was given to the distributed algorithms for consensus (synchronization) in the multiagent networks with identical agents of arbitrary order and unknown nonlinear couplings satisfying the sector inequalities or their multidimensional counterparts. The network topology may be unknown and varying in time. A frequency synchronization criterion was proposed which is a generalization of the circular criterion for absolute stability of the Lur'e systems.

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1. INTRODUCTION

The problems of *consensus* or *synchronization* (terms such as coordination, agreement, and so on are used) in the network multiagent systems over the last decade attracts attention of the experts in various areas of science and technology. The question of synchronism (consensus) between heterogeneous system parts (agents) as the result of local interactions—the agents do not have or use information about the system as a whole and interact only with a relatively small number of their "neighbors"—is of greatest interest. The so-reached synchronization underlies many natural phenomena and technical designs and is closely related also with the law of motion of complex biological and technical formations such as insect swarms, bird flocks, ensemble of interrelated oscillators, group of mobile robots, and so on. The reader can find examples of the problems of control of formations and their allied problems of multiagent control in the monographs and papers [1–10].

The basic ideas of many distributed consensus algorithms are rooted in the Markov theory of chains and theory of positive matrices, as well as in the procedures for coordinated decision making studied in the applied statistics. The detailed history of this issue can be found in the remarkable review of [11] and also in [2, 3, 12] and their references. For the time being, the consensus protocols with linear couplings between the agents are most accepted and studied. For the fixed-topology networks, attainment of synchronization is defined by the spectral properties of the network Laplacian matrix [2, 8, 12, 13]; for the agents with the integrator model, by the availability of oriented spanning tree. For the changeable network topology, it is the case of the first-order agents that is studied most thoroughly. The results obtained on convergence of the products of stochastic matrices (see [2, 11]) and the Lyapunov methods [4, 14] are the main tools of study. It deserves noting that some consensus algorithms for higher-order agents are organized similar to the first-order algorithms [2, 3] based directly on them [15]. At the same time, many applications require synchronization using nonlinear couplings between the agents. One of the important examples is offered by the networks of coupled oscillators [16, 17] where the coupling functions usually are periodic. The nonlinear couplings arise in a natural way if the communication between the mobile agents is limited by the spatial distance [1, 10, 18]. Additionally, in practice the linear algorithms can become nonlinear as the result of data distortion, quantization effects, and so on.

The results obtained on the nonlinear synchronization algorithms refer mostly to the control of special agents. A technique to study the compressing discrete-time consensus algorithms where each agent moves within a convex hull of its neighbors was developed in one of the basic papers [14]. It uses the convex hull as the multivalued Lyapunov function. Similar methods are applicable to the continuous-time agents [18]. An alternative of the Lyapunov methods is represented by the modified method of compressing maps is the [19]. We note that conceptually allied topological methods were used by V.I. Opoitsev [20] in the problem of existence and stability of equilibria in the nonlinear networks of the first-order agents. If the agents are passive control plants, another Lyapunov function equal to the full energy of the system (sum of the supply rates of individual agents) [4, 21] can be used to study convergence of the synchronization algorithms. In the case of networks of nonpassive agents with nonlinear couplings, the results on attainment of consensus refer as a rule to very specific situations such as second-order agents with hyperbolic tangent couplings [2, 3].

Therefore, the majority of the existing results on synchronization in the networks with nonlinear couplings presume either a special dynamics of the agents such as the integrators of the first or second order, passive agents, and so on or a special condition structure. In particular, the assumption of permanent topology or existence of the dwell time between of its switchings is popular. In the present paper, these constraints are appreciably relaxed. Consideration is given to the networks with arbitrary-order agents and varying topology which is not assumed to be a piecewise constant time function but just Lebesgue-measurable. It is also assumed that the topology retains connectivity. Consensus algorithms of two types are considered. In the first case, the topology is assumed to be undirected. The coupling functions can be at that nonlinear and indefinite, it is only assumed that satisfied is the quadratic constraint which in the scalar case comes to the sector inequality (see [22] and Section 3.2 below) and also the coupling symmetry condition similar to the third Newton law. Another class of the consensus protocols considered in the present paper has linear coupling functions and an directed interaction graph meeting the balance condition.

The present paper proposes a criterion for consensus under the aforementioned conditions. We note that it can be regarded as the *robust* synchronization criterion in the sense that it provides under any uncertain functions of coupling from the above class. It is a direct generalization of the well-known circular criterion for stability of the Lur'e systems (see, for example, [22]) colligating some previous results on synchronization on the multiagent networks. For example, it extends ti results of [4] to some classes of the nonpassive linear agents and generalizes the result of [3, Section 3.2.2.] on synchronization of the harmonic oscillators. The criterion obtained in the present paper allows one to obtain also new consensus criteria for the networks of double integrators and agents with nonlinear couplings passifiable in the sense of [23, 24].

2. PROBLEM FORMULATION AND BASIC ASSUMPTIONS

2.1. Auxiliary Notions of the Graph Theory

By the (directed) graph is meant an ordered pair $G = (V, E)$ of two finite sets V (sets of *nodes* or *vertices* and $E \subseteq V \times V$ (set of graph *arcs* or *edges*). The node v is connected with the node w in the graph G if $(v, w) \in E$. A finite sequence of nodes $v_1, \dots, v_k$ where any term is connected to the next one ($(v_i, v_{i+1}) \in E$ for $i < k$) is called the *route* from v_1 to v_k. The graph is *strongly connected* if there exists in it a route between any two vertices. For the graph $G = (V, E)$, its mirror graph $\hat{G} = (V, \hat{E})$ is obtained by adding all arcs inverse to the arcs G, that is, $\hat{E} = E \cup \{(w, v) \colon (v, w) \in E\}$. The graph coinciding with its mirror graph is called the *undirected* one. In this case, a pair of counter-directed edges between two vertices is often regarded as one undirected edge. Everywhere below $\mathbb{G}_N$ denotes the set of graphs $G = (V_N, E)$ with the vertex set $V_N = \{1, \dots, N\}$ without "loops": $(v, v) \notin E$ for all $v \in V_N$. The adjacency matrix $(a_{jk}(G))$ of the graph $G \in \mathbb{G}_N$ obeys $a_{jk}(G) := 1$ for $(j, k) \in E$ and $a_{jk}(G) := 0$, otherwise. The *Laplacian* of the graph G is given by

$$L(G) := \begin{bmatrix} \sum_{j=1}^N a_{1j} & -a_{12} & \dots & -a_{1N} \\ -a_{21} & \sum_{j=1}^N a_{2j} & \dots & -a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N1} & -a_{N2} & \dots & \sum_{j=1}^N a_{Nj} \end{bmatrix}. \tag{1}$$

For the undirected graph G, we get $L(G) = L(G)^T$ and $\lambda_1(G)$, the least eigenvalue of $L(G)$, is equal to 0; in particular, $L(G) \geqslant 0$ [12, 13]. Here and below $L \geqslant 0$ denotes nonnegative definiteness of the symmetrical matrix L. The second eigenvalue of the Laplacian $\lambda_2(G)$ called the *algebraic connectivity* of G [25] has the following extremal property:

$$\begin{gathered} \lambda_2(G) = N \min_{z \in \mathfrak{Z}} \frac{\sum_{j,k=1}^N a_{ij}(G)(z_j - z_k)^2}{\sum_{j,k=1}^N (z_k - z_j)^2}, \\ \mathfrak{Z} := \left\{ z \in \mathbb{R}^N : \sum_{j,k=1}^N (z_k - z_j)^2 \neq 0 \right\}. \end{gathered} \tag{2}$$

Whence the well-known fact follows [2, 13, 26] stating that $\lambda_2(G) > 0$ if and only if the undirected graph G is connected. Additionally, the minimal sufficient number of edges that may be removed to violate connectivity of the nonoriented graph $e(G) \geqslant 0$ and $\lambda_2(G)$ are related by the following *Fiedler inequality* [25]:

$$\lambda_2(G) \geqslant 2e(G) \left(1 - \cos \frac{\pi}{N}\right). \tag{3}$$

We follow [26] in defining the algebraic connectivity of the directed graph G by formula (2) and emphasize that in this case $\lambda_2(G)$ is not any more the eigenvalue of the Laplace matrix; moreover, all nonzero numbers of this matrix need not to be real. Since $a_{jk}(G) + a_{kj}(G) \geqslant a_{jk}(\hat{G}) = a_{kj}(\hat{G})$, the inequality is satisfied $\lambda_2(G) \geqslant \frac{1}{2}\lambda_2(\hat{G})$; in particular, for the strongly connected graph $G \in \mathbb{G}_N$ we have

$$\lambda_2(G) \geqslant e(\hat{G}) \left(1 - \cos \frac{\pi}{N}\right) \geqslant 1 - \cos \frac{\pi}{N}. \tag{4}$$

The estimates (3), (4), and others [27] can be used where precise calculation of λ_2 is impossible (for example, the graph is either unknown or too large).

2.2. Formulation of the Problem

In what follows, we consider a group of identical agents numerated from 1 to $N \geqslant 2$ and described by the model of the state space

$$\dot{x}_j(t) = Ax_j(t) + Bu_j(t), \quad y_j(t) = Cx_j(t), \quad t \geqslant 0, \quad j = 1, \dots, N, \tag{5}$$

where $x_j \in \mathbb{R}^d$, $u_j \in \mathbb{R}^m$, and $y_j \in \mathbb{R}^l$ are, respectively, the state, control, and output of the jth agent. System (5) is assumed to be controllable and observable. The control inputs are generated as the result of agents' interaction (physical, informational, and so on). The interaction graph (network topology) $G(t) = [V_N, E(t)] \in \mathbb{G}_N$ can vary in time. The output $y_k(t)$ of the kth agent participates in generation of the control $u_j(t)$ if and only if $(k, j) \in E(t)$.

To put it more appropriately, we study the control algorithms (protocols) like

$$u_j(t) = \sum_{k:(k,j)\in E(t)} \varphi_{jk}(t, y_k(t) - y_j(t)). \tag{6}$$

The maps $\varphi_{jk} : [0; +\infty) \times \mathbb{R}^l \to \mathbb{R}^m$ are called the *coupling functions* or simply *couplings* defining "intensity" of agent interactions. The present paper aims at determining the sufficient conditions under which the protocol (6) assures *consensus* between the agents in the following sense.

Definition. Protocol (6) assures output consensus if the following conditions are satisfied under any initial data $(x_j(0))$ and some constant $m > 0$:

$$\lim_{t\to+\infty} |y_j(t) - y_k(t)| = 0 \quad \forall k, j; \tag{7}$$

$$\mathcal{W}(x_1(t), \ldots, x_N(t)) := \sum_{j\neq k} |x_j(t) - x_k(t)|^2 \leqslant m\mathcal{W}(x_1(0), \ldots, x_N(0)) \quad \forall t \geqslant 0. \tag{8}$$

If $\mathcal{W}(x_1(t), \ldots, x_N(t)) \to 0$ for $t \to +\infty$, then protocol (6) assures state consensus. It is said that the exponential state consensus is assured if $\mathcal{W}(x_1(t), \ldots, x_N(t)) \leqslant ce^{-\alpha t}\mathcal{W}(x_1(0), \ldots, x_N(0))$ for some constant $c, \alpha > 0$.

Obviously, the state consensus entails the output consensus, the inverse being true under additional conditions such as the uniform continuity of couplings in zero.

Remark 1. Attainment of output consensus amounts to attainment of state consensus if $\lim_{y\to 0} \sup_{t\geqslant 0} |\varphi_{jk}(t, y)| = 0$ for all j, k.

Indeed, according to (6) and (7), it follows from the condition $\lim_{y\to 0} \sup_{t\geqslant 0} |\varphi_{jk}(t, y)| = 0$ that $|u_j(t)| \to 0$ for $t \to +\infty$. By taking by definition $X_{jk} := x_k - x_j$, $U_{jk} := u_k - u_j$ and $Y_{jk} := y_k - y_j$, we get $\dot{X}_{jk} = AX_{jk} + BU_{jk}$ and $Y_{jk} = CX_{jk}$, and $U_{jk}(t) \to 0$ and $Y_{jk}(t) \to 0$ for $t \to +\infty$. In virtue of controllability of the pair (A, B) and observability of (A, C), we obtain $X_{jk}(t) \to 0$ for all k, j which implies the state consensus.

2.3. Basic Assumptions

The basic assumptions come to the connectivity of the interaction graph (Assumption 1), coupling symmetry or balance (Assumption 2) and quadratic inequalities on the coupling function which are multidimensional generalizations of the scalar sector inequalities (see [22] and Section 3.2 below).

Assumption 1. The function $G(\cdot)$ is Lebesgue-measurable, that is, $G^{-1}(\Gamma) \subset \mathbb{R}$, the preimage of the arbitrary graph $\Gamma \in \mathbb{G}_N$, is a measurable set. The graph $G(t)$ is strongly connected almost for all $t \geqslant 0$.

It is impossible to reject the assumption of network connectivity in a sense because it is necessary to eliminate division of the agent group into several disconnected clusters which cannot by synchronized with each other. The next assumption is the condition for coupling symmetry or balance.

Assumption 2. For any subscripts $j \neq k$ and the tie instant $t \geqslant 0$, either of the following relations holds:

(a) the graph $G(t)$ is undirected and the condition $\varphi_{jk}(t, y) = -\varphi_{kj}(t, -y)$ is satisfied;

(b) the coupling function is linear in the second variable $\varphi_{jk}(t,y) = \mathfrak{W}_{jk}(t)y$, the gains $\mathfrak{W}_{jk}(t) \in \mathbb{R}^{l\times m}$ satisfying the balance condition

$$\sum_{k:(k,j)\in E(t)} \mathfrak{W}_{jk}(t) = \sum_{k:(j,k)\in E(t)} \mathfrak{W}_{kj}(t)\ \forall j. \tag{9}$$

We notice that the symmetry condition in item (a) resembles the third Newton law because $\varphi_{jk}(t, y_k - y_j) = -\varphi_{kj}(t, y_j - y_k)$, whereas the balance condition is close conceptually to the conditions for mass or charge conservation (the total income equals the total consumption). Additionally, in some physical applications such as the networks of coupled oscillators, Assumption 2 is satisfied namely by virtue of the aforementioned laws. We notice that if Assumption 2 is satisfied, then the condition for output consensus (7) means that $y_j(t) - Ce^{tA}\tilde{x}_0 \to 0$ for $t \to \infty$ for all j, where $\tilde{x}_0 = \frac{1}{N}\sum_{j=1}^N x_j(0)$. Indeed, $\sum_{j=1}^N u_j = 0$ in virtue of Assumption 2. By summing Eqs. (5) one can easily see that $\sum_{j=1}^N y_j(t) = Ce^{tA}\tilde{x}_0$, whence follows the assumption.

The present paper considers the case where the coupling functions need not be known, but satisfy special quadratic constraints which will be shown below are a natural generalization of the sector inequalities (see [22] and Section 3.2 below) to the multidimensional case Namely, it is assumed that $\varphi_{jk} \in \mathfrak{S}(\mathcal{F})$, where $\mathcal{F} : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ is a quadratic form and $\mathfrak{S}(\mathcal{F})$ denotes the set of maps $\varphi : [0; +\infty) \times \mathbb{R}^l \to \mathbb{R}^m$ with the properties

(i) the map φ satisfies the Carathéodori conditions, that is, the function $\varphi(t,\cdot)$ is continuous almost for all $t \geqslant 0$, and the function $\varphi(\cdot, y)$ is measurable for any y;

(ii) $\varphi(\cdot, 0) \equiv 0$ and for any compact $K \subset \mathbb{R} \setminus \{0\}$ satisfied is the inequality

$$\inf_{y\in K, t\geqslant 0} \mathcal{F}(\varphi(t,y), y) > 0. \tag{10}$$

In particular, the graph of the function $\varphi(t,\cdot)$ lies strictly within the cone the cone $\mathcal{F}(\varphi, y) > 0$, except for the origin.

Therefore, the consensus condition below is formulated only in terms of the quadratic form $\mathcal{F}$ and the coefficients A, B, C, but not the coupling functions themselves. Thus condition mechanically ensures synchronization for any coupling functions from the class $\mathfrak{S}(\mathcal{F})$ that satisfy the above conditions.

3. BASIC RESULTS

The main result of the present paper, the frequency consensus condition for the agents like (5) is given in Section 3.1. This condition has the simplest and convenient form in the scalar case ($k = m = 1$) where the quadratic constraint takes on the form of the sector inequality. This special case is discussed in Section 3.2.

3.1. Consensus Criterion in the Multidimensional Case

We introduce for convenience the following notation. Let $W_x(\lambda) := (\lambda I - A)^{-1}B$ and $W_y(\lambda) := CW_x(\lambda)$ be the transfer functions of system (5) from u to x and y, respectively, and θ denote the minimal algebraic connectivity of the network graph:

$$\theta = \min_{t\geqslant 0} \lambda_2(G(t)). \tag{11}$$

Let the quadratic form constraining the coupling functions be given by

$$\mathcal{F}(\varphi, y) = \varphi^* L y - y^* Q y - \varphi^* R \varphi, \quad y \in \mathbb{R}^l, \quad \varphi \in \mathbb{R}^m, \tag{12}$$

where $L \in \mathbb{R}^{m\times l}$, $Q = Q^* \in \mathbb{R}^{l\times l}$, $R = R^* \in \mathbb{R}^{m\times m}$. We assume for the matrix $\Lambda \in \mathbb{C}^{l\times m}$ that

$$\Pi_{\mathcal{F}}(\Lambda) := L\Lambda + \Lambda^* L^* + \theta\Lambda^* Q\Lambda + \frac{1}{2(N-1)}R. \tag{13}$$

In the case of complex matrices, Λ^* as usual denotes the Hermitian conjugation. Theorem 2 represents the main result of the present paper.

Theorem 1. *Let satisfied be Assumptions 1 and 2 and $\varphi_{jk} \in \mathfrak{S}(\mathcal{F})$ $\forall j,k$, where $\mathcal{F}$ is the quadratic form* (12) *and $Q \geqslant 0, R \geqslant 0$. We assume that*

(i) *there exists the matrix $K \in \mathbb{R}^{m\times k}$ such that $A - NBKC$ is Hurwitzian*[1] *matrix and $\mathcal{F}(Ky, y) > 0$ $\forall y \neq 0$;*

(ii) *for all $\omega \in \mathbb{R}$ such that* $\det(i\omega I - A) \neq 0$, *satisfied is the inequality*

$$\Pi_{\mathcal{F}}(\Lambda) \geqslant 0 \quad \text{for} \quad \Lambda := W_y(i\omega). \tag{14}$$

Then algorithm (6) *ensures output consensus. If $\Pi(i\omega) \geqslant \varepsilon|W_x(i\omega)|^2$, $\forall\omega \in \mathbb{R}$, where $\varepsilon > 0$, the exponential state consensus is reached.*

Theorem 1 is proved in the Appendix.

According to Theorem 1, conditions (i) and (ii) entail in fact the "robust" consensus in the sense that condition (7) is satisfied for all coupling functions $\varphi_{jk} \in \mathfrak{S}(\mathcal{F})$ and variable interaction graphs $G(t)$ for which the theorem assumptions hold. In practice, condition (i) is necessary for reaching robust consensus. The assumption existence of the matrix K for which the map $y \mapsto Ky$ belongs to the class $\mathfrak{S}(\mathcal{F})$ is usually nonrestrictive and, in particular, is always satisfied in the case of scalar sector inequalities (see Section 3.2 below). The following remark shows that any such matrix must satisfy (i).

Remark 2. If the robust, in the above sense, consensus is reached, then (i) is satisfied for any matrix $K \in \mathbb{R}^{m\times k}$ such that $\mathcal{F}(Ky, y) > 0$ $\forall y \neq 0$.

Indeed, for a network with *complete graph* $G(t) \equiv G$ and linear couplings $\varphi_{jk}(t, y) = Ky$ the output consensus entails in virtue of Remark 1 the state consensus. Consequently, according to [8, Theorem 3] for the eigenvalue $\lambda := N$ of the Laplace matrix $L(G)$ the matrix $A - \lambda BKC$ is Hurwitz.

Remark 3. Under the conditions of Theorem 1, the network topology is involved in inequality (13) only as the multiplier θ. Since $Q \geqslant 0$ by assumption, replacement of θ by a smaller value retains sufficiency of the condition for consensus. This observation is useful if it is impossible to calculate θ explicitly. In such cases one can use the estimates of the algebraic connectivity (3), (4) and others, see [27]. In the case of $Q = 0$, the value of θ is of no importance.

3.2. Consensus for the Nonlinear Scalar Coupling Functions

In this section we discuss application of Theorem 1 for the agents with scalar outputs and inputs and scalar coupling functions satisfying the strict sector inequalities. Stated differently, we assume that $k = m = 1$ and $\varphi_{jk} \in S[\alpha;\beta]$, where the class $S[\alpha;\beta]$ consists of the functions $\varphi : [0;+\infty) \times \mathbb{R} \to \mathbb{R}$ such that $\varphi(t, 0) \equiv 0$, the Carathéodori conditions are valid, and

$$\alpha < \inf_{\sigma\in K,\, t\geqslant 0} \frac{\varphi(t,\sigma)}{\sigma} \leqslant \sup_{\sigma\in K,\, t\geqslant 0} \frac{\varphi(t,\sigma)}{\sigma} < \beta \tag{15}$$

for any compact $K \subset \mathbb{R}\setminus\{0\}$. Inequality (15) means that for the given $t \geqslant 0$ the graph of the function $\xi = \varphi(t,\sigma)$ is everywhere, except for the origin, lies within the sector between the straight lines $\xi = \alpha\sigma$ and $\xi = \beta\sigma$, which accounts for the term "sector inequalities."

[1] Matrix is *Hurwitz* if all its eigenvalues have negative real parts.

We note that if A is not a Hurwitz matrix, then for $\alpha < 0$ and $\beta > 0$ algorithm (6) knowingly does not provide consensus under zero couplings $\varphi_{jk} \equiv 0$ lying in $S[\alpha;\beta]$ and satisfying Assumption 2. This implies that for $\alpha\beta < 0$ it is impossible to establish a criterion for robust consensus similar to Theorem 1 and doing without the information about particular nonlinearities. Therefore, we can assume that either $0 \leqslant \alpha < \beta$ or $\alpha < \beta \leqslant 0$. We consider for definiteness the first case, the second case being reducible to it by the replacement $(\alpha, \beta, B) \mapsto (-\beta, -\alpha, -B)$.

We rearrange inequalities (15) in a quadratic constraint and introduce for that the constants

$$\gamma = \frac{1}{\beta + \alpha} \geqslant 0, \quad \delta = \frac{\alpha}{1 + \alpha\beta^{-1}} \geqslant 0 \tag{16}$$

and the quadratic form

$$\mathcal{F}_{\alpha;\beta}(\varphi, y) := \varphi y - \delta y^2 - \gamma\varphi^2, \quad \varphi, y \in \mathbb{R}, \tag{17}$$

$$\Pi_{\alpha;\beta}(\lambda) := Re\,\lambda + \theta\delta|\lambda|^2 + \frac{\gamma}{2(N-1)}, \quad \lambda \in \mathbb{C}. \tag{18}$$

One can readily verify that $\Pi_{\mathcal{F}_{\alpha;\beta}} \equiv \Pi_{\alpha;\beta}$. Since the sector inequalities (15) are equivalent to $(\varphi - \alpha y)(y - \beta^{-1}\varphi) = (1 + \alpha\beta^{-1})\mathcal{F}_{\alpha;\beta}(\varphi(t, y), y) > 0$, the last inequality being satisfied uniformly over all $t \geqslant 0$ and $y \in K$ (for any compact $K \subset \mathbb{R} \setminus \{0\}$), we come to the following simple proposition.

Proposition. *$S[\alpha;\beta] = \mathfrak{S}(\mathcal{F}_{\alpha;\beta})$ and $\Pi_{\mathcal{F}_{\alpha;\beta}} \equiv \Pi_{\alpha;\beta}$.*

Proposition enables one to determine a criterion for consensus among the scalar agents.

Theorem 2. *Let Assumptions 1 and 2 be satisfied, $\varphi_{jk} \in S[\alpha;\beta]$ $\forall j, k$, where $0 \leqslant \alpha < \beta \leqslant \infty$, and two following conditions are satisfied:*

(a) *there exists $\mu \in (\alpha;\beta)$ for which $A - \mu NBC$ is a Hurwitz matrix;*

(b) *for all $\omega \in \mathbb{R}$ such that $\det(i\omega I - A) \neq 0$, the inequality holds*

$$\Pi_{\alpha;\beta}(W_y(i\omega)) \geqslant 0. \tag{19}$$

Then, algorithm (6) ensures output consensus. If there exists $\varepsilon > 0$ such that $\Pi_{\alpha;\beta}(W_y(i\omega)) \geqslant \varepsilon|W_x(i\omega)|^2$, then the exponential state consensus is reached.

Theorem 2 follows from Theorem 1 as applied to $\mathcal{F} := \mathcal{F}_{\alpha;\beta}$, $L := 1$, $R := \gamma$, $Q := \delta$ and the proposition because condition (a) in Theorem 2 coincides with (i) from the condition of Theorem 1 and the frequency inequality (b) is nothing but (ii).

Inequality (19) is nontrivial because the Hermitian form $\Pi_{\alpha;\beta}$ is not negative definite, its discriminant being $\frac{1}{4}\left[\frac{2\delta\gamma\theta}{N-1} - 1\right] \leqslant 0$ because $\theta \leqslant N$ in virtue of (2) and $\delta\gamma \leqslant 1/4$.

Remark 4. Condition (b) implies that the locus of the frequency characteristic $\{W_y(i\omega)\}$ lies outside the set $\mathcal{D}$ which is the half-plane $\{z \in \mathbb{C} : Re\,z < -\frac{\gamma}{2(N-1)}\}$ for $\alpha = 0$ and circle $\mathcal{D} = \{z \in \mathbb{C} : |z - z_0| < \rho_0\}$ for $\alpha > 0$, where

$$z_0 = -\frac{1}{2\delta\theta}, \quad \rho_0 = \frac{1}{2\delta\theta}\sqrt{1 - \frac{2\theta\delta\gamma}{N-1}}.$$

Indeed, $\alpha = 0 \Rightarrow \delta = 0$ by virtue of (16), whence it follows the first statement. For $\alpha > 0$, we get $\delta > 0$ and $Re\,z + \delta\theta|z|^2 + \frac{\gamma}{2(N-1)} \geqslant 0 \Leftrightarrow |z|^2 - 2z_0\,Re\,z + |z_0|^2 - \rho_0^2 \geqslant 0 \Leftrightarrow z \notin \mathcal{D}$.

Geometrical interpretation of the frequency inequality in Remark 4 allows one to parallel the consensus criterion of Theorem 2 and the famous circle criterion for the Lur'e systems (see, for example, [22]). Theorem 2 is in fact a direct generalization of the circle criterion in the following sense. Let us consider a system of two agents ($N = 2$) using protocol (6) with bidirectional interaction ($E(t) = \{(1, 2), (2, 1)\}$) and the coupling functions $\varphi_{12} = -\varphi_{21} \in S[\alpha; \beta]$. Assuming that $X(t) := x_2(t) - x_1(t)$, $Y(t) := y_2(t) - y_1(t)$, $U(t) := u_2(t) - u_1(t)$ and $\Phi(t, Y) = -2\varphi_{12}(t, Y)$, we obtain $\Phi \in S[-2\beta; -2\alpha]$ and

$$\dot{X}(t) = AX(t) + BU(t), \quad U(t) = \Phi(t, Y(t)). \tag{20}$$

According to the circle criterion, the equilibrium $X = 0$ of system (20) is stable if $A + kBC$ is a Hurwitz matrix for some $k \in [-2\beta; -2\alpha]$, which is equivalent to condition (a) with $\theta = 2$) and the locus $\{W_y(i\omega) : \omega \in \mathbb{R}\} \subset \mathbb{C}$ lies beyond the circle constructed on the segment $[(-2\alpha)^{-1} + 0i; (-2\beta)^{-1} + 0i]$ as the diameter or the half-plane $\{z : Re\, z < (-2\beta)^{-1}\}$ for $\alpha = 0$. These circle or half-plane coincide with the aforementioned set $\mathcal{D}$.

4. EXAMPLES

This section illustrates application of Theorems 1 and 2 to groups of agents with special dynamics.

4.1. Synchronization of Passive Agents

We assume that the transfer matrix W_y is square ($k = m$) satisfies the frequency inequality

$$W_y(i\omega) + W_y(i\omega)^* \geqslant 0 \qquad \forall\omega \in \mathbb{R}. \tag{21}$$

It is common knowledge that if the easy additional conditions such as the assumptions of Theorem 3 below are satisfied, then condition (21) entails passivity [4, 21, 23] of agent (5), that is, existence of a function $V : \mathbb{R}^d \to [0; +\infty)$ such that for any solution (5) and any time moments $t_1 \geqslant t_0 \geqslant 0$ we have

$$V(x(t_1)) - V(x(t_0)) \leqslant \int\limits_{t_0}^{t_1} y(t)^T u(t)dt.$$

Synchronization in the networks of passive agents, including the nonlinear ones, was considered in a number of publications, see, for example, [4, 21]. In the majority of the existing results it is assumed that the graph is either constant or has dwell time between switchings. The main result of this section (Theorem 3) refers to the case of linear agents, but the network topology is assumed to be only a measurable time function; in addition, nonstationary coupling functions are considered.

Theorem 3. *Assume that Assumptions 1 and 2 and condition* (21), $\varphi_{jk} \in \mathfrak{S}(\mathcal{F})$ *are satisfied, where* $F(\varphi, y) = \varphi^T y$, *and there exists a matrix* $K \in \mathbb{R}^{m\times m}$ *such that* $K + K^T > 0$ *and* $A - BKC$ *is a Hurwitz matrix. Then, protocol* (6) *establishes the output consensus.*

To prove Theorem 3, it suffices to use Theorem 1 assuming that $\mathcal{F}(\varphi, y) := \varphi^T y$ and noticing that condition (21) entails (14) for any $\theta \geqslant 0$; additionally, $\mathcal{F}(Ky, y) = \frac{1}{2}y^T(K + K^T)y/2\ \forall K \in \mathbb{R}^{m\times m}$. In the scalar case, Theorem 3 assumes the simplest form.

Corollary 1. *Let* $k = m = 1$, *Assumptions 1 and 2 and condition* (21) *be satisfied,* $\varphi_{jk} \in S[0; +\infty]$, *and* $A - \mu BC$ *be a Hurwitz matrix for some* $\mu > 0$. *Then, protocol* (6) *ensures the output consensus.*

Corollary 1 generalizes, in particular, the results of [4, Theorems 2 and 7] stating that upon satisfying Assumptions 1 and 2 protocol (6) establishes consensus in the network of first-order agents $\dot{y}_j = u_j$, where $W_y(i\omega) = (i\omega)^{-1}$. Another example is represented by the considered network of harmonic oscillators $\dot{q}_j = y_j \in \mathbb{R}$, $\dot{y}_j = -\omega_0^2 q_j + u_j$. According to [3, Theorem 3.8], protocol (6) upon satisfaction of Assumptions 1 and 2b (with constant coefficients $\mathfrak{W}_{jk} > 0$) establishes consensus between the agents. Corollary 1 generalizes this result to the case of variable gains $\mathfrak{W}_{jk}(t)$, as well as to the case of nonlinear couplings where Assumption 2a is satisfied. Indeed, $W_y(z) = z/(z^2 - \omega_0^2)$, whence we get $W_y(i\omega) + W_y(i\omega)^* = 2\,Re\,W_y(i\omega) = 0$; additionally, the jth agent is stabilized by the feedback $u_j = -\mu y_j$ for any $\mu > 0$.

4.2. Consensus of Strictly Passifiable Agents

Passifiability [23, 24], that is, possibility of making the control plant passive with the use of feedback, is a natural generalization of the notion of passivity. In this section we consider one of the most popular types of passifiable agents, the strictly passifiable ones, or, which is the same, minimal-phase agents. We note that the problem of reaching consensus in the networks of such agents is little-investigated. The paper [24] where consideration is given to synchronization in the networks of linear agents with linear couplings and consensus is shown to be reached for sufficiently strong couplings is one of the most important results along this line of research. The results of [24] are extended below to the case of variable graph and couplings satisfying Assumption 2. To simplify our reasoning, we confine our consideration to the case of scalar input and output: $k = m = 1$. Under this assumption, the strict passifiability is equivalent to the fact that $CB > 0$ and the polynomial $\psi(\lambda) = \det(\lambda I - A) W_y(\lambda)$ is Hurwitz [23, 24]. The following theorem shows for the strictly passifiable agents consensus is reached under sufficiently strong couplings.

Theorem 4. *Let the agents be strictly passifiable, and satisfied be Assumptions 1 and 2 and* $\varphi_{jk} \in S[\alpha; +\infty]$, *where the inequality*

$$\alpha \geqslant \alpha_* := \frac{1}{\theta} \sup_{\omega \in \mathbb{R}} f(\omega), \quad f(\omega) := -\frac{Re\,W_y(i\omega)}{|W_y(i\omega)|^2} \tag{22}$$

is satisfied for $\theta = \min_{t \geqslant 0} \lambda_2(G(t))$. *Then, the protocol* (6) *ensures output consensus. Additionally, the exponential state consensus is reached for* $\alpha > \alpha_*$.

We note that the supremum in (22) is finite because the function $f(\omega) = |\psi(i\omega)|^{-2} \times (\psi(i\omega)\chi(-i\omega) + \psi(-i\omega)\chi(i\omega))$, where $\chi(z) := \det(zI - A)$, is definite and continuous for $\omega \in \mathbb{R}$; additionally, $f(\omega) \to -CAB/(CB)^2$ for $\omega \to \pm\infty$. The number α_* is the threshold value of the coupling "intensity" above which synchronization is ensured.

Proof of Theorem 4. The first assertion follows immediately from Theorem 2 for $\beta = \infty$ (consequently, $\delta = \alpha$, $\gamma = 0$) because condition (a) of this theorem is satisfied for a sufficiently great $\mu > 0$ [23] and inequality (19) is satisfied in virtue of (22). Now, $\Pi_{\alpha;\infty}(W_y(i\omega)) \geqslant (\alpha - \alpha_*)\theta |W_y(i\omega)|^2$. Since $CB > 0$ and $\det(i\omega I - A) \neq 0$, there exists a constant $\xi > 0$ such that $|W_y(i\omega)| \geqslant \xi |W_x(i\omega)|$ and, consequently, $\Pi_{\alpha;\infty}(W_y(i\omega)) \geqslant \varepsilon |W_x(i\omega)|^2$ for all $\omega \in \mathbb{R}$, where $\varepsilon := \xi^2 \theta (\alpha - \alpha_*)$.

4.3. Synchronization of Double Integrators

The problems of consensus for the second-order agents have recently attracted significant attention owing to numerous applications to control of robot formations, see [2, 10] and others. Each agent is described by the model

$$\ddot{z}_j = u_j, \quad y_j = q_0 z_j + q_1 \dot{z}_j, \tag{23}$$

where $q_0, q_1 \in \mathbb{R}$ are constants. Therefore, $W_y(i\omega) = q_0 (i\omega)^{-2} + q_1 (i\omega)^{-1}$.

It is easy to verify that an agent is strictly passifiable for $q_0, q_1 > 0$, (19) taking on form $Kq_0^2\omega^{-4}+(Kq_1^2 - q_0)\omega^{-2} \geqslant 0$, where $K := \alpha\theta$. This fact allows one to use Theorem 4 which in this case assumes the following form.

Corollary 2. *We assume that the agents are described by the model* (23)*, where* $q_0, q_1 > 0$*, Assumptions 1 and 2 are satisfied, and* $\varphi_{jk} \in S\,[\alpha; +\infty]$*, where* $\theta := \inf_{t\geqslant 0} \lambda_2\big(G(t)\big)$ *and* $\alpha \geqslant \theta^{-1}q_1^{-2}q_0$*. Then, protocol* (6) *establishes the output consensus. If* $\alpha > \theta^{-1}q_1^{-2}q_0$*, then exponential state consensus is reached as well.*

We notice that the assumptions of Corollary 2 exclude the use of limited nonlinearities of the kind of hyperbolic tangent or saturation because they are not contained in the class $S[\alpha; +\infty]$ with $\alpha > 0$. This disadvantage can be overcome if the jth agent has access not only to the relative speeds $\dot{z}_j - \dot{z}_k$, but also its own speed $\dot{z}_j$. In this case, a modified protocol

$$u_j = -p\dot{z}_j + \sum_{(k,j)\in E(t)} \varphi_{jk}(t, y_k(t) - y_j(t)) \tag{24}$$

can be used. As the result of applying protocol (23) to agents (24), a closed system results with the same solutions as in the case of applying algorithm (6) to the modified agents

$$\ddot{z}_j + p\dot{z}_j = u_j, \quad y_j = q_0 z_j + q_1\dot{z}_j. \tag{25}$$

We notice that for $p > 0$, $q_0 > 0$, $q_1 \geqslant 0$ the feedback $u_j = -\mu y_j$ stabilizes plant (25) for any $\mu > 0$. Since we have for such agent that $W_y(i\omega) = (q_0 + q_1(i\omega))[(i\omega)^2 + p(i\omega)]^{-1}$ and $Re\, W_y(i\omega) = (q_1 p - q_0)(p^2 + \omega^2)^{-2}$, (19) is satisfied for $\alpha = \theta = 0$:

$$\inf_{\omega\in\mathbb{R}} Re\, W_y(i\omega) = \min\left(\frac{q_1 p - q_0}{p^2}, 0\right) \geqslant -\frac{\beta^{-1}}{2(N-1)}.$$

In virtue of Theorem 2, we establish the following result.

Corollary 3. *Let the agents have the form* (23) *with* $q_1 \geqslant 0$*,* $q_0 > 0$*, Assumptions 1 and 2 be satisfied,* $p > 0$ *and* $\varphi_{jk} \in S\,[0; \beta]$*, where* $\beta \leqslant +\infty$*, and* $p^2\beta^{-1} + 2(N-1)(q_1 p - q_0) \geqslant 0$*. Then, protocol* (24) *established output consensus.*

Corollary 3 guarantees consensus for $q_1 p \geqslant q_0$ and $\beta = +\infty$ and for $q_1 = 0$, $\beta \leqslant p^2/(2q_0(N-1))$ (the algorithm does not use the relative speeds and only the absolute one). One can also see easily from Eqs. (25) that if the assumption of Remark 1 is satisfied, protocol (24) ensures not only the state consensus but also the condition $\dot{z}_j(t) \to 0$.

5. CONCLUSIONS

The question was considered of reaching the output consensus (synchronization) in the networks of identical linear agents of arbitrary order with variable topology. the couplings between the agents are not known and satisfy the quadratic constraints of the of the sector kind. The frequency-domain consensus condition was established which is a direct generalization of the circle criterion for the Lur'e systems with one nonlinearity. It is planned to extend in what follows the results obtained to wider classes of agents, as well as to consider more complex problems of multiagent control (consensus with leaders and so on).

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APPENDIX

This Appendix is devoted to proving Theorem 1 along the following lines. Making use of the individual quadratic constraints on the coupling functions (10), we generate the "global" quadratic constraint for solution of the closed system (5) and (6) (Lemma 1 below). The further reasoning relies on the frequency theorem (Kalman–Yakubovich–Popov lemma) [22, 28] enabling one to prove existence of the quadratic Lyapunov function in virtue of the aforementioned quadratic constraint. The proof of Theorem 1 is based on existence of such function.

We assume in what follows that the assumptions of Theorem 1 are satisfied. In particular, given is a quadratic form $\mathcal{F}$ like (12) such that $\varphi_{jk} \in \mathfrak{S}(\mathcal{F})$. Let $\bar{\xi} := (\xi_1^T, \dots, \xi_N^T)^T$ denote the unified column vector for any sequence of the column vectors $\xi_1, \xi_2, \dots, \xi_N$. Everywhere below $a_{jk}(t) := a_{jk}(G(t))$ denotes the adjacency matrix of the graph $G(t)$.

For convenience of the further reasoning, we introduce also the Hermitian forms

$$P(y,u) = -Re(u^* Ly) - \theta y^* Qy - \frac{1}{2(N-1)} u^* Ry, \quad y \in \mathbb{C}^l, \quad u \in \mathbb{C}^m,$$

$$\mathfrak{P}(\bar{y}, \bar{u}) = \sum_{j,k=1}^{N} P(y_j - y_k, u_j - u_k).$$

Lemma 1. *The inequality $\mathfrak{P}(\bar{y}(t), \bar{u}(t)) \geqslant 0 \ \forall t \geqslant 0$ is satisfied for any solution of system* (5) *and* (6). *Moreover, for any $\nu > \mu > 0$ there exists $\rho = \rho(\nu, \mu) > 0$ such that $\mathfrak{P}(\bar{y}(t), \bar{u}(t)) > \rho$ for $\nu > \max_{1 \leqslant j,k \leqslant N} |y_k(t) - y_j(t)| > \mu$.*

Proof of Lemma 1. Let $\xi_{jk}(t) := a_{kj}(t)\varphi_{jk}(t, y_k(t) - y_j(t))$ $(\xi_{jj} := 0)$ and $\eta_{jk}(t) := a_{kj}(t)[y_k(t) - y_j(t)]$. In virtue of (6) we have $u_j(t) = \sum_{k=1}^{N} \xi_{jk}(t)$; consequently,

$$|u_j(t)|^2 \leqslant (N-1) \sum_{k=1}^{N} |\xi_{jk}(t)|^2 \tag{A.1}$$

in virtue of the Cauchy–Buniakowsky–Schwarz inequality, and

$$\sum_{j,k=1}^{N} \xi_{jk}^T L y_j = \sum_{j=1}^{N} u_j^T L y_j. \tag{A.2}$$

We also prove validity of the relation

$$\sum_{j,k=1}^{N} \xi_{jk}^T L y_k = -\sum_{j=1}^{N} u_j^T L y_j. \tag{A.3}$$

If (a) of Assumption 2 is satisfied, this follows immediately from (A.2) because $\xi_{jk} = -\xi_{kj}$. Consider the case (b). We note that we get by virtue of (9) that

$$\sum_{j,k=1}^{N} y_k^T \mathfrak{W}_{jk}^T L y_k = \sum_{j,k=1}^{N} y_k^T \frac{\mathfrak{W}_{jk}^T L + L^T \mathfrak{W}_{jk}}{2} y_k = \frac{1}{2} \sum_{j,k=1}^{N} y_k^T \mathfrak{W}_{jk}^T L y_k + \frac{1}{2} \sum_{j,k=1}^{N} y_k^T L^T \mathfrak{W}_{kj} y_k\ ;$$

$$\sum_{j,k=1}^{N} y_j^T \mathfrak{W}_{jk}^T L y_k = \sum_{j,k=1}^{N} y_j^T \frac{\mathfrak{W}_{jk}^T L + L^T \mathfrak{W}_{jk}}{2} y_k = \frac{1}{2} \sum_{j,k=1}^{N} y_j^T \mathfrak{W}_{jk}^T L y_k + \frac{1}{2} \sum_{j,k=1}^{N} y_k^T L^T \mathfrak{W}_{kj} y_j$$

(without loss of generality, we assume that $\mathfrak{W}_{jk}(t) := 0$ for $a_{kj}(t) = 0$). Subtraction results in

$$\sum_{j,k=1}^{N} \xi_{jk}^T L y_k = \sum_{j,k=1}^{N} (y_k - y_j)^T \mathfrak{W}_{jk}^T L y_k$$
$$= \frac{1}{2} \sum_{j,k=1}^{N} \Big[(y_k - y_j)^T \mathfrak{W}_{jk}^T L y_k + y_k^T L^T \mathfrak{W}_{kj} (y_k - y_j) \Big]$$
$$= \frac{1}{2} \sum_{j,k=1}^{N} \xi_{jk}^T L y_k - \frac{1}{2} \sum_{j,k=1}^{N} y_k^T L^T \xi_{kj},$$

whence the desired relation (A.3) follows. Indeed,

$$\sum_{j,k=1}^{N} \xi_{jk}^T L y_k = - \sum_{j,k=1}^{N} y_k^T L^T \xi_{kj} \overset{\text{(A.2)}}{=} - \sum_{k=1}^{N} u_k^T L y_k.$$

By definition of the class of functions $\mathfrak{S}(\mathcal{F})$ we get $\mathcal{F}(\varphi_{jk}(t, y), y) \geqslant \varkappa(\mu, \nu) \geqslant 0$ for $0 \leqslant \mu \leqslant |\sigma| \leqslant \nu < \infty$, where $\varkappa(\mu, \nu) > 0$ for $\mu > 0$. By assuming in the last inequality that $y = y_k - y_j$ and carrying out simple transformations we establish that

$$\overbrace{\xi_{jk}^T L (y_k - y_j)}^{f} - \overbrace{a_{jk}(t) (y_k - y_j)^T Q (y_k - y_j)}^{g} - \overbrace{\xi_{jk}^T R \xi_{jk}}^{h} \geqslant a_{kj}(t) \varkappa(\mu_{kj}, \nu_{kj})$$

for $\mu_{kj} \leqslant |y_k - y_j| \leqslant \nu_{kj}$. Here, $f = -2 \sum_{j=1}^{N} u_j^T L y_j$ in virtue of (A.2) and (A.3). One can readily see from (2) and condition $Q \geqslant 0$ that $g = \sum_{j,k=1}^{N} a_{kj}(t) \left| Q^{1/2}(y_k - y_j) \right|^2 \geqslant N^{-1} \theta \sum_{j,k=1}^{N} (y_k - y_j)^T \times Q(y_k - y_j)$ and $h \geqslant (N-1)^{-1} \sum_{j=1}^{N} u_j^2$ by inequality (A.1). Consequently, $N^{-1} \mathfrak{P}(\bar{y}(t), \bar{u}(t)) \geqslant \sum_{j,k} a_{kj}(t) \varkappa(\mu_{kj}, \nu_{kj})$. By taking $\mu_{jk} := 0$ and $\nu_{jk} := |y_j - y_k|$, we get the first statement of Lemma 1. To prove the second statement, it suffices to notice that for $\nu \geqslant \max_{j,k} |y_k - y_j| \geqslant \mu$ we have, first, $|y_k - y_j| \leqslant \nu_{jk} := \nu \; \forall j, k$ and, second, $|y_k - y_j| \geqslant \mu_{jk} := \mu/(2N)$ at least for one edge $(j, k) \in E(t)$ because the graph $G(t)$ by Assumption 1 strongly connected, and for the rest of the pairs (j, k) one can assume that $\mu_{jk} := 0$.

Lemma 2. *Let* $\Pi_{\mathcal{F}}(W_y(i\omega)) \geqslant \varepsilon |W_x(i\omega)|^2$ *for some* $\varepsilon \geqslant 0$ *for all* $\omega \in \mathbb{R}$ *such that* $\det(i\omega I - A) \neq 0$. *Then, there exists matrix* $H = H^T > 0$ *such that* $\dot{V}(\bar{x}(t)) + \mathfrak{P}(\bar{y}(t), \bar{u}(t)) \leqslant -\varepsilon \mathcal{W}(\bar{x}(t))$ *for all solutions of system* (5), *where* $V(\bar{x}) := \sum_{j,k=1}^{N} (x_k - x_j)^* H (x_k - x_j)$ *and* $\mathcal{W}(\bar{x})$ *is defined by* (8).

Proof of Lemma 2. Direct computation gives that $\dot{V}(\bar{x}) + \mathfrak{P}(\bar{y}, \bar{u}) + \varepsilon \mathcal{W}(\bar{x}) \leqslant 0$ amounts to satisfying the linear matrix inequality $2x^* H(Ax + Bu) + P(Cx, u) + \varepsilon |x|^2 \leqslant 0 \; \forall x, u$. In virtue of the frequency theorem of [22, 28], this inequality is solvable with respect to the matrix $H = H^T$ if and only if $P(C(i\omega I - A)^{-1} B u, u) \overset{(13)}{=} -\Pi_{\mathcal{F}}(W_y(i\omega)) \leqslant -\varepsilon |W_x(i\omega)|^2$ for all $u \in \mathbb{C}^m$ and $\omega \in \mathbb{R}$ for which $\det(i\omega I - A)^{-1} \neq 0$. The last condition is satisfied in virtue of the assumptions of the lemma, whence it follows that H exists. We prove that $H > 0$.

Let (x_j^+, u_j^+, y_j^+) be solution of system (5) and (6) with the graph $G(t) \equiv G_0$, where G_0 is the complete graph, thee couplings $\varphi_{jk}(t, y) = Ky$, where K is the matrix from (i) of Theorem 1. By Lemma 1, we get $\mathfrak{P}(\bar{y}^+(t), \bar{u}^+(t)) \geqslant 0$; at the same time, $\lim_{t \to +\infty} |x_j^+(t) - x_k^+(t)| = 0$ by definition of the matrix K (see the substantiation of Remark 2). By integrating the inequality $\dot{V}(\bar{x}^+(t)) + \mathfrak{P}(\bar{y}^+(t), \bar{u}^+(t)) \leqslant 0$ we obtain $V(\bar{x}^+(0)) \geqslant 0 \; \forall \bar{x}^+(0) \Rightarrow H \geqslant 0$. Moreover, $V(\bar{x}^+(0)) = 0 \Rightarrow \mathfrak{P}(\bar{y}^+(\cdot), \bar{u}^+(\cdot)) \equiv 0 \Rightarrow y_1^+(\cdot) \equiv \ldots \equiv y_N^+(\cdot) \Rightarrow u_j(\cdot) \equiv 0$; consequently, in virtue of observability $x_1^+(0) = \ldots = x_N^+(0)$. Therefore, $H > 0$.

Proof of Theorem 1. For the function $V(\bar{x}) = x^* H x$ from Lemma 2 (for $\varepsilon := 0$) and any solution of the closed system (5) and (6) we obtain

$$0 \leqslant V(\bar{x}(t)) \leqslant V(\bar{x}(0)) - \int_0^t \mathfrak{P}(\bar{y}(\xi), \bar{u}(\xi)) d\xi, \tag{A.4}$$

where $\mathfrak{P}(\ldots) \geqslant 0$ by Lemma 1 and $H > 0$ by Lemma 2. Whence it follows that $\mathcal{W}(\bar{x}(t)) = \sum_{j,k=1}^N |x_j(t) - x_k(t)|^2 \leqslant \frac{M(H)}{m(H)} \mathcal{W}(\bar{x}(0))$, where $M(H)$ and $m(H)$ are the maximal and minimal eigenvalues of H, respectively. Consequently, (8) is valid. To prove (7), we assume the contrary. Then, there are $\zeta > 0$ and an increasing sequence $t_n \to +\infty$ such that $\max_{j,k} |y_j(t_n) - y_k(t_n)| > 2\zeta$. Since $\sup_{t \geqslant 0} \|\dot{x}_j(t)\| < \infty$, there is $\Delta > 0$ such that $\max_{j,k} |y_j(t) - y_k(t)| > \zeta$ for $|t - t_n| \leqslant \Delta$. By Lemma 1, there exists $\rho > 0$ such that $\mathfrak{P}(\bar{y}(t), \bar{u}(t)) > \rho$ for $|t - t_n| \leqslant \Delta$, which contradicts (A.4) and proves (7) and the first assertion of the theorem guaranteeing the output consensus. To prove the second assertion, we again use Lemma 2, where $\varepsilon > 0$ is the same as in Theorem 1, and notice that apart from (A.4) there exists the inequality $\dot{V}(\bar{x}) + \mathfrak{P} \leqslant -\mu V$ for small $\mu > 0$ from which we get $V(\bar{x}(t)) \leqslant m e^{-\mu t}$. Since $H > 0$, the last estimate implies exponential state consensus.

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